10-2015

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DOI: http://dx.doi.org/10.1080/00949655
Journal of Statistical Computation and Simulation
Vol. 00, No. 00, February 2013, 1–18

Manuscript

Bias-Correction for Weibull Common Shape Estimation

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(*v0.0 released February 2013*)

A general method for correcting the bias of the maximum likelihood estimator (MLE) of the common shape parameter of Weibull populations, allowing a general right censorship, is proposed in this paper. Extensive simulation results show that the new method is very effective in correcting the bias of the MLE, regardless of censoring mechanism, sample size, censoring proportion and number of populations involved. The method can be extended to more complicated Weibull models.

Keywords: Bias correction; Bootstrap; Right censoring; Stochastic expansion; Weibull models

AMS Subject Classification: 62NXX

1. Introduction

The Weibull distribution is a parametric model popular in reliability and biostatistics. It plays a particularly important role for the analysis of failure time data. Suppose the failure time of an item, denoted by T, follows a Weibull distribution $\mathrm{WB}(\alpha, \beta)$. Then the probability density function (pdf) of T has the form: $f(t) = \alpha^{-\beta}\beta t^{\beta-1}\exp\{-(t/\alpha)^{\beta}\}$, $t \geq 0$, where $\alpha > 0$ is the *scale parameter* and $\beta > 0$ is the *shape parameter*. The flexibility (e.g., pdf has many different shapes) and simplicity (e.g., cumulative distribution function has a closed form) are perhaps the main reasons for the popularity of the Weibull distribution. As we know, the shape of the Weibull distribution is determined primarily by its shape parameter β. Therefore, how to estimate β accurately has been one of the most important research focuses since the Weibull literature begun in 1951 (see, e.g., [1–10]). A more interesting and general problem may be the estimation of the common shape parameter of several Weibull populations. As indicated in [11], equality of Weibull shape parameters across different groups of individuals is an important and simplifying assumption in many applications. In Weibull regression models, such an assumption is analogous to the constant variance assumption in normal regression models.

The most common method to estimate the Weibull shape parameter is the maximum likelihood method. However, it is widely recognized that the maximum likelihood estimator (MLE) can be quite biased, in particular when the sample size is small, data are heavily censored, or many Weibull populations are involved. To deal with this problem, Hirose [3] proposed a bias-correction method for a single Weibull population with small

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DOI: http://dx.doi.org/10.1080/00949655

complete samples by expanding the bias as a nonlinear function. In [5], a modified MLE (MMLE) is proposed for the shape parameter of a Weibull population through modifying the profile likelihood. This study was further generalized to the common shape parameter of several Weibull populations [6]. While these methods work well as shown by the Monte Carlo results, they can only be applied to complete, Type I, or Type II censored data. Furthermore, under Type I censored data, it is seen that their methods have room for further improvements, in particular when sample size is small and censorship is heavy.

In this paper, we propose a general method of bias-correction for the MLE of the Weibull common shape parameter that allows a general right censoring mechanism, including Type I censoring, Type II censoring, random censoring, progressive Type II censoring, adaptive Type II progressive censoring, etc. [11, 12]. The method is based on a third-order stochastic expansion for the MLE of the shape parameter [13] and a simple bootstrap procedure for estimating various expectations involved in the expansion [14]. Besides its simplicity, the method is also quite general as it is essentially applicable to any situation where a smooth estimating equation (not necessarily the concentrated score function) for the parameter of interest (common shape in this case) is available. Extensive simulation experiments are designed and carried out to assess the performance of the new method under different types of data. The results show that the new method is generally very effective in correcting the bias of the MLE of β, regardless of censoring mechanism, sample size, censoring proportion and number of groups. Compared with the methods of [6], we see that the proposed method performs equally well under Type II censored data, but better under Type I censored data. Furthermore, the proposed method performs very well under the random censoring mechanism; in contrast, the methods of [6] may not perform satisfactorily when sample size is small and censorship is heavy. This is because they are developed particularly under either Type I or Type II censoring mechanisms. With the new method, the bias-correction can be easily made up to third-order, and more complicated Weibull models can be handled in a similar fashion.

The paper is organized as follows. Section 2 describes the general methodology. Section 3 presents the bias-correction method for the MLE of the common shape parameter of several Weibull populations. Section 4 presents Monte Carlo results. Section 5 presents a real data example and a discussion on some immediate subsequent inference problems. Section 6 concludes the paper.

2. The Method

In studying the finite sample properties of the parameter estimator, say $\hat{\theta}_n$, defined as $\hat{\theta}_n = \arg\{\psi_n(\theta) = 0\}$, where $\psi_n(\theta)$ is a function of the data and the parameter θ with ψ_n and θ having the same dimension (e.g., normalized score function), Rilstone et al. [13] and Bao and Ullah [15] developed a stochastic expansion from which bias-correction on $\hat{\theta}_n$ can be made. Often, the vector of parameters θ contains a set of *linear parameters*, say α, and one *nonlinear parameter*, say β, in the sense that given β, the constrained estimator $\hat{\alpha}_n(\beta)$ of the vector α possesses an explicit expression and the estimation of β has to be done through numerical optimization. In this case, Yang [14] argued that it is more effective to work with the *concentrated estimating equation*: $\tilde{\psi}_n(\beta) = 0$, where $\tilde{\psi}_n(\beta) \equiv \psi_n(\hat{\alpha}_n(\beta), \beta)$, and to perform stochastic expansion and hence bias correction only on the nonlinear estimator defined by

$$\hat{\beta}_n = \arg\{\tilde{\psi}_n(\beta) = 0\}. \quad (1)$$

Doing so, a multi-dimensional problem is reduced to a one-dimensional problem, and the additional variability from the estimation of the 'nuisance' parameters α is taken

DOI: http://dx.doi.org/10.1080/00949655

into account in bias-correcting the estimation of the nonlinear parameter β. Let β_0 be the true value of β and θ_0 the true value of θ. Let $\tilde{\psi}_n \equiv \tilde{\psi}_n(\beta_0)$. For $r = 1, 2, 3$, let $H_{rn}(\beta) = \frac{d^r}{d\beta^r}\tilde{\psi}_n(\beta)$, $H_{rn} \equiv H_{rn}(\beta_0)$, $H^\circ_{rn} = H_{rn} - \mathrm{E}(H_{rn})$, and $\Omega_n = -1/\mathrm{E}(H_{1n})$, where E denotes the expectation corresponding to θ_0. Under some general smoothness conditions on $\tilde{\psi}_n(\beta)$, [14] presented a third-order stochastic expansion for $\hat{\beta}_n$ at β_0,

$$\hat{\beta}_n - \beta_0 = a_{-1/2} + a_{-1} + a_{-3/2} + O_p(n^{-2}), \tag{2}$$

where $a_{-s/2}$, $s = 1, 2, 3$, represent terms of order $O_p(n^{-s/2})$, having the forms

$$a_{-1/2} = \Omega_n\tilde{\psi}_n,$$
$$a_{-1} = \Omega_n H^\circ_{1n} a_{-1/2} + \frac{1}{2}\Omega_n \mathrm{E}(H_{2n})(a^2_{-1/2}), \text{ and}$$
$$a_{-3/2} = \Omega_n H^\circ_{1n} a_{-1} + \frac{1}{2}\Omega_n H^\circ_{2n}(a^2_{-1/2}) + \Omega_n \mathrm{E}(H_{2n})(a_{-1/2}a_{-1}) + \frac{1}{6}\Omega_n \mathrm{E}(H_{3n})(a^3_{-1/2}).$$

The above stochastic expansion leads immediately to a second-order bias $b_2 \equiv b_2(\theta_0) = \mathrm{E}(a_{-1/2} + a_{-1})$, and a third-order bias $b_3 \equiv b_3(\theta_0) = \mathrm{E}(a_{-3/2})$, which may be used for performing bias corrections on $\hat{\beta}_n$, provided that analytical expressions for the various expected quantities in the expansion can be derived so that they can be estimated through a plug-in method. Several applications of this plug-in method to some simple models have appeared in the literature: [15] for a pure spatial autoregressive process, [16] for time-series models, [17] for a Poisson regression model, and [18] for an exponential regression. Except [14], all these works used the joint estimation function $\psi_n(\theta)$ under which $\mathrm{E}(a_{-1/2}) = 0$. In contrast, under the concentrated estimation function $\tilde{\psi}_n(\beta)$, $\mathrm{E}(a_{-1/2}) = O(n^{-1})$ which constitutes an important element in the second-order bias correction. See Section 3 below and [14] for details.

However, for slightly more complicated models such as the Weibull model considered in this paper, $b_2(\theta_0)$ and $b_3(\theta_0)$ typically do not possess analytical expressions and the plug-in method cannot be applied. To overcome this major difficulty, a general nonparametric bootstrap method was proposed in [14] to estimate those expectations, which sheds light on the parametric bootstrap procedure designed in this work. Kundhi and Rilstone [18] considered standard bootstrap correction: bootstrapping $\hat{\beta}_n$ directly for bias-reduction. However, their Monte Carlos results showed that this method does not work as well compared with the analytical method they proposed.

It was argued that in many situations there is a sole nonlinear parameter (like the shape parameter in Weibull models) that is the main source of bias in model estimation, and that given this parameter the estimation of other parameters incurs much less bias and usually can be done analytically too [14]. Thus, for the purpose of bias-correction, it may only be necessary to focus on the estimation of this parameter.

3. Bias-Correction for Weibull Common Shape Estimation

Consider the case of estimating the common shape parameter of several Weibull populations based on the maximum likelihood estimation (MLE) method. For the ith Weibull population $\mathrm{WB}(\alpha_i, \beta)$, $i = 1, 2, \ldots, k$, let t_{ij} ($j = 1, 2, \ldots, n_i$) be the observed failure times or censoring times of n_i randomly selected 'items' from $\mathrm{WB}(\alpha_i, \beta)$, $\delta_{ij}(j = 1, 2, \ldots, n_i)$ be the failure indicators with $\delta_{ij} = 1$ for the actual failure time and $\delta_{ij} = 0$ for the censored time, $r_i = \sum_{j=1}^{n_i} \delta_{ij}$ be the number of observed failure times.

DOI: http://dx.doi.org/10.1080/00949655

Also let $m = \sum_{i=1}^{k} r_i$ be the total number of observed failure times in all k samples and $n = \sum_{i=1}^{k} n_i$ be the total number of items. Let $\theta = (\alpha_1, \ldots, \alpha_k, \beta)'$. The log-likelihood function can be written as

$$\ell_n(\theta) = m \log\ \beta + (\beta - 1) \sum_{i=1}^{k} \sum_{j=1}^{n_i} \delta_{ij} \log t_{ij} - \beta \sum_{i=1}^{k} r_i \log \alpha_i - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left\{ \frac{t_{ij}}{\alpha_i} \right\}^{\beta}. \quad (3)$$

Maximizing $\ell_n(\theta)$ with respect to α_i $(i = 1, 2, \ldots, k)$ gives the constrained MLEs:

$$\hat{\alpha}_{n,i}(\beta) = \left\{ \frac{1}{r_i} \sum_{j=1}^{n_i} t_{ij}^{\beta} \right\}^{1/\beta}, i = 1, \ldots, k. \quad (4)$$

Substituting the $\hat{\alpha}_{n,i}(\beta)$ back into (3) yields the concentrated log-likelihood function

$$\ell_n^c(\beta) = \sum_{i=1}^{k} r_i \log r_i - m + m \log \beta + (\beta - 1) \sum_{i=1}^{k} \sum_{j=1}^{n_i} \delta_{ij} \log t_{ij} - \sum_{i=1}^{k} r_i \log \sum_{j=1}^{n_i} t_{ij}^{\beta}. \quad (5)$$

Maximizing $\ell_n^c(\beta)$, or equivalently solving $\tilde{\psi}_n(\beta) \equiv n^{-1} \frac{d}{d\beta} \ell_n^c(\beta) = 0$, where

$$\tilde{\psi}_n(\beta) \equiv \frac{m}{n\beta} + \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \delta_{ij} \log t_{ij} - \frac{1}{n} \sum_{i=1}^{k} r_i \left\{ \frac{\sum_{j=1}^{n_i} t_{ij}^{\beta} \log t_{ij}}{\sum_{j=1}^{n_i} t_{ij}^{\beta}} \right\}, \quad (6)$$

gives the unconstrained MLE $\hat{\beta}_n$ of β, and hence the unconstrained MLEs of α_i as $\hat{\alpha}_{n,i} \equiv \hat{\alpha}_{n,i}(\hat{\beta}_n)$, $i = 1, 2, \ldots, k$. See [11] for details on maximum likelihood estimation of censored failure time models, including Weibull models.

It is well known that the MLE $\hat{\beta}_n$ can be significantly biased for small sample sizes, heavy censorship, or complicated Weibull models. Such a bias would make the subsequent statistical inferences inaccurate. Various attempts have been made to reduce the bias of the Weibull shape estimation. The approach adopted by [5] and [6] is to modify the concentrated log-likelihood defined in (5). Alternative methods can be found in, e.g., [2, 7, 9].

As discussed in the introduction, the approach of [5] and [6] applies only to Type I and Type II right censored data. Apparently, the log-likelihood function (3) is not restricted to these two types of censoring. Random censoring and progressive Type II censoring, etc, are also included [11]. It is thus of a great interest to develop a general method that works for more types of censoring mechanisms. In theory, the method outlined in Section 2 indeed works for any type of censoring mechanism as long as the log-likelihood function possesses explicit derivatives up to fourth-order. In this paper, we focus on the general right censoring mechanism for estimating the common shape parameter of Weibull populations, where the concentrated estimating function for the common shape parameter is defined in (6).

DOI: http://dx.doi.org/10.1080/00949655

3.1 *The 2nd and 3rd-order bias corrections*

Note that the function $\tilde{\psi}_n(\beta)$ defined in (6) is of order $O_p(n^{-1/2})$ for regular MLE problems. Let $H_{rn}(\beta) = \frac{d^r}{d\beta^r}\tilde{\psi}_n(\beta), r = 1, 2, 3$. We have after some algebra,

$$H_{1n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(-\frac{1}{\beta^2} - \frac{\Lambda_{2i}}{T_i} + \frac{\Lambda_{1i}^2}{T_i^2}\right),$$

$$H_{2n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(\frac{2}{\beta^3} - \frac{\Lambda_{3i}}{T_i} + \frac{3\Lambda_{1i}\Lambda_{2i}}{T_i^2} - \frac{2\Lambda_{1i}^3}{T_i^3}\right),$$

$$H_{3n}(\beta) = \sum_{i=1}^{k} \frac{r_i}{n}\left(-\frac{6}{\beta^4} - \frac{\Lambda_{4i}}{T_i} + \frac{4\Lambda_{1i}\Lambda_{3i}}{T_i^2} + \frac{3\Lambda_{2i}^2}{T_i^2} - \frac{12\Lambda_{1i}^2\Lambda_{2i}}{T_i^3} + \frac{6\Lambda_{1i}^4}{T_i^4}\right),$$

where $T_i \equiv T_i(\beta) = \sum_{j=1}^{n_i} t_{ij}^{\beta}$, and $\Lambda_{si} \equiv \Lambda_{si}(\beta) = \sum_{j=1}^{n_i} t_{ij}^{\beta}(\log t_{ij})^s$, $s = 1, 2, 3, 4$, $i = 1, \ldots, k$. The validity of the stochastic expansion (2) depends crucially on the $\sqrt{n}$-consistency of $\hat{\beta}_n$, and the proper stochastic behavior of the various ratios of the quantities T_i and Λ_{si}. Along the lines of the general results of [14], the following set of simplified regularity conditions is sufficient.

Assumption 1. *The true β_0 is an interior point of an open subset of the real line.*

Assumption 2. *$\frac{1}{n}\ell_n^c(\beta)$ converges in probability to a nonstochastic function $\ell(\beta)$ uniformly in β in an open neighborhood of β_0, and $\ell(\beta)$ attains the global maximum at β_0.*

Assumption 3. *(i) $\lim_{n\to\infty} \mathrm{Var}\left(\sqrt{n}\tilde{\psi}_n(\beta_0)\right)$ exists and (ii) $H_{1n}(\tilde{\beta}_n) \xrightarrow{p} c(\beta_0)$, $-\infty < c(\beta_0) < 0$, for any sequence $\tilde{\beta}_n$ such that $\tilde{\beta}_n \xrightarrow{p} \beta_0$.*

Assumptions 1-3 are sufficient conditions for the $\sqrt{n}$-consistency of $\hat{\beta}_n$ (see [19]). Clearly, Assumption 3 (*ii*) requires that the (expected) number of observed failures times ($\mathrm{E}(m)$ or m) approaches infinity at rate n as $n \to \infty$ ([11], p. 62). The assumptions given below ensure the proper behaviors of the higher order terms.

Assumption 4. *For each $i \in (1, \ldots, k)$ and $s \in (1, 2, 3, 4)$, (i) $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{1i}^5(\beta_0)}{T_i^5(\beta_0)}]$, $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{2i}^2(\beta_0)}{T_i^2(\beta_0)}]$, $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{3i}^2(\beta_0)}{T_i^2(\beta_0)}]$, and $\mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{4i}(\beta_0)}{T_i(\beta_0)}]$ exist; (ii) $\frac{r_i}{n}\frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)} = \mathrm{E}[\frac{r_i}{n}\frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)}] + O_p(n^{-\frac{1}{2}})$; and (iii) $\frac{r_i}{n}\left|\frac{\Lambda_{si}(\beta)}{T_i(\beta)} - \frac{\Lambda_{si}(\beta_0)}{T_i(\beta_0)}\right| = |\beta-\beta_0|X_{n,is}$, for β in a neighborhood of β_0 and $\mathrm{E}|X_{n,is}| < c_{is} < \infty$.*

Theorem 3.1. *Under Assumptions 1-4, we have, respectively, the 2nd-order ($O(n^{-1})$) bias and the 3rd-order ($O(n^{-3/2})$) bias for the MLE $\hat{\beta}_n$ of the shape parameter β_0:*

$$b_2(\theta_0) = 2\Omega_n\mathrm{E}(\tilde{\psi}_n) + \Omega_n^2\mathrm{E}(H_{1n}\tilde{\psi}_n) + \frac{1}{2}\Omega_n^3\mathrm{E}(H_{2n})\mathrm{E}(\tilde{\psi}_n^2), \tag{7}$$

$$\begin{aligned} b_3(\theta_0) = {}& \Omega_n\mathrm{E}(\tilde{\psi}_n) + 2\Omega_n^2\mathrm{E}(H_{1n}\tilde{\psi}_n) + \Omega_n^3\mathrm{E}(H_{2n})\mathrm{E}(\tilde{\psi}_n^2) + \Omega_n^3\mathrm{E}(H_{1n}^2\tilde{\psi}_n), \\ & + \frac{1}{2}\Omega_n^3\mathrm{E}(H_{2n}\tilde{\psi}_n^2) + \frac{3}{2}\Omega_n^4\mathrm{E}(H_{2n})\mathrm{E}(H_{1n}\tilde{\psi}_n^2) + \frac{1}{2}\Omega_n^5(\mathrm{E}(H_{2n}))^2\mathrm{E}(\tilde{\psi}_n^3) \\ & + \frac{1}{6}\Omega_n^4\mathrm{E}(H_{3n})\mathrm{E}(\tilde{\psi}_n), \end{aligned} \tag{8}$$

where $\tilde{\psi}_n \equiv \tilde{\psi}_n(\beta_0)$, $H_{rn} \equiv H_{rn}(\beta_0)$, $r = 1, 2, 3$, and $\Omega_n = -1/\mathrm{E}(H_{1n})$.

DOI: http://dx.doi.org/10.1080/00949655

The proof of Theorem 3.1 is given in Appendix A. As noted in [14], $\tilde{\psi}_n(\beta)$ represents the concentrated estimating equation, which incorporates the extra variability resulted from the estimation of the nuisance parameters α_i's, hence, $\mathrm{E}[\tilde{\psi}_n(\beta_0)] \neq 0$ at the true value β_0 of β. We show in the proof of Theorem 3.1 in Appendix A that $\mathrm{E}[\tilde{\psi}_n(\beta_0)] = O(n^{-1})$. This is in contrast to the case of using the joint estimating equation, $\psi_n(\theta) = 0$, introduced at the beginning of Section 2, for which we have $\mathrm{E}[\psi_n(\theta_0)] = 0$.

The above equations (7) and (8) lead immediately to the second- or third-order bias-corrected MLEs of β as

$$\hat{\beta}_n^{bc2} = \hat{\beta}_n - \hat{b}_2 \qquad \textit{and} \qquad \hat{\beta}_n^{bc3} = \hat{\beta}_n - \hat{b}_2 - \hat{b}_3, \tag{9}$$

provided that the estimates, $\hat{b}_2$ and $\hat{b}_3$, of the bias terms $b_2 \equiv b_2(\theta_0)$ and $b_3 \equiv b_3(\theta_0)$ are readily available, and that they are valid in the sense that the estimation of the biases does not introduce extra variability that is higher than the remainder. Obviously, the analytical expressions of b_2 and b_3 are not available, and hence the usual 'plug-in' method does not work. We introduce a parametric bootstrap method to overcome this difficulty and give formal justifications on its validity in next section.

Substituting (9) back into (4) gives the corresponding estimators of scale parameters as $\hat{\alpha}_n^{bc2} \equiv \hat{\alpha}_n(\hat{\beta}_n^{bc2})$ and $\hat{\alpha}_n^{bc3} \equiv \hat{\alpha}_n(\hat{\beta}_n^{bc3})$. If there is only one complete sample composed of n failure times, denoted by $t_j, j = 1, \ldots, n$, the associated concentrated estimating function $\tilde{\psi}_n$ reduces to

$$\tilde{\psi}_n(\beta) = \frac{1}{\beta} + \frac{1}{n}\sum_{j=1}^{n} \log\ t_j - \Big(\sum_{j=1}^{n} t_j^{\beta}\Big)^{-1} \sum_{j=1}^{n} t_j^{\beta} \log t_j.$$

Similarly, the quantities $H_{1n}(\beta)$,$H_{2n}(\beta)$ and $H_{3n}(\beta)$ reduce to

$$\begin{aligned}
H_{1n}(\beta) &= -\frac{1}{\beta^2} - \frac{\Lambda_2}{T} + \frac{\Lambda_1^2}{T^2}, \\
H_{2n}(\beta) &= \frac{2}{\beta^3} - \frac{\Lambda_3}{T} + \frac{3\Lambda_1\Lambda_2}{T^2} - \frac{2\Lambda_{1n}^3}{T_n^3}, \\
H_{3n}(\beta) &= -\frac{6}{\beta^4} - \frac{\Lambda_4}{T} + \frac{4\Lambda_1\Lambda_3}{T^2} + \frac{3\Lambda_2^2}{T^2} - \frac{12\Lambda_1^2\Lambda_2}{T^3} + \frac{6\Lambda_1^4}{T^4},
\end{aligned}$$

respectively, where $T \equiv T(\beta) = \sum_{j=1}^{n} t_j^{\beta}$, and $\Lambda_s \equiv \Lambda_s(\beta) = \sum_{j=1}^{n} t_j^{\beta}(\log t_j)^s$, $s = 1, 2, 3, 4$.

3.2 *The bootstrap method for practical implementations*

A typical way of obtaining the estimate of the bias term is to find its analytical expression, and then plug-in the estimates for the parameters [13, 15, 16]. However, this approach often runs into difficulty if more complicated models are considered, simply because this analytical expression is either unavailable, or difficult to obtain, or too tedious to be practically tractable. Apparently, the problem we are considering falls into the first category. This indicates that the usefulness of stochastic expansions in conducting bias correction is rather limited if one does not have a general method for estimating the expectations of various quantities involved in the expansions. Thus, alternative methods are desired. In working with the bias-correction problem for a general spatial autoregressive model, [14] proposed a simple but rather general nonparametric bootstrap method,

DOI: http://dx.doi.org/10.1080/00949655

leading to bias-corrected estimators of the spatial parameter that are nearly unbiased.

The situation we are facing now is on one hand simpler than that of [14] in that the distribution of the model is completely specified, but on the other hand more complicated in that various censoring mechanisms are allowed. Following the general idea of [14] and taking advantage of a known distribution, we propose a parametric bootstrap method for estimating the expectations involved in (7) and (8):

(1) Compute the MLEs $\hat{\beta}_n$ and $\hat{\alpha}_{n,i}, i = 1, \ldots, k$, based on the original data;
(2) From each Weibull population $\mathrm{WB}(\hat{\alpha}_{n,i}, \hat{\beta}_n), i = 1, \ldots, k$, generate n_i random observations, censor them according to the original censoring mechanism, and denote the generated (bootstrapped) data as $\{t^b_{ij}, j = 1, \ldots, n_i, i = 1, \ldots, k\}$;
(3) Compute $\tilde{\psi}_{n,b}(\hat{\beta}_n)$, $H_{1n,b}(\hat{\beta}_n)$, $H_{2n,b}(\hat{\beta}_n)$, and $H_{3n,b}(\hat{\beta}_n)$ based on the bootstrapped data $\{t^b_{ij}, j = 1, \ldots, n_i, i = 1, \ldots, k\}$;
(4) Repeat the steps (2)-(3) B times ($b = 1, \ldots, B$) to get sequences of bootstrapped values for $\tilde{\psi}_n(\hat{\beta}_n)$, $H_{1n}(\hat{\beta}_n)$, $H_{2n}(\hat{\beta}_n)$, and $H_{3n}(\hat{\beta}_n)$.

The bootstrap estimates of various expectations in (7) and (8) thus follow. For example, the bootstrap estimates for $\mathrm{E}(\tilde{\psi}_n^2)$ and $\mathrm{E}(H_{1n}\tilde{\psi}_n)$ are, respectively,

$$\hat{\mathrm{E}}(\tilde{\psi}_n^2) = \tfrac{1}{B}\textstyle\sum_{b=1}^B [\tilde{\psi}_{n,b}(\hat{\beta}_n)]^2 \quad \text{and} \quad \hat{\mathrm{E}}(H_{1n}\tilde{\psi}_n) = \tfrac{1}{B}\textstyle\sum_{b=1}^B H_{1n,b}(\hat{\beta}_n)\tilde{\psi}_{n,b}(\hat{\beta}_n).$$

The estimates of other expectations can be obtained in a similar fashion, and hence the bootstrap estimates of the second- and third-order biases, denoted as $\hat{b}_2$ and $\hat{b}_3$. The step (2) above aims to make bootstrapped data mimic the original data with respect to censoring pattern, such as censoring time, number of censored data, etc.

Corollary 3.2. *Under Assumptions 1-4, if further (i)* $\frac{\partial^k}{\partial \theta_0^r} b_j(\theta_0) \sim b_j(\theta_0), r = 1, 2, j = 2, 3$, *(ii)* r_i *or* $\mathrm{E}(r_i)$ *approaches infinity at rate* n *as* $n \to \infty$, $i = 1 \ldots, k$, *and (iii) a quantity bounded in probability has a finite expectation, then the bootstrap estimates of the 2nd- and 3rd-order biases for the MLE* $\hat{\beta}_n$ *are such that:*

$$\hat{b}_2 = b_2 + O_p(n^{-2}) \ \ \textit{and} \ \ \hat{b}_3 = b_3 + O_p(n^{-5/2}),$$

where $\sim$ *indicates that the two quantities are of the same order of magnitude. It follows that Bias*$(\hat{\beta}_n^{\mathrm{bc2}}) = O(n^{-3/2})$ *and Bias*$(\hat{\beta}_n^{\mathrm{bc3}}) = O(n^{-2})$.

The results of Corollary 3.2 say that estimating the bias terms using the bootstrap method only (possibly) introduces additional bias of order $O_p(n^{-2})$ or higher. This makes the third-order bootstrap bias correction valid. Thus, the validity of the second-order bootstrap bias correction follows. Assumption *(ii)* stated in the corollary ensures that each $\hat{\alpha}_{n,i}$ is $\sqrt{n}$-consistent, and Assumption *(iii)* is to ensure $\mathrm{E}[O_p(1)] = O(1)$, $\mathrm{E}[O_p(n^{-2})] = O(n^{-2})$, etc., so that the expectation of a 'stochastic' remainder is of proper order. See the proof of Corollary 3.2 given in the Appendix B.

4. Monte Carlo Simulation

To investigate the finite sample performance of the proposed method of bias-correcting the MLE of the Weibull common shape parameter, extensive Monte Carlo simulations are performed. Tables 1-12 summarize the empirical mean, root-mean-square-error (rmse) and standard error (se) of the original and bias-corrected MLEs under various combinations of models, censoring schemes, and the values of n_i, α, β and p, where p denotes the

DOI: http://dx.doi.org/10.1080/00949655

Table 1. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
10	0.5	0.584 [.198](.179)	0.500 [.154](.154)	0.500 [.154](.154)	0.508 [.155](.155)
	0.8	0.934 [.311](.281)	0.800 [.241](.241)	0.800 [.241](.241)	0.812 [.244](.244)
	1.0	1.164 [.384](.347)	0.997 [.298](.298)	0.997 [.298](.298)	1.012 [.302](.301)
	2.0	2.344 [.795](.717)	2.008 [.615](.615)	2.008 [.615](.615)	2.037 [.622](.621)
	5.0	5.859 [1.95](1.75)	5.020 [1.50](1.50)	5.020 [1.50](1.50)	5.094 [1.52](1.52)
20	0.5	0.539 [.111](.104)	0.502 [.097](.097)	0.502 [.097](.097)	0.505 [.098](.097)
	0.8	0.863 [.177](.166)	0.803 [.154](.154)	0.804 [.154](.154)	0.809 [.155](.155)
	1.0	1.074 [.215](.201)	0.999 [.188](.188)	1.000 [.188](.188)	1.006 [.189](.189)
	2.0	2.152 [.435](.407)	2.001 [.379](.379)	2.004 [.379](.379)	2.016 [.382](.381)
	5.0	5.375 [1.10](1.03)	4.998 [.959](.959)	5.004 [.960](.960)	5.035 [.966](.965)
50	0.5	0.514 [.060](.058)	0.499 [.057](.057)	0.500 [.057](.057)	0.501 [.057](.057)
	0.8	0.823 [.097](.095)	0.800 [.092](.092)	0.800 [.092](.092)	0.802 [.092](.092)
	1.0	1.028 [.122](.118)	1.000 [.115](.115)	1.000 [.115](.115)	1.003 [.115](.115)
	2.0	2.060 [.244](.236)	2.003 [.230](.230)	2.004 [.230](.230)	2.009 [.230](.230)
	5.0	5.141 [.595](.579)	4.998 [.563](.563)	5.000 [.563](.563)	5.014 [.564](.564)
100	0.5	0.507 [.041](.040)	0.500 [.040](.040)	0.500 [.040](.040)	0.501 [.040](.040)
	0.8	0.812 [.066](.065)	0.800 [.064](.064)	0.801 [.064](.064)	0.802 [.064](.064)
	1.0	1.015 [.082](.080)	1.001 [.079](.079)	1.001 [.079](.079)	1.002 [.079](.079)
	2.0	2.030 [.163](.160)	2.002 [.158](.158)	2.002 [.158](.158)	2.005 [.158](.158)
	5.0	5.070 [.410](.404)	5.000 [.398](.398)	5.000 [.398](.398)	5.008 [.399](.399)

non-censoring proportion. We consider four scenarios: (i) complete samples, (ii) Type I censored samples, (iii) Type II censored samples, and (iv) randomly censored samples. Under each scenario, the numbers of groups considered are $k = 1, 2$ and 8; accordingly, the values of α_i's are set to be 1, (1, 2) and (1, 2, 3, 4, 5, 6, 7, 8). Furthermore, we also compare the proposed method with the modified MLE (MMLE) discussed in [5] and [6].

In the entire simulation study, the parametric bootstrapping procedure is adopted, which (i) fits original data to Weibull model, (ii) draws random samples from this fitted distribution with the size being the same as the original sample size, and then (iii) censors the data in the identical way as the original data. For all the experiments, 10,000 replications are run in each simulation and the number of bootstrap B is set to be 699, following, e.g., [20]. Also, for convenience, the values of p are set as 0.3, 0.5 and 0.7, respectively, so that $n_i p$ are integers for n_i = 10, 20, 50 and 100, respectively.

4.1 *Complete samples*

Tables 1-3 present the results corresponding to the cases of complete samples with $k = 1, 2, 8$, respectively. From the tables, we see that the second-order and third-order bias-corrected MLEs, $\hat{\beta}^{bc2}_n$ and $\hat{\beta}^{bc3}_n$, are generally nearly unbiased and are much superior to the original MLE $\hat{\beta}_n$ regardless of the values of n and k. Some details are: (i) $\hat{\beta}_n$ always over-estimates the shape parameter, (ii) $\hat{\beta}^{bc2}_n$ and $\hat{\beta}^{bc3}_n$ have smaller rmses and ses compared with those of $\hat{\beta}_n$, (iii) the second-order bias-correction seems sufficient and a higher order bias correction may not be necessary, at least for the cases considered in this work, (iv) $\hat{\beta}^{bc2}_n$ and $\hat{\beta}^{bc3}_n$ are generally better than the MMLE of [6], except $n_i = 10$, $k = 8$, and (v) the estimation results do not depend on the true values of α_i's.

4.2 *Type I censoring*

Type I censoring is a type of right censoring that has a predetermined time C such that T_j is observed if $T_j \leq C$, otherwise only $T_j > C$ is known. In our experiment, the way to generate original Type I censored data in one replication is as follows. For a Weibull population with given parameters and a given non-censoring proportion p, generate a random sample $\{T_1, \ldots, T_n\}$ and set the censoring time C as the pth quantile of Weibull distribution. Then the desired sample data, either observed failure time or censored time, are obtained by $\min(T_j, C)$ and the failure indicators are $\delta_j = 1$ if $T_j < C$

DOI: http://dx.doi.org/10.1080/00949655

Table 2. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 2$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
10	0.5	0.556 [.123](.110)	0.497 [.099](.098)	0.495 [.098](.098)	0.502 [.099](.099)
	0.8	0.895 [.204](.180)	0.800 [.161](.161)	0.797 [.161](.161)	0.808 [.163](.163)
	1.0	1.117 [.250](.221)	0.999 [.198](.198)	0.995 [.197](.197)	1.008 [.199](.199)
	2.0	2.230 [.492](.435)	1.994 [.390](.390)	1.986 [.388](.388)	2.013 [.392](.392)
	5.0	5.588 [1.26](1.11)	4.995 [.995](.995)	4.975 [.992](.991)	5.043 [1.00](1.00)
20	0.5	0.528 [.075](.070)	0.501 [.066](.066)	0.501 [.066](.066)	0.503 [.066](.066)
	0.8	0.845 [.119](.110)	0.802 [.104](.104)	0.801 [.104](.104)	0.805 [.105](.105)
	1.0	1.053 [.149](.140)	0.999 [.133](.133)	0.999 [.132](.132)	1.003 [.133](.133)
	2.0	2.110 [.297](.276)	2.002 [.262](.262)	2.001 [.262](.262)	2.011 [.263](.263)
	5.0	5.257 [.720](.673)	4.988 [.639](.639)	4.985 [.639](.638)	5.009 [.641](.641)
50	0.5	0.511 [.043](.041)	0.501 [.040](.040)	0.501 [.040](.040)	0.502 [.040](.040)
	0.8	0.817 [.067](.065)	0.801 [.064](.064)	0.801 [.064](.064)	0.802 [.064](.064)
	1.0	1.021 [.085](.082)	1.000 [.081](.081)	1.000 [.081](.081)	1.002 [.081](.081)
	2.0	2.040 [.169](.164)	1.999 [.161](.161)	1.999 [.161](.161)	2.003 [.161](.161)
	5.0	5.105 [.414](.400)	5.003 [.393](.393)	5.002 [.393](.393)	5.011 [.393](.393)
100	0.5	0.505 [.028](.028)	0.500 [.028](.028)	0.500 [.028](.028)	0.500 [.028](.028)
	0.8	0.808 [.046](.045)	0.800 [.045](.045)	0.800 [.045](.045)	0.801 [.045](.045)
	1.0	1.010 [.057](.056)	1.000 [.055](.055)	1.000 [.055](.055)	1.001 [.055](.055)
	2.0	2.019 [.114](.113)	1.999 [.112](.112)	1.999 [.112](.112)	2.001 [.112](.112)
	5.0	5.051 [.287](.282)	5.000 [.280](.280)	5.000 [.280](.280)	5.004 [.279](.279)

Table 3. Empirical mean [rmse](se) of MLE-type estimators of β, complete data, $k = 8$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
10	0.5	0.540 [.065](.051)	0.498 [.047](.047)	0.497 [.047](.047)	0.501 [.047](.047)
	0.8	0.865 [.104](.082)	0.797 [.075](.075)	0.795 [.075](.075)	0.802 [.076](.076)
	1.0	1.080 [.129](.102)	0.995 [.094](.094)	0.993 [.094](.094)	1.001 [.095](.095)
	2.0	2.162 [.263](.207)	1.993 [.191](.191)	1.989 [.191](.191)	2.004 [.192](.192)
	5.0	5.395 [.642](.506)	4.973 [.468](.467)	4.964 [.468](.467)	5.002 [.469](.469)
20	0.5	0.519 [.038](.034)	0.499 [.032](.032)	0.499 [.032](.032)	0.500 [.032](.032)
	0.8	0.830 [.061](.053)	0.799 [.051](.051)	0.799 [.051](.051)	0.801 [.051](.051)
	1.0	1.038 [.077](.067)	1.000 [.065](.065)	0.999 [.065](.065)	1.001 [.065](.065)
	2.0	2.076 [.153](.133)	1.998 [.128](.128)	1.998 [.128](.128)	2.002 [.128](.128)
	5.0	5.188 [.382](.333)	4.995 [.321](.321)	4.993 [.321](.321)	5.004 [.321](.321)
50	0.5	0.507 [.021](.020)	0.500 [.020](.020)	0.500 [.020](.020)	0.500 [.020](.020)
	0.8	0.812 [.034](.032)	0.800 [.032](.032)	0.800 [.032](.032)	0.800 [.032](.032)
	1.0	1.014 [.043](.040)	1.000 [.040](.040)	1.000 [.040](.040)	1.000 [.040](.040)
	2.0	2.030 [.087](.081)	2.001 [.080](.080)	2.000 [.080](.080)	2.002 [.080](.080)
	5.0	5.074 [.216](.203)	5.001 [.200](.200)	5.001 [.200](.200)	5.004 [.200](.200)
100	0.5	0.504 [.014](.014)	0.500 [.014](.014)	0.500 [.014](.014)	0.500 [.014](.014)
	0.8	0.806 [.023](.023)	0.800 [.023](.023)	0.800 [.023](.023)	0.800 [.022](.022)
	1.0	1.007 [.029](.028)	1.000 [.028](.028)	1.000 [.028](.028)	1.000 [.028](.028)
	2.0	2.015 [.058](.056)	2.000 [.056](.056)	2.000 [.056](.056)	2.001 [.056](.056)
	5.0	5.035 [.145](.141)	4.999 [.140](.140)	4.999 [.140](.140)	5.000 [.140](.140)

and 0 otherwise. Based on the sample data, the estimates for Weibull parameters can be calculated.

The bootstrap samples are generated in a similar way, but from the estimated Weibull distribution obtained from the original sample data, and with the censoring time set to be the (r_i/n_i)th quantile of the estimated distribution rather than the given censoring time in the original sample. This is because, for the bootstrap procedure to work, the bootstrap world must be set up so that it mimics the real world. Following this fundamental principle, the same censoring mechanism should be followed for generating the bootstrap sample.

Tables 4-6 reports Monte Carlo results for comparing the four estimators under Type I censoring, where p denotes the proportion of data that are observed in a sample (non-censoring proportion). From Tables 4-6, we observe that the two bias-corrected estimators $\hat{\beta}^{bc2}_n$ and $\hat{\beta}^{bc3}_n$, in particular the former, have much better finite sample performances in terms of the empirical mean, rmse and se, than the original MLE $\hat{\beta}_n$ in almost all combinations simulated. $\hat{\beta}^{bc2}_n$ dramatically reduces the bias of $\hat{\beta}_n$ and thus provides much more accurate estimation. $\hat{\beta}^{bc3}_n$ may not perform as well as $\hat{\beta}^{bc2}_n$ in cases of small samples and heavy censorship due to its higher demand on numerical stability. Generally, the

DOI: http://dx.doi.org/10.1080/00949655

sample size, the non-censoring proportion and the number of groups do not affect the performance of $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$ much. Furthermore, the two bias-corrected estimators have smaller variability compared with $\hat{\beta}_n$.

As pointed out in [6], with several Type I censored samples, the MMLE does not perform as well as with complete and Type II censored data. Here we see again that the MMLE can have an obvious positive bias for $k = 1, 2$. In contrast, the proposed bias-corrected MLEs, especially $\hat{\beta}_{bc2}$, show a very good performance under the Type I censoring scheme, unless many small and heavily censored samples are involved. Thus, to deal with Type I censored data, the bias-corrected MLEs is strongly recommended.

4.3 *Type II censoring*

Type II censoring schemes arise when all items involved in an experiment start at the same time, and the experiment terminates once a predetermined number of failures is observed. To generate Type II censored data, first a random sample is drawn from each of k Weibull distributions. Consider $k = 1$. For a given sample $\{T_1, \ldots, T_n\}$ from a population, set the censoring time $C = T_{(np)}$, the (np)th smallest value of $\{T_1, \ldots, T_n\}$ with p being the non-censoring proportion which in this case is the ratio of the number of observed failure times r to the sample size n. Thus, the observed Type II censored lifetimes are $\{\min(T_j, C), j = 1 \ldots, n\}$, or $\{T_{(1)}, \ldots, T_{(np)}, T_{(np)}, \ldots\}$, and the associated failure indicators are such that $\delta_j = 1$ if $T_j < C$ and 0 otherwise. Following a similar manner as the original data being generated, Type II censored bootstrap samples are obtained based on the estimated Weibull populations and with the same predetermined number of failures np. The bootstrap censoring time C^* is the (np)th smallest order statistic of the n random data generated from the estimated Weibull distribution.

The experiment results reported in Tables 7-9 show that the estimation is in general greatly improved after bias correction, as the bias-corrected estimators $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$ have means much closer to the true β_0 than the MLE $\hat{\beta}_n$, as well as the smaller rmses and standard errors. Again, the simulation results indicate that the use of the second-order bias-corrected MLE $\hat{\beta}_n^{bc2}$ seems sufficient for this situation as well. Although the third-order estimator $\hat{\beta}_n^{bc3}$ has the smallest errors, its performance in terms of bias may not be as good as that for the cases of small sample and heavy censorship.

The two bias-corrected estimators are superior to the original MLE, but they do not outperform the MMLE in regard of bias, in particular when $k = 2, 8$ and $n_i = 10, 20$. This implies that the MMLE is preferred in the case of Type II censoring. Occasionally, the MMLE would be slightly less efficient, but overall, it is a better choice than others in cases of Type II censored data.

4.4 *Random censoring*

We also consider the case of samples with random censoring, which extends Type I censoring by treating censoring time as a random variable rather than a fixed time. In random censoring schemes, each item is subject to a different censoring time. To simplify the descriptions for the data generating procedure, we assume $k = 1$. For each Monte Carlo replication, two sets of observations $T = \{T_1, \ldots, T_n\}$ and $C = \{C_1, \ldots, C_n\}$ are generated, with T_j from a Weibull distribution and C_j from any proper distribution. In this paper, a Uniform distribution $\text{U}(0.5q_p, 1.5q_p)$, with q_p the pth quantile of the Weibull population involved, is chosen to describe the randomness of censoring times, considering its simple formulation and easy-handling. Then the observed lifetimes $Y = \{\min(T_j, C_j), j = 1 \ldots, n\}$ and the failure indicators $\{\delta_j\}$ are recorded.

To generate a bootstrap sample, $Y_j^*, j = 1, \ldots, n$, we follow a procedure given in [21]:

DOI: http://dx.doi.org/10.1080/00949655

1) Generate $T_1^*, \ldots, T_n^*$ independently from $WB(\hat{\alpha}, \hat{\beta})$ where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs;
2) Sort the observed lifetimes Y as $(Y_{(1)}, ..., Y_{(n)})$ and denote the corresponding failure indicators by $(\delta_{(1)}, ..., \delta_{(n)})$;
3) If $\delta_{(j)} = 0$, then the bootstrap censoring time $C_j^* = Y_{(j)}$; otherwise C_j^* is a value randomly chosen from $(Y_{(j)}, ..., Y_{(n)})$;
4) Set $Y_j^* = \min(T_j^*, C_j^*)$, for $j = 1, \ldots, n$.

Monte Carlo results are summarized in Tables 10-12. From the results we see that the two bias-corrected MLEs $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$ can reduce greatly the bias as well as the variability of $\hat{\beta}_n$ in all combinations under the random censoring mechanism. Different from the previous censoring scenarios, $\hat{\beta}_n^{bc3}$ offers improvements over $\hat{\beta}_n^{bc2}$ for the cases of small n and $k = 1$, and performs well in general.

Since there is no specific MMLE designed for the randomly censored data, the MMLEs designed for Type I and Type II censoring schemes are included in the Monte Carlo experiments for comparison purpose. The two estimators are denoted, respectively, by MMLE-I and MMLE-II. Indeed, from the results we see that under a 'wrong censoring scheme', the MMLEs do not seem to be able to provide a reliable estimation of β_0. In general, MMLE-I always over-estimates β_0 when it is large, whereas MMLE-II tends to under-estimate β_0 when it is small. Based on these observations, we may conclude that the proposed method is a good choice when dealing with randomly censored data.

We end this section by offering some remarks.

Remark 4.1. Since Type I censoring can be treated as a special case of random censoring with a degenerated censoring time distribution, we also used random censoring mechanism to generate bootstrap samples from Type I censored data. The Monte Carlo results (not reported for brevity) show that the bias-corrected MLEs still perform well, comparable with those in Section 4.2 when Type I censored bootstrap samples are provided, and that they also outperform the MMLE-I.

Remark 4.2. Theoretically, $\hat{\beta}_n^{bc3}$ should perform better than $\hat{\beta}_n^{bc2}$. This is clearly observed from the Monte Carlo results corresponding to the random censoring mechanism. However, in cases of Type I and II censored data with a very small sample and a very heavy censorship, the performance of $\hat{\beta}_n^{bc3}$ may deteriorate slightly due to the numerical instability in the third-order correction term. In these situations, it is recommended to use the second-order bias-corrected MLEs when the sample size is very small and the censorship is very heavy.

Remark 4.3. Apparently, the proposed bootstrap bias-corrected MLEs are computationally more demanding than the regular MLE or MMLEs. However, our results (available from the authors upon request) show that computation time is not at all an issue, as the time required to compute one value of $\hat{\beta}_n^{bc2}$ using a regular desktop computer never exceeds 2 seconds for all the cases considered in our Monte Carlo experiments, though much larger than 0.005 seconds which is the time needed for computing one value of $\hat{\beta}_n$.

Remark 4.4. In terms of bias and rmse, the proposed estimators are clearly favored under the Type I and random censoring schemes; with complete data, one can use either the proposed estimators or the existing one although a slight advantage goes to the proposed ones; and with Type II censored data, the choice may go to the existing estimator but the proposed ones perform almost equally well.

5. A Real Data Example and A Discussion on Subsequent Inferences

In this section, we present a real data example to give a clear guidance on the practical implementations of the proposed bias correction method. We also discuss some immediate

DOI: http://dx.doi.org/10.1080/00949655

inference problems following a bias-corrected shape estimation to show the potential usefulness of the proposed method.

5.1 *A real data example*

A set of real data from Lawless ([11], p.240) is used to illustrate the applications of the proposed bias-corrected method under different censoring schemes. The complete data given in the table below are the failure voltages (in kV/mm) of 40 specimens (20 each type) of electrical cable insulation.

The MLEs of the individual shape parameters are $\hat{\beta}_1 = 9.3833$ and $\hat{\beta}_2 = 9.1411$, respectively. Since these two estimates are close, an equal-shape assumption is considered to be reasonable [6]. The MLE of the common shape β_0 is $\hat{\beta}_n = 9.2611$, and the MLEs of α_{0i}'s are $\hat{\alpha}_n = (48.05, 59.54)$. The MMLE of β_0 is $\hat{\beta}_{\text{MMLE}} = 8.8371$. Drawing $B = 699$ random samples from each of the estimated populations WB$(48.05, 9.2611)$ and WB$(59.54, 9.2611)$, leads to the 2nd- and 3rd-order bias-corrected MLEs of β_0 as: $\hat{\beta}_n^{bc2} = 8.7917$ and $\hat{\beta}_n^{bc3} = 8.7867$. The three bias-corrected estimates of β_0 are seen to be similar and are all quite different from the MLE, showing the need for bias correction. Changing the value of B does not yield much change in the values of $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$.

Failure voltages (in kV/mm) of 40 specimens of electrical cable insulation: 20 *Type I insulation* (T-I) and 20 *Type II insulation* (T-II)

Complete Data		Type I censoring		Type II censoring		Random censoring	
T-I	T-II	T-I	T-II	T-I	T-II	T-I	T-II
32.0	39.4	32.0	39.4	32.0	39.4	32.0	39.4
35.4	45.3	35.4	45.3	35.4	45.3	35.4	39.6+
36.2	49.2	36.2	49.2	36.2	49.2	36.2	49.2
39.8	49.4	39.8	49.4	39.8	49.4	26.5+	49.4
41.2	51.3	41.2	51.3	41.2	51.3	41.2	51.3
43.3	52.0	43.3	52.0	43.3	52.0	43.3	52.0
45.5	53.2	45.5	53.2	45.5	53.2	45.5	30.5+
46.0	53.2	46.0	53.2	46.0	53.2	46.0	53.2
46.2	54.9	46.2	54.9	46.2	54.9	25.3+	54.9
46.4	55.5	46.4	55.5	46.4	55.5	46.4	55.5
46.5	57.1	46.5	57.1	46.5	57.1	29.9+	57.1
46.8	57.2	46.8	57.2	46.8	57.2	46.8	57.2
47.3	57.5	47.3	57.5	47.3	57.5	24.5+	54.7+
47.3	59.2	47.3	59.2	47.3	59.2	47.3	59.2
47.6	61.0	47.6	60.7+	47.3+	59.2+	47.6	35.1+
49.2	62.4	49.0+	60.7+	47.3+	59.2+	39.5+	38.7+
50.4	63.8	49.0+	60.7+	47.3+	59.2+	46.4+	63.8
50.9	64.3	49.0+	60.7+	47.3+	59.2+	50.9	64.3
52.4	67.3	49.0+	60.7+	47.3+	59.2+	52.4	67.3
56.3	67.7	49.0+	60.7+	47.3+	59.2+	56.3	53.9+

Now, we consider various ways of censoring the data to further demonstrate the practical implementations of the proposed bootstrap method for bias correction.

Type I censoring: Suppose a decision had been made before the start of the experiment that tests were to be stopped at times (49.0, 60.7), roughly the 70th quantiles of WB$(48.05, 9.2611)$ and WB$(59.54, 9.2611)$. In this case, the failure times of some items would have been censored (indicated by "+"); see the table above (columns 3 & 4).

The MLEs for β_0 and α_{0i}'s are $\hat{\beta}_n = 9.5613$ and $\hat{\alpha}_n = (47.61, 58.99)$. As the censoring times are fixed at (49.0, 60.7), this is a Type I censoring scheme with $r_1 = 15$, $r_2 = 14$. Thus the bootstrap samples are generated as follows:

1) Draw a pair of independent random samples $\{T_{i1}^*, \ldots, T_{in_i}^*\}$, $i = 1, 2$, $n_i = 20$, from the estimated distributions WB$(47.61, 9.5613)$ and WB$(58.99, 9.5613)$, respectively;

DOI: http://dx.doi.org/10.1080/00949655

2) Calculate the 15/20 quantile C_1^* of WB(47.61, 9.5613) and the 14/20 quantile C_2^* of WB(58.99, 9.5613), to be used as the bootstrap censoring times;
3) Set the $i$the bootstrap sample as $\{Y_{ij}^* = \min(T_{ij}^*, C_i^*), j = 1, \ldots, n_i\}$, and the failure indicators as $\{\delta_{ij} = 1$ if $T_{ij}^* \le C_i^*$ and 0 otherwise, $j = 1, \ldots, n_i\}$, $i = 1, 2$.

With $B = 699$ bootstrap samples, we obtain the bias-corrected MLEs $\hat{\beta}_n^{bc2} = 9.1370$ and $\hat{\beta}_n^{bc3} = 9.1352$. The MMLE is $\hat{\beta}_{\rm MMLE-I} = 9.1853$. Again, the three bias-corrected MLEs are similar and are all quite different from the original MLE, showing the need for bias correction. The value of B does not affect much the values of $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$.

Type II censoring: Suppose the number of failures to be observed is predetermined to be 14 for both samples. Then the data obtained are again censored (columns 5 & 6 in the above table). The MLEs for β_0 and α_{0i}'s are $\hat{\beta}_n = 10.7545$ and $\hat{\alpha}_n = (47.17, 58.12)$. As the data have the form "47.3, 47.3+" and "59.2, 59.2+", this indicates a Type II censoring scheme, which has the censored time equal to the largest observed failure time. Hence we adopt the following procedure to generate bootstrap samples:

1) Draw a pair of independent random samples $\{T_{i1}^*, \ldots, T_{in_i}^*\}, i = 1, 2, n_i = 20$, from the estimated distributions WB(47.17, 10.7545) and WB(58.12, 10.7545), respectively;
2) Sort $\{T_{i1}^*, \ldots, T_{in_i}^*\}$ in ascending order, for $i = 1, 2$, denoted the sorted samples by $\{T_{(i1)}^*, \ldots, T_{(in_i)}^*\}, i = 1, 2$;
3) For $i = 1, 2$, set the ith bootstrap samples $Y_{ij}^* = T_{(ij)}^*$, $\delta_{ij} = 1$, for $j \le r_i$, and $Y_{ij}^* = T_{(i,r_i)}^*$, $\delta_{ij} = 0$, for $j > r_i$.

In this example, $r_1 = r_2 = 14$. With $B = 699$, we obtain the bias-corrected MLEs, $\hat{\beta}_n^{bc2} = 9.6179$ and $\hat{\beta}_n^{bc3} = 9.5750$. The MMLE is $\hat{\beta}_{\rm MMLE-II} = 9.8139$. The bias-corrected or modified MLEs are seen to differ from the MLE more substantially, compared with the cases of complete or Type I censored data.

Random censoring: Suppose the two samples are randomly censored according to a Uniform$(0.5 * 49.0, 1.5 * 49.0)$ and a Uniform$(0.5 * 60.7, 1.5 * 60.7)$, respectively, where 49.0 and 60.7 are the fixed censoring times used in generating the Type-I censored data. A typical set of generated data is given in the above table (the last two columns). The MLEs for β and α_i's are $\hat{\beta}_n = 8.9824$ and $\hat{\alpha}_n = (48.18, 58.77)$.

Since the censoring times vary across the individuals, a random censoring mechanism should be considered. For illustration, to generate a randomly censored bootstrap sample for Type I insulation, (i) draw a random sample $\{T_1^*, \ldots, T_{20}^*\}$ from WB(48.18, 8.9824), (ii) sort the observed data in ascending order to give $\{Y_{(1)}, \ldots, Y_{(20)}\} = \{$24.5+, 25.3+, 26.5+, 29.9+, 32.0, 35.4, 36.2, 39.5+, 41.2, 43.3, 45.5, 46.0, 46.4+, 46.4, 46.8, 47.3, 47.6, 50.9, 52.4, 56.3$\}$, and $\{\delta_{(1)}, \ldots, \delta_{(20)}\} = \{0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1\}$, (iii) generate the bootstrap censoring time as $C_j^* = Y_{(j)}$ if $\delta_{(j)} = 0$ otherwise a random draw from $\{Y_{(j)}, \ldots, Y_{(20)}\}$, $j = 1, \ldots, 20$, e.g., $\delta_{(4)} = 0$ then $C_4^* = Y_{(4)} = 29.9$, $\delta_{(16)} = 1$ then C_{16}^* is a random draw from (47.3, 47.6, 50.9, 52.4, 56.3), (iv) set $Y_j = \min(T_j^*, C_j^*)$, $j = 1, \ldots, 20$. Similarly, the bootstrap samples for Type II insulation are generated.

Applying the bootstrap procedure with $B = 699$, we obtained the bias-corrected MLEs as $\hat{\beta}_n^{bc2} = 8.3680$ and $\hat{\beta}_n^{bc3} = 8.3696$, which are quite different from the MLE $\hat{\beta}_n = 8.9824$, and are also noticeably different from the MMLEs $\hat{\beta}_{\rm MMLE-I} = 8.7325$ and $\hat{\beta}_{\rm MMLE-II} = 8.4423$. Clearly in this case $\hat{\beta}_n^{bc2}$ and $\hat{\beta}_n^{bc3}$ are more reliable as they are based on the correct censoring scheme.

DOI: http://dx.doi.org/10.1080/00949655

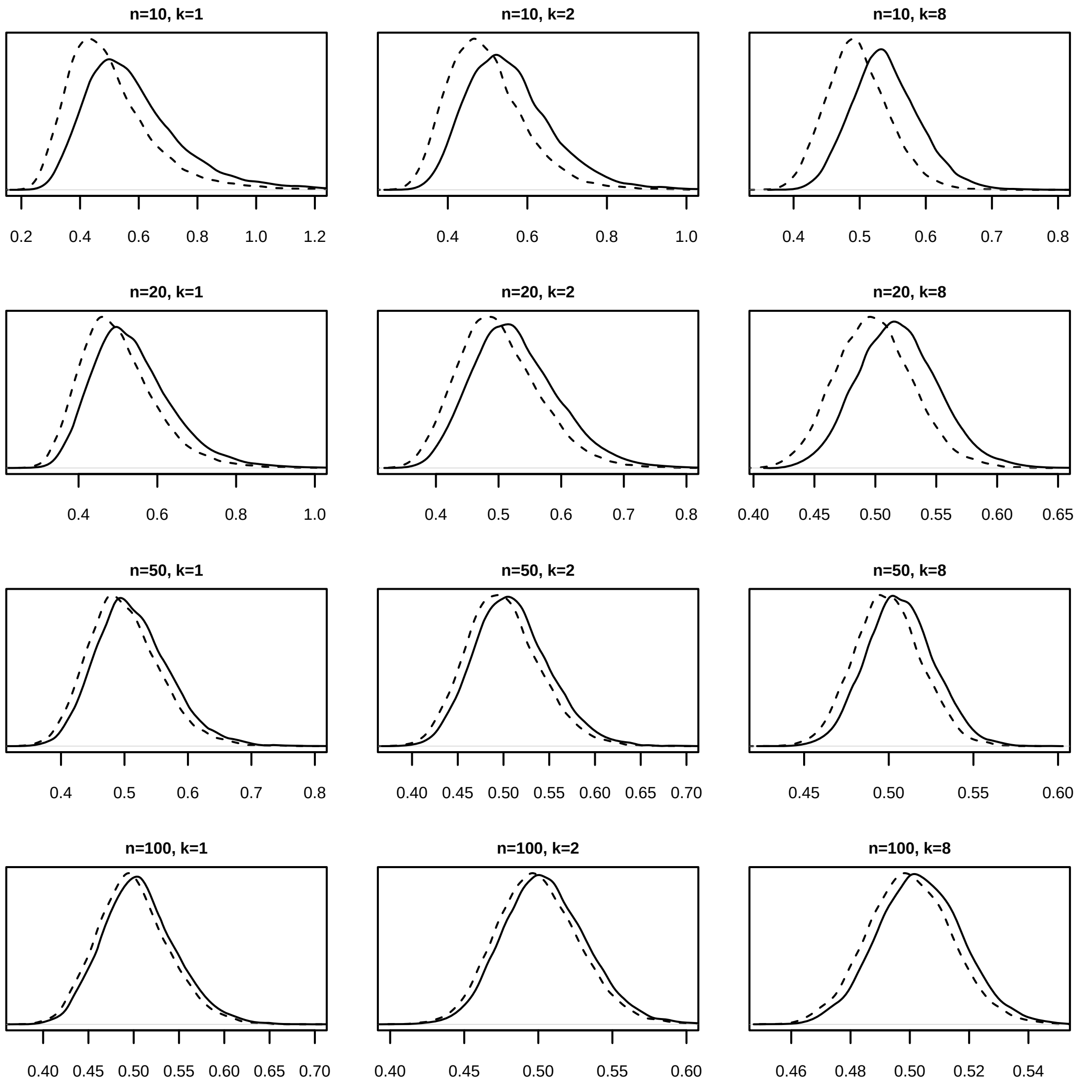

Figure 1. Empirical distributions for $\hat{\beta}_n$ (solid line) and $\hat{\beta}_n^{bc2}$ (dashed line) with complete data, $\beta_0 = 0.5$.

5.2 *A discussion on subsequent inferences*

Once the common shape estimator is bias-corrected, a natural question is on its impact on the subsequent inferences such as interval estimation for the shape parameter, and the estimation of the scale parameters, percentile lives, survival functions, etc.

Empirical distributions. Figure 1 presents some plots of the empirical distributions of $\hat{\beta}_n$ and $\hat{\beta}_n^{bc2}$. The plots show that the finite sample distributions of the two estimators can be substantially different, showing the potential of an improved inference using the bias-corrected estimator.

Confidence intervals for β_0. Under a general right censorship, the confidence interval (CI) for β_0 is constructed based on the large sample result $\hat{\beta}_n \sim N(0, J_n^{-1}(\hat{\beta}_n))$, where $J_n(\beta) = -nH_{1n}(\beta)$. Thus the large sample $100(1-\gamma)\%$ CI for β_0 takes the form

$$\text{CI}_1 = \left\{\hat{\beta}_n - z_{\gamma/2} J_n^{-\frac{1}{2}}(\hat{\beta}_n), \hat{\beta}_n + z_{\gamma/2} J_n^{-\frac{1}{2}}(\hat{\beta}_n)\right\},$$

DOI: http://dx.doi.org/10.1080/00949655

where $z_{\gamma/2}$ is the upper $\gamma/2$-quantile of the standard normal distribution. To construct a CI for β_0 using $\hat{\beta}_n^{bc2}$, we can use the second-order variance of $\hat{\beta}_n^{bc2}$:

$$V_2(\hat{\beta}_n^{bc2}) = 4\Omega_n^2\mathrm{E}(\tilde{\psi}_n^2) + 4\Omega_n^3\mathrm{E}(H_{1n}\tilde{\psi}_n^2) + \Omega_n^4\mathrm{E}(H_{1n}^2\tilde{\psi}_n^2) + 2\Omega_n^4\mathrm{E}(H_{2n})\mathrm{E}(\tilde{\psi}_n^3)$$
$$+\Omega_n^5\mathrm{E}(H_{2n})\mathrm{E}(H_{1n}\tilde{\psi}_n^3) + \frac{1}{4}\Omega_n^6(\mathrm{E}(H_{2n}))^2\mathrm{E}(\tilde{\psi}_n^4) + O(n^{-2}), \tag{10}$$

which is obtained by applying the expansion (2) to the Weibull model, making use of the fact that $b_2(\theta_0)$ defined in (7) is $O(n^{-1})$. Thus, a second-order corrected, $100(1-\gamma)\%$ CI for β_0 based on $\hat{\beta}_n^{bc2}$ and $V_2(\hat{\beta}_n^{bc2})$ takes the form:

$$\mathrm{CI}_2 = \left\{\hat{\beta}_n^{bc2} - z_{\gamma/2}V_2^{\frac{1}{2}}(\hat{\beta}_n^{bc2}),\ \ \hat{\beta}_n^{bc2} + z_{\gamma/2}V_2^{\frac{1}{2}}(\hat{\beta}_n^{bc2})\right\}.$$

The quantities in $V_2(\hat{\beta}_n^{bc2})$ are bootstrapped in the same manner as for, e.g., $b_2(\theta_0)$.

Table 13 presents a small part of Monte Carlo results for the coverage probability and average length of the two CIs with $k = 8$ and different combinations of β_0 and n_i. The results do show a significant improvements of CI_2 over CI_1, and thus a positive impact of bias-correction on the common Weibull shape estimation. However, more comprehensive investigation along this line is beyond the scope of the current paper.

Estimation of percentile lives. Intuitively, with the bias-corrected common shape estimator, say $\hat{\beta}_n^{\mathrm{bc2}}$, the estimation of other parameters (α_i's) in the model, and the estimation of reliability related quantities such as percentile lives should be improved automatically, as the common shape parameter β is the key parameter as far as estimation bias is concerned. To illustrate, take the p-quantile of the ith Weibull population:

$$x_{p,i} = \alpha_i[-\ln(1-p)]^{1/\beta},$$

for $0 < p < 1$. The MLE of $x_{p,i}$ is $\hat{x}_{p,i} = \hat{\alpha}_{n,i}(\hat{\beta}_n)[-\log(1-p)]^{1/\hat{\beta}_n}$, which is compared with $\hat{x}_{p,i}^{\mathrm{bc2}} = \hat{\alpha}_{n,i}(\hat{\beta}_n^{\mathrm{bc2}})[-\log(1-p)]^{1/\hat{\beta}_n^{\mathrm{bc2}}}$, obtained by replacing $\hat{\beta}_n$ by $\hat{\beta}_n^{\mathrm{bc2}}$, noting that $\hat{\alpha}_{n,i}(\beta) = \{\frac{1}{r_i}\sum_{j=1}^{n_i} t_{ij}^{\beta}\}^{1/\beta}$. Intuitively, $\hat{x}_{p,i}^{\mathrm{bc2}}$ should be less biased than $\hat{x}_{p,i}$ as $\hat{\beta}_n^{\mathrm{bc2}}$ is less biased than $\hat{\beta}_n$. However, a formal study along this line is beyond the scope of this paper.

Alternatively, one may consider to perform bias correction jointly on $\hat{\beta}_n$ and $\hat{x}_{p,i}, i = 1,\ldots k$, based on the reparameterized Weibull probability density function,

$$f(t_{ij}) = -\ln(1-p)x_{p,i}^{-\beta}\beta t_{ij}^{\beta-1}\exp[\log(1-p)(t_{ij}/x_{p,i})^{\beta}], t_{ij} \geq 0,$$

for $i = 1,\ldots,k; j = 1,\ldots,n_i$, the stochastic expansion based on joint estimating equation of [13], and the bootstrap method of [14]. Once again, this study is quite involved, and will be dealt with in a future research.

6. Conclusion

In this paper, we proposed a general method for correcting the bias of the MLE of the common Weibull shape parameter that allows a general censoring mechanism. The method is based a third-order stochastic expansion for the MLE of the common shape parameter, and a simple bootstrap procedure that allows easy estimation of various expected quantities involved in the expansions for bias. Extensive Monte Carlo simulation experiments are conducted and the results show that the proposed method performs

DOI: http://dx.doi.org/10.1080/00949655

very well in general. Even for very small or heavy censored data, the proposed bias-corrected MLEs of the Weibull shape parameter, especially the second-order one, perform rather satisfactorily. When compared with the MMLE of [5, 6], the results show that the proposed estimators are more robust against the underlying data generating mechanism, i.e., the various types of censoring.

Although four censoring mechanisms are examined in our simulation studies, it seems reasonable to believe that our asymptotic-expansion and bootstrap-based method should work well for the estimation of the shape parameter with most types of data.

Moreover, we may infer that the proposed method should be a simple and good method that can be applied to other parametric distributions, as long as the concentrated estimating equations for the parameter(s) of interest, and their derivatives (up to third order) can be expressed in analytical forms.

In the literature, there are various other likelihood-related approaches available for the type of estimation problems discussed in this paper, such as modified profile likelihood method [22], profile-kernel likelihood inference [23], marginal likelihood approach [24] and penalized maximum likelihood approach [25], etc. Thus it would be interesting for a possible future work to compare the existing approaches with our asymptotic-expansion and bootstrap-based method.

Moreover, as known, another commonly used method for the Weibull shape parameter is the least-square estimation (LSE), which is also biased. Thus, it would be interesting to consider the stochastic expansion of the LSE, and then develop a LSE-based bias-corrected method and compare it to some existing approaches, such as [7].

Acknowledgement

The authors are grateful to an Associate Editor and three anonymous referees for the helpful comments. This work is supported by research grants from National Bureau of Statistics of China (Grant No.: 2012006), and from MOE (Ministry of Education) Key Laboratory of Econometrics and Fujian Key Laboratory of Statistical Sciences, China.

Appendix A. Proof of Theorem 3.1

The proof of $\sqrt{n}$-consistency of $\hat{\beta}_n$ amounts to check the conditions of Theorem 4.1.2 and Theorem 4.1.3 of [19]. The differentiability and measureability of $\ell_n^c(\beta)$ are obvious. It is easy to see that $\ell_n^c(\beta)$ is globally concave, and thus $\ell_n^c(\beta)$ attains a unique global maximum at $\hat{\beta}_n$ which is the unique solution of $\tilde{\psi}_n(\beta) = 0$. These and Assumptions 1-3 lead to the $\sqrt{n}$-consistency of $\hat{\beta}_n$.

Now, the differentiability of $\tilde{\psi}_n(\beta)$ leads to the Taylor series expansion:

$$0 = \tilde{\psi}_n(\hat{\beta}_n) = \tilde{\psi}_n + H_{1n}(\hat{\beta}_n - \beta_0) + \frac{1}{2}H_{2n}(\hat{\beta}_n - \beta_0)^2 + \frac{1}{6}H_{3n}(\hat{\beta}_n - \beta_0)^3 + \frac{1}{6}[H_{3n}(\bar{\beta}_n) - H_{3n})(\hat{\beta}_n - \beta_0)^3,$$

where $\bar{\beta}_n$ lies between $\hat{\beta}_n$ and β_0. As $\hat{\beta}_n = \beta_0 + O_p(n^{-1/2})$, we have $\bar{\beta}_n = \beta_0 + O_p(n^{-1/2})$. Assumptions 3-4 lead to the following:

1) $\tilde{\psi}_n = O_p(n^{-1/2})$ and $\mathrm{E}(\tilde{\psi}_n) = O(n^{-1})$;
2) $\mathrm{E}(H_{rn}) = O(1)$ and $H_{rn}^{\circ} = O_p(n^{-\frac{1}{2}})$, $r = 1,2,3$;
3) $\mathrm{E}(H_{1n})^{-1} = O(1)$ and $H_{1n}^{-1} = O_p(1)$;

DOI: http://dx.doi.org/10.1080/00949655

4) $|H_{rn}(\beta) - H_{rn}| \le |\beta - \beta_0| X_n$, for β in a neighborhood of β_0, $r = 1, 2, 3$, and $\mathrm{E}|X_n| < c < \infty$ for some constant c;
5) $H_{3n}(\bar{\beta}_n) - H_{3n} = O_p(n^{-1/2})$.

The proofs of these results are straightforward. The details parallel those of [14] and are available from the authors. These make the stochastic expansion (2) valid. Some details are as follows. The $\sqrt{n}$-consistency of $\hat{\beta}_n$ and the results 1) to 5) ensure that the above Taylor expansion yields $\hat{\beta}_n - \beta_0 = -H_{1n}^{-1}\tilde{\psi}_n - \frac{1}{2}H_{1n}^{-1}H_{2n}(\hat{\beta}_n - \beta_0)^2 - \frac{1}{6}H_{1n}^{-1}H_{3n}(\hat{\beta}_n - \beta_0)^3 + O_p(n^{-2})$, or $-H_{1n}^{-1}\tilde{\psi}_n - \frac{1}{2}H_{1n}^{-1}H_{2n}(\hat{\beta}_n - \beta_0)^2 + O_p(n^{-3/2})$, or $-H_{1n}^{-1}\tilde{\psi}_n + O_p(n^{-1})$. The results 2) and 3) lead to $-H_{1n}^{-1} = (\Omega_n^{-1} - H_{1n}^{\circ})^{-1} = (1 - \Omega_n H_{1n}^{\circ})^{-1}\Omega_n = \Omega_n + \Omega_n^2 H_{1n}^{\circ} + \Omega_n^3 H_{1n}^{\circ 2} + O_p(n^{-3/2})$, or $\Omega_n + \Omega_n^2 H_{1n}^{\circ} + O_p(n^{-1})$, or $\Omega_n + O_p(n^{-1/2})$. Combining these expansions and their reduced forms, we obtain (2). Finally, Assumptions 4(i) and 4(iii) guarantee the transition from the stochastic expansion (2) to the results of Theorem 3.1.

Appendix B. Proof of Corollary 3.2

Note that the second-order bias $b_2 \equiv b_2(\theta_0)$ is of order $O(n^{-1})$, and the third-order bias $b_3 = b_3(\theta_0)$ is of order $O(n^{-3/2})$. If explicit expressions of $b_2(\theta_0)$ and $b_3(\theta_0)$ exist, then the "plug-in" estimates of b_2 and b_3 would be, respectively, $\hat{b}_2 = b_2(\hat{\theta}_n)$ and $\hat{b}_3 = b_3(\hat{\theta}_n)$, where $\hat{\theta}_n$ is the MLE of θ_0 defined at the beginning of Section 3. Assumption (ii) stated in the corollary guarantees the $\sqrt{n}$-consistency of $\hat{\alpha}_{n,i}$, and hence the $\sqrt{n}$-consistency of $\hat{\theta}_n$. We have under the additional assumptions in the corollary,

$$b_2(\hat{\theta}_n) = b_2(\theta_0) + \frac{\partial}{\partial \theta_0} b_2(\theta_0)(\hat{\theta}_n - \theta_0) + O_p(n^{-2}),$$

and $\mathrm{E}[b_2(\hat{\theta}_n)] = b_2(\theta_0) + \frac{\partial}{\partial\theta_0} b_2(\theta_0)\mathrm{E}(\hat{\theta}_n - \theta_0) + \mathrm{E}[O_p(n^{-2})] = b_2(\theta_0) + O(n^{-2})$, noting that $\frac{\partial}{\partial\theta_0} b_2(\theta_0) = O(n^{-1})$ and $\mathrm{E}(\hat{\theta}_n - \theta_0) = O(n^{-1})$. Similarly, $\mathrm{E}[b_3(\hat{\theta}_n)] = b_3(\theta_0) + O(n^{-5/2})$. These show that replacing θ_0 by $\hat{\theta}_n$ only (possibly) imposes additional bias of order $O_p(n^{-2})$ for $b_2(\hat{\theta}_n)$, and an additional bias of order $O_p(n^{-5/2})$ for $b_3(\hat{\theta}_n)$, leading to $\mathrm{Bias}(\hat{\beta}_n^{\mathrm{bc2}}) = O(n^{-3/2})$ and $\mathrm{Bias}(\hat{\beta}_n^{\mathrm{bc3}}) = O(n^{-2})$.

Clearly, our bootstrap estimate has two step approximations, one is that described above, and the other is the bootstrap approximations to the various expectations in (7) and (8), given $\hat{\theta}_n$. For example,

$$\hat{\mathrm{E}}(H_{1n}\tilde{\psi}_n) = \frac{1}{B}\sum_{b=1}^{B} H_{1n,b}(\hat{\beta}_n)\tilde{\psi}_{n,b}(\hat{\beta}_n).$$

However, these approximations can be made arbitrarily accurate, for a given $\hat{\theta}_n$, by choosing an arbitrarily large B. The results of Corollary 3.2 thus follow.

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DOI: http://dx.doi.org/10.1080/00949655

Table 4. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.579 [.246](.233)	0.496 [.199](.199)	0.509 [.206](.205)	0.533 [.216](.214)
	0.8	0.937 [.417](.394)	0.803 [.337](.337)	0.823 [.344](.343)	0.863 [.367](.361)
	1.0	1.160 [.508](.482)	0.994 [.413](.413)	1.020 [.425](.425)	1.067 [.447](.442)
	2.0	2.322 [.974](.919)	1.990 [.783](.783)	2.041 [.800](.799)	2.136 [.854](.843)
	5.0	5.823 [2.43](2.28)	4.992 [1.96](1.96)	5.120 [2.01](2.01)	5.357 [2.12](2.09)
20	0.5	0.535 [.136](.131)	0.499 [.124](.124)	0.502 [.124](.124)	0.515 [.127](.126)
	0.8	0.858 [.223](.216)	0.800 [.202](.202)	0.804 [.203](.203)	0.825 [.209](.207)
	1.0	1.065 [.275](.267)	0.992 [.251](.251)	0.997 [.252](.252)	1.023 [.259](.257)
	2.0	2.148 [.563](.543)	2.002 [.510](.510)	2.012 [.511](.511)	2.065 [.527](.523)
	5.0	5.372 [1.41](1.37)	5.005 [1.28](1.28)	5.031 [1.29](1.29)	5.162 [1.32](1.31)
50	0.5	0.515 [.080](.078)	0.502 [.076](.076)	0.502 [.076](.076)	0.507 [.077](.077)
	0.8	0.822 [.130](.128)	0.800 [.125](.125)	0.800 [.125](.125)	0.809 [.126](.126)
	1.0	1.028 [.160](.157)	1.001 [.154](.154)	1.002 [.154](.154)	1.012 [.155](.155)
	2.0	2.062 [.325](.319)	2.007 [.312](.312)	2.009 [.312](.312)	2.030 [.316](.314)
	5.0	5.133 [.796](.785)	4.995 [.768](.768)	4.999 [.768](.768)	5.053 [.775](.774)
100	0.5	0.507 [.055](.054)	0.500 [.054](.054)	0.501 [.054](.054)	0.503 [.054](.054)
	0.8	0.809 [.087](.086)	0.799 [.085](.085)	0.799 [.085](.085)	0.803 [.086](.086)
	1.0	1.015 [.111](.110)	1.001 [.109](.109)	1.002 [.109](.109)	1.007 [.109](.109)
	2.0	2.026 [.215](.214)	1.999 [.211](.211)	2.000 [.211](.211)	2.011 [.212](.212)
	5.0	5.072 [.543](.538)	5.004 [.532](.532)	5.005 [.532](.532)	5.032 [.535](.534)
		$p = 0.5$			
10	0.5	0.649 [.510](.487)	0.506 [.306](.304)	0.549 [.340](.335)	0.595 [.463](.452)
	0.8	1.030 [.769](.733)	0.803 [.477](.475)	0.871 [.539](.534)	0.943 [.699](.683)
	1.0	1.287 [1.02](.973)	1.004 [.626](.625)	1.087 [.702](.696)	1.179 [.928](.910)
	2.0	2.557 [1.94](1.86)	1.996 [1.20](1.20)	2.166 [1.36](1.35)	2.343 [1.77](1.74)
	5.0	6.368 [4.47](4.26)	4.982 [2.80](2.80)	5.403 [3.11](3.09)	5.829 [4.01](3.92)
20	0.5	0.555 [.197](.190)	0.501 [.171](.171)	0.513 [.176](.176)	0.531 [.184](.181)
	0.8	0.887 [.322](.310)	0.801 [.274](.274)	0.819 [.283](.283)	0.849 [.301](.297)
	1.0	1.105 [.379](.365)	0.998 [.331](.331)	1.021 [.338](.337)	1.058 [.354](.349)
	2.0	2.212 [.778](.748)	1.998 [.671](.671)	2.044 [.691](.690)	2.118 [.726](.716)
	5.0	5.538 [1.94](1.86)	5.003 [1.68](1.68)	5.119 [1.72](1.72)	5.302 [1.81](1.78)
50	0.5	0.520 [.102](.100)	0.500 [.097](.097)	0.502 [.097](.097)	0.511 [.099](.099)
	0.8	0.830 [.163](.161)	0.798 [.155](.155)	0.801 [.155](.155)	0.815 [.159](.158)
	1.0	1.039 [.205](.202)	1.000 [.195](.195)	1.003 [.195](.195)	1.021 [.199](.198)
	2.0	2.077 [.408](.401)	1.999 [.388](.388)	2.005 [.388](.388)	2.041 [.396](.394)
	5.0	5.196 [1.03](1.02)	5.000 [.980](.980)	5.015 [.982](.982)	5.107 [1.00](.997)
100	0.5	0.510 [.069](.069)	0.500 [.068](.068)	0.501 [.068](.068)	0.505 [.068](.068)
	0.8	0.816 [.111](.110)	0.801 [.108](.108)	0.802 [.108](.108)	0.809 [.109](.109)
	1.0	1.020 [.135](.134)	1.001 [.132](.132)	1.001 [.132](.132)	1.011 [.133](.133)
	2.0	2.040 [.275](.272)	2.002 [.268](.268)	2.003 [.268](.268)	2.022 [.271](.270)
	5.0	5.097 [.690](.683)	5.001 [.672](.672)	5.005 [.672](.672)	5.053 [.679](.677)
		$p = 0.3$			
10	0.5	0.858 [1.17](1.10)	0.485 [.491](.469)	0.543 [.543](.522)	0.789 [1.10](1.05)
	0.8	1.352 [1.81](1.72)	0.763 [.778](.758)	0.859 [.873](.854)	1.240 [1.69](1.62)
	1.0	1.695 [2.32](2.20)	0.937 [.930](.911)	1.055 [1.06](1.05)	1.553 [2.17](2.09)
	2.0	3.334 [4.51](4.30)	1.850 [1.84](1.84)	2.092 [2.09](2.09)	3.054 [4.22](4.08)
	5.0	8.266 [11.2](10.7)	4.604 [4.61](4.59)	5.182 [5.20](5.19)	7.568 [10.4](10.1)
20	0.5	0.619 [.423](.406)	0.498 [.266](.265)	0.540 [.293](.289)	0.590 [.401](.390)
	0.8	1.003 [.739](.710)	0.806 [.447](.446)	0.876 [.499](.493)	0.958 [.704](.686)
	1.0	1.239 [.856](.822)	1.000 [.557](.556)	1.088 [.622](.615)	1.183 [.811](.790)
	2.0	2.516 [1.99](1.92)	2.008 [1.12](1.12)	2.184 [1.23](1.22)	2.402 [1.90](1.86)
	5.0	6.180 [4.30](4.13)	4.969 [2.61](2.61)	5.395 [2.92](2.89)	5.899 [4.08](3.98)
50	0.5	0.536 [.151](.146)	0.501 [.136](.136)	0.508 [.139](.138)	0.526 [.146](.143)
	0.8	0.858 [.241](.234)	0.801 [.218](.218)	0.813 [.222](.222)	0.842 [.234](.230)
	1.0	1.077 [.310](.301)	1.006 [.280](.279)	1.020 [.285](.285)	1.057 [.301](.295)
	2.0	2.143 [.601](.584)	2.001 [.544](.544)	2.030 [.554](.553)	2.104 [.582](.573)
	5.0	5.375 [1.54](1.49)	5.018 [1.39](1.39)	5.091 [1.41](1.41)	5.276 [1.49](1.46)
100	0.5	0.516 [.095](.093)	0.499 [.090](.090)	0.501 [.091](.091)	0.511 [.093](.092)
	0.8	0.826 [.153](.151)	0.800 [.146](.146)	0.802 [.147](.147)	0.819 [.151](.149)
	1.0	1.038 [.196](.192)	1.005 [.186](.186)	1.008 [.186](.186)	1.029 [.192](.190)
	2.0	2.068 [.387](.381)	2.001 [.369](.369)	2.007 [.371](.371)	2.049 [.381](.378)
	5.0	5.170 [.953](.938)	5.004 [.910](.910)	5.020 [.913](.912)	5.123 [.937](.929)

DOI: http://dx.doi.org/10.1080/00949655

Table 5. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 2$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.548 [.149](.141)	0.498 [.128](.128)	0.499 [.128](.128)	0.506 [.130](.130)
	0.8	0.876 [.235](.222)	0.795 [.202](.202)	0.796 [.202](.202)	0.808 [.206](.206)
	1.0	1.095 [.296](.281)	0.995 [.256](.255)	0.996 [.255](.255)	1.010 [.260](.260)
	2.0	2.201 [.594](.559)	1.999 [.508](.508)	2.001 [.507](.507)	2.031 [.519](.518)
	5.0	5.467 [1.47](1.39)	4.964 [1.27](1.27)	4.969 [1.26](1.26)	5.043 [1.29](1.29)
20	0.5	0.523 [.094](.091)	0.500 [.087](.087)	0.500 [.087](.087)	0.503 [.088](.088)
	0.8	0.834 [.150](.146)	0.798 [.141](.141)	0.798 [.140](.140)	0.801 [.141](.141)
	1.0	1.048 [.187](.181)	1.002 [.174](.174)	1.002 [.174](.174)	1.007 [.175](.174)
	2.0	2.087 [.372](.362)	1.996 [.347](.347)	1.997 [.347](.347)	2.006 [.348](.348)
	5.0	5.223 [.938](.911)	4.996 [.874](.874)	4.997 [.874](.874)	5.019 [.878](.877)
50	0.5	0.509 [.056](.055)	0.500 [.054](.054)	0.500 [.054](.054)	0.501 [.054](.054)
	0.8	0.814 [.088](.087)	0.800 [.086](.086)	0.800 [.086](.086)	0.801 [.086](.086)
	1.0	1.017 [.109](.108)	1.000 [.106](.106)	1.000 [.106](.106)	1.001 [.106](.106)
	2.0	2.034 [.218](.216)	2.000 [.213](.213)	2.000 [.213](.213)	2.002 [.212](.212)
	5.0	5.081 [.548](.542)	4.995 [.534](.534)	4.995 [.534](.534)	5.002 [.534](.534)
100	0.5	0.505 [.038](.038)	0.500 [.037](.037)	0.500 [.037](.037)	0.501 [.037](.037)
	0.8	0.807 [.061](.061)	0.800 [.060](.060)	0.800 [.060](.060)	0.801 [.060](.060)
	1.0	1.009 [.077](.076)	1.000 [.076](.076)	1.000 [.076](.076)	1.001 [.076](.076)
	2.0	2.019 [.153](.152)	2.002 [.151](.151)	2.002 [.151](.151)	2.003 [.151](.151)
	5.0	5.045 [.384](.382)	5.002 [.379](.379)	5.002 [.379](.379)	5.006 [.379](.379)
		$p = 0.5$			
10	0.5	0.569 [.208](.192)	0.506 [.175](.171)	0.514 [.179](.174)	0.522 [.182](.177)
	0.8	0.901 [.320](.301)	0.801 [.270](.268)	0.813 [.275](.272)	0.826 [.279](.276)
	1.0	1.125 [.403](.381)	0.998 [.338](.336)	1.014 [.347](.344)	1.030 [.352](.349)
	2.0	2.248 [.779](.738)	1.996 [.654](.654)	2.027 [.666](.665)	2.058 [.679](.676)
	5.0	5.616 [1.99](1.90)	4.985 [1.68](1.68)	5.064 [1.71](1.71)	5.142 [1.74](1.74)
20	0.5	0.530 [.119](.115)	0.501 [.109](.109)	0.502 [.109](.109)	0.507 [.110](.110)
	0.8	0.844 [.188](.183)	0.797 [.173](.173)	0.800 [.174](.174)	0.808 [.175](.175)
	1.0	1.060 [.239](.232)	1.001 [.220](.220)	1.005 [.220](.220)	1.014 [.222](.222)
	2.0	2.122 [.478](.462)	2.006 [.439](.439)	2.013 [.440](.440)	2.032 [.444](.443)
	5.0	5.305 [1.22](1.18)	5.014 [1.12](1.12)	5.031 [1.12](1.12)	5.080 [1.13](1.13)
50	0.5	0.510 [.070](.069)	0.499 [.068](.068)	0.500 [.068](.068)	0.502 [.068](.068)
	0.8	0.818 [.111](.109)	0.801 [.107](.107)	0.801 [.107](.107)	0.804 [.107](.107)
	1.0	1.022 [.138](.136)	0.999 [.134](.134)	1.000 [.134](.134)	1.004 [.134](.134)
	2.0	2.047 [.278](.274)	2.003 [.269](.269)	2.004 [.269](.269)	2.012 [.270](.269)
	5.0	5.107 [.698](.690)	4.995 [.676](.676)	4.998 [.677](.677)	5.019 [.678](.678)
100	0.5	0.505 [.048](.047)	0.499 [.047](.047)	0.499 [.047](.047)	0.501 [.047](.047)
	0.8	0.807 [.075](.074)	0.798 [.074](.074)	0.799 [.074](.074)	0.800 [.074](.074)
	1.0	1.012 [.095](.094)	1.001 [.093](.093)	1.001 [.093](.093)	1.003 [.094](.094)
	2.0	2.020 [.189](.188)	1.998 [.186](.186)	1.998 [.186](.186)	2.003 [.186](.186)
	5.0	5.056 [.477](.474)	5.001 [.470](.470)	5.002 [.470](.470)	5.013 [.470](.470)
		$p = 0.3$			
10	0.5	0.640 [.426](.344)	0.534 [.351](.280)	0.573 [.381](.310)	0.587 [.390](.318)
	0.8	0.989 [.587](.506)	0.822 [.462](.401)	0.882 [.507](.445)	0.905 [.525](.460)
	1.0	1.226 [.801](.733)	1.014 [.578](.531)	1.089 [.639](.590)	1.121 [.719](.671)
	2.0	2.399 [1.48](1.43)	1.974 [1.08](1.08)	2.124 [1.19](1.19)	2.188 [1.31](1.29)
	5.0	5.873 [3.43](3.32)	4.833 [2.55](2.55)	5.206 [2.85](2.84)	5.349 [3.00](2.98)
20	0.5	0.548 [.188](.178)	0.499 [.163](.160)	0.510 [.170](.167)	0.523 [.175](.170)
	0.8	0.882 [.295](.280)	0.804 [.258](.254)	0.820 [.265](.261)	0.842 [.274](.267)
	1.0	1.102 [.367](.352)	1.004 [.320](.319)	1.025 [.329](.327)	1.052 [.341](.336)
	2.0	2.206 [.729](.699)	2.010 [.634](.634)	2.052 [.652](.650)	2.105 [.675](.667)
	5.0	5.486 [1.82](1.77)	4.999 [1.59](1.59)	5.104 [1.64](1.64)	5.236 [1.69](1.69)
50	0.5	0.520 [.096](.094)	0.502 [.091](.091)	0.503 [.092](.092)	0.510 [.093](.093)
	0.8	0.831 [.156](.153)	0.802 [.147](.147)	0.804 [.148](.148)	0.816 [.151](.150)
	1.0	1.034 [.194](.191)	0.998 [.185](.185)	1.001 [.185](.185)	1.015 [.188](.187)
	2.0	2.078 [.384](.376)	2.006 [.363](.363)	2.011 [.364](.364)	2.040 [.371](.369)
	5.0	5.179 [.956](.939)	4.999 [.908](.908)	5.013 [.911](.911)	5.084 [.926](.922)
100	0.5	0.510 [.064](.064)	0.501 [.063](.063)	0.501 [.063](.063)	0.505 [.063](.063)
	0.8	0.815 [.104](.103)	0.801 [.101](.101)	0.801 [.101](.101)	0.808 [.102](.102)
	1.0	1.018 [.131](.130)	1.000 [.128](.128)	1.001 [.128](.128)	1.009 [.129](.128)
	2.0	2.037 [.261](.259)	2.002 [.255](.255)	2.003 [.255](.255)	2.018 [.257](.256)
	5.0	5.094 [.654](.647)	5.006 [.637](.637)	5.010 [.638](.638)	5.048 [.643](.641)

DOI: http://dx.doi.org/10.1080/00949655

Table 6. Empirical mean [rmse](se) of MLE-type estimators of β, Type-I data, $k = 8$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.526 [.070](.065)	0.497 [.061](.061)	0.496 [.061](.061)	0.486 [.062](.060)
	0.8	0.845 [.113](.104)	0.798 [.097](.097)	0.797 [.097](.097)	0.779 [.098](.096)
	1.0	1.054 [.141](.130)	0.995 [.122](.122)	0.994 [.122](.122)	0.972 [.124](.121)
	2.0	2.109 [.279](.257)	1.991 [.242](.242)	1.989 [.241](.241)	1.945 [.245](.238)
	5.0	5.275 [.701](.645)	4.980 [.607](.606)	4.974 [.606](.605)	4.865 [.613](.598)
20	0.5	0.513 [.046](.044)	0.500 [.043](.043)	0.500 [.043](.043)	0.493 [.043](.042)
	0.8	0.820 [.072](.069)	0.799 [.067](.067)	0.799 [.067](.067)	0.788 [.068](.067)
	1.0	1.026 [.090](.086)	1.000 [.084](.084)	1.000 [.084](.084)	0.986 [.084](.083)
	2.0	2.049 [.183](.176)	1.996 [.172](.171)	1.996 [.171](.171)	1.969 [.172](.169)
	5.0	5.119 [.448](.432)	4.988 [.421](.421)	4.987 [.421](.421)	4.920 [.424](.416)
50	0.5	0.505 [.027](.027)	0.500 [.027](.027)	0.500 [.027](.027)	0.497 [.027](.026)
	0.8	0.808 [.044](.044)	0.800 [.043](.043)	0.800 [.043](.043)	0.796 [.043](.043)
	1.0	1.011 [.055](.054)	1.001 [.054](.054)	1.001 [.054](.054)	0.995 [.054](.054)
	2.0	2.017 [.110](.108)	1.997 [.107](.107)	1.997 [.107](.107)	1.985 [.108](.107)
	5.0	5.048 [.272](.268)	4.999 [.265](.265)	4.998 [.265](.265)	4.969 [.265](.264)
100	0.5	0.502 [.019](.019)	0.500 [.019](.019)	0.500 [.019](.019)	0.499 [.019](.019)
	0.8	0.804 [.031](.031)	0.800 [.031](.031)	0.800 [.031](.031)	0.797 [.031](.030)
	1.0	1.005 [.038](.038)	1.000 [.038](.038)	1.000 [.038](.038)	0.997 [.038](.037)
	2.0	2.010 [.077](.076)	2.001 [.076](.076)	2.001 [.076](.076)	1.995 [.075](.075)
	5.0	5.024 [.191](.190)	4.999 [.189](.189)	4.999 [.189](.189)	4.984 [.189](.188)
		$p = 0.5$			
10	0.5	0.530 [.123](.090)	0.502 [.117](.087)	0.501 [.117](.087)	0.486 [.117](.086)
	0.8	0.845 [.158](.128)	0.800 [.146](.122)	0.799 [.146](.121)	0.775 [.146](.118)
	1.0	1.054 [.194](.159)	0.998 [.179](.151)	0.996 [.179](.151)	0.966 [.179](.146)
	2.0	2.102 [.348](.333)	1.988 [.314](.313)	1.986 [.313](.313)	1.926 [.312](.303)
	5.0	5.231 [.842](.810)	4.946 [.757](.755)	4.941 [.757](.755)	4.791 [.757](.728)
20	0.5	0.513 [.056](.054)	0.499 [.053](.053)	0.499 [.053](.053)	0.491 [.053](.052)
	0.8	0.822 [.090](.087)	0.801 [.085](.085)	0.801 [.085](.085)	0.788 [.084](.083)
	1.0	1.025 [.112](.109)	0.998 [.106](.106)	0.998 [.106](.106)	0.981 [.106](.104)
	2.0	2.055 [.223](.216)	2.002 [.211](.211)	2.001 [.211](.211)	1.968 [.210](.207)
	5.0	5.134 [.560](.543)	5.000 [.530](.530)	4.999 [.530](.530)	4.916 [.527](.521)
50	0.5	0.505 [.034](.034)	0.500 [.034](.034)	0.500 [.034](.034)	0.497 [.033](.033)
	0.8	0.809 [.055](.054)	0.800 [.054](.054)	0.800 [.054](.054)	0.795 [.053](.053)
	1.0	1.011 [.067](.067)	1.000 [.066](.066)	1.000 [.066](.066)	0.993 [.066](.066)
	2.0	2.020 [.135](.133)	1.999 [.132](.132)	1.999 [.132](.132)	1.985 [.132](.131)
	5.0	5.053 [.336](.332)	5.001 [.329](.329)	5.001 [.329](.329)	4.966 [.328](.326)
100	0.5	0.503 [.023](.023)	0.500 [.023](.023)	0.500 [.023](.023)	0.498 [.023](.023)
	0.8	0.804 [.038](.038)	0.800 [.038](.038)	0.800 [.038](.038)	0.797 [.038](.037)
	1.0	1.005 [.047](.047)	1.000 [.047](.047)	1.000 [.047](.047)	0.996 [.047](.047)
	2.0	2.011 [.094](.093)	2.001 [.093](.093)	2.001 [.093](.093)	1.994 [.093](.092)
	5.0	5.025 [.237](.235)	5.000 [.234](.234)	5.000 [.234](.234)	4.982 [.234](.233)
		$p = 0.3$			
10	0.5	0.633 [.471](.213)	0.606 [.469](.223)	0.606 [.469](.223)	0.595 [.468](.227)
	0.8	0.887 [.480](.166)	0.842 [.474](.167)	0.843 [.474](.167)	0.825 [.473](.167)
	1.0	1.054 [.500](.196)	0.999 [.492](.182)	1.000 [.492](.182)	0.978 [.491](.177)
	2.0	1.906 [.610](.603)	1.797 [.581](.545)	1.798 [.582](.545)	1.753 [.577](.522)
	5.0	4.495 [1.13](1.01)	4.219 [1.03](.664)	4.220 [1.03](.667)	4.109 [1.01](.471)
20	0.5	0.519 [.109](.082)	0.503 [.107](.081)	0.503 [.107](.081)	0.496 [.106](.080)
	0.8	0.827 [.143](.119)	0.801 [.137](.115)	0.801 [.137](.115)	0.790 [.136](.114)
	1.0	1.031 [.166](.147)	0.998 [.159](.143)	0.998 [.159](.143)	0.985 [.158](.141)
	2.0	2.063 [.315](.308)	1.997 [.298](.298)	1.997 [.298](.298)	1.970 [.295](.293)
	5.0	5.144 [.762](.748)	4.978 [.721](.721)	4.979 [.721](.721)	4.910 [.713](.707)
50	0.5	0.507 [.045](.045)	0.500 [.044](.044)	0.500 [.044](.044)	0.497 [.044](.044)
	0.8	0.810 [.072](.072)	0.800 [.071](.071)	0.800 [.071](.071)	0.795 [.070](.070)
	1.0	1.013 [.091](.090)	1.000 [.089](.089)	1.000 [.089](.089)	0.995 [.089](.088)
	2.0	2.025 [.181](.179)	1.999 [.177](.177)	1.999 [.177](.177)	1.988 [.176](.176)
	5.0	5.070 [.453](.447)	5.004 [.443](.443)	5.005 [.443](.443)	4.977 [.440](.439)
100	0.5	0.503 [.031](.031)	0.500 [.031](.031)	0.500 [.031](.031)	0.498 [.031](.031)
	0.8	0.805 [.051](.051)	0.800 [.050](.050)	0.800 [.050](.050)	0.798 [.050](.050)
	1.0	1.006 [.063](.063)	1.000 [.063](.063)	1.000 [.063](.063)	0.997 [.063](.062)
	2.0	2.011 [.124](.124)	1.998 [.123](.123)	1.998 [.123](.123)	1.993 [.123](.123)
	5.0	5.034 [.317](.316)	5.001 [.314](.314)	5.001 [.314](.314)	4.988 [.313](.313)

DOI: http://dx.doi.org/10.1080/00949655

Table 7. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.658 [.334](.294)	0.501 [.224](.224)	0.491 [.220](.220)	0.507 [.226](.226)
	0.8	1.058 [.528](.460)	0.804 [.350](.350)	0.789 [.343](.343)	0.815 [.355](.355)
	1.0	1.312 [.659](.581)	0.998 [.442](.442)	0.979 [.435](.434)	1.011 [.447](.447)
	2.0	2.650 [1.33](1.16)	2.015 [.883](.883)	1.977 [.868](.868)	2.040 [.893](.892)
	5.0	6.610 [3.31](2.89)	5.023 [2.20](2.20)	4.928 [2.16](2.16)	5.087 [2.22](2.22)
20	0.5	0.570 [.170](.155)	0.502 [.136](.136)	0.501 [.136](.136)	0.505 [.137](.137)
	0.8	0.907 [.261](.238)	0.800 [.210](.210)	0.797 [.209](.209)	0.804 [.211](.211)
	1.0	1.139 [.338](.308)	1.004 [.272](.272)	1.000 [.271](.271)	1.009 [.273](.273)
	2.0	2.275 [.672](.613)	2.006 [.541](.541)	1.999 [.539](.539)	2.017 [.543](.543)
	5.0	5.666 [1.66](1.52)	4.995 [1.34](1.34)	4.977 [1.33](1.33)	5.022 [1.34](1.34)
50	0.5	0.524 [.086](.083)	0.499 [.079](.079)	0.499 [.079](.079)	0.500 [.079](.079)
	0.8	0.840 [.137](.131)	0.800 [.125](.125)	0.800 [.125](.125)	0.802 [.125](.125)
	1.0	1.049 [.171](.163)	1.000 [.156](.156)	1.000 [.155](.155)	1.002 [.156](.156)
	2.0	2.101 [.343](.328)	2.002 [.313](.313)	2.001 [.313](.313)	2.006 [.314](.314)
	5.0	5.250 [.857](.820)	5.003 [.782](.782)	5.000 [.781](.781)	5.013 [.783](.783)
100	0.5	0.513 [.057](.056)	0.501 [.054](.054)	0.501 [.054](.054)	0.502 [.055](.055)
	0.8	0.820 [.091](.089)	0.801 [.087](.087)	0.801 [.087](.087)	0.802 [.087](.087)
	1.0	1.026 [.115](.112)	1.002 [.110](.110)	1.001 [.110](.110)	1.003 [.110](.110)
	2.0	2.049 [.225](.219)	2.001 [.214](.214)	2.001 [.214](.214)	2.003 [.214](.214)
	5.0	5.114 [.567](.555)	4.994 [.542](.542)	4.993 [.542](.542)	4.999 [.543](.543)
		$p = 0.5$			
10	0.5	0.777 [.556](.482)	0.497 [.309](.309)	0.471 [.295](.294)	0.505 [.318](.318)
	0.8	1.262 [.946](.826)	0.807 [.529](.529)	0.765 [.503](.502)	0.820 [.542](.542)
	1.0	1.572 [1.16](1.00)	1.006 [.643](.643)	0.953 [.612](.610)	1.022 [.652](.652)
	2.0	3.132 [2.33](2.03)	2.005 [1.31](1.31)	1.900 [1.24](1.24)	2.036 [1.32](1.32)
	5.0	7.858 [5.86](5.12)	5.029 [3.28](3.28)	4.762 [3.11](3.10)	5.108 [3.41](3.41)
20	0.5	0.611 [.242](.215)	0.503 [.177](.177)	0.498 [.175](.175)	0.505 [.178](.177)
	0.8	0.967 [.370](.330)	0.795 [.272](.272)	0.788 [.269](.269)	0.800 [.273](.273)
	1.0	1.214 [.470](.418)	0.998 [.344](.344)	0.988 [.341](.341)	1.004 [.346](.346)
	2.0	2.440 [.964](.858)	2.007 [.705](.705)	1.987 [.699](.699)	2.018 [.710](.709)
	5.0	6.075 [2.39](2.14)	4.994 [1.76](1.76)	4.945 [1.74](1.74)	5.022 [1.77](1.77)
50	0.5	0.537 [.111](.105)	0.499 [.098](.098)	0.499 [.098](.098)	0.500 [.098](.098)
	0.8	0.859 [.179](.169)	0.799 [.158](.158)	0.797 [.157](.157)	0.800 [.158](.158)
	1.0	1.078 [.228](.214)	1.002 [.199](.199)	1.001 [.199](.199)	1.004 [.199](.199)
	2.0	2.159 [.456](.427)	2.007 [.398](.398)	2.004 [.397](.397)	2.010 [.398](.398)
	5.0	5.371 [1.11](1.05)	4.991 [.976](.976)	4.984 [.974](.974)	5.001 [.977](.977)
100	0.5	0.517 [.072](.070)	0.498 [.068](.068)	0.498 [.068](.068)	0.499 [.068](.068)
	0.8	0.832 [.116](.112)	0.803 [.108](.108)	0.802 [.108](.108)	0.803 [.108](.108)
	1.0	1.037 [.146](.141)	1.001 [.136](.136)	1.000 [.136](.136)	1.002 [.136](.136)
	2.0	2.074 [.287](.278)	2.001 [.268](.268)	2.000 [.268](.268)	2.002 [.268](.268)
	5.0	5.172 [.719](.699)	4.990 [.674](.674)	4.989 [.674](.674)	4.995 [.674](.674)
		$p = 0.3$			
10	0.5	1.302 [1.64](1.43)	0.475 [.524](.524)	0.341 [.414](.382)	0.506 [.719](.719)
	0.8	2.140 [2.71](2.35)	0.782 [.861](.861)	0.561 [.672](.628)	0.829 [1.14](1.14)
	1.0	2.672 [3.41](2.97)	0.977 [1.09](1.09)	0.700 [.847](.793)	1.040 [1.48](1.48)
	2.0	5.250 [6.65](5.80)	1.917 [2.13](2.13)	1.374 [1.68](1.55)	2.053 [3.01](3.01)
	5.0	13.21 [16.7](14.5)	4.819 [5.34](5.33)	3.447 [4.17](3.87)	5.123 [7.17](7.17)
20	0.5	0.731 [.481](.422)	0.501 [.289](.289)	0.483 [.279](.278)	0.505 [.292](.292)
	0.8	1.172 [.745](.646)	0.804 [.444](.444)	0.775 [.428](.428)	0.810 [.446](.446)
	1.0	1.450 [.911](.793)	0.995 [.545](.545)	0.958 [.528](.526)	1.002 [.548](.548)
	2.0	2.891 [1.80](1.56)	1.984 [1.07](1.07)	1.911 [1.04](1.03)	1.997 [1.08](1.08)
	5.0	7.248 [4.65](4.07)	4.975 [2.79](2.79)	4.793 [2.70](2.69)	5.009 [2.84](2.84)
50	0.5	0.570 [.171](.157)	0.499 [.137](.137)	0.496 [.137](.137)	0.500 [.137](.137)
	0.8	0.915 [.274](.249)	0.801 [.218](.218)	0.797 [.217](.217)	0.802 [.218](.218)
	1.0	1.142 [.349](.318)	1.000 [.279](.279)	0.996 [.278](.278)	1.002 [.279](.279)
	2.0	2.285 [.693](.632)	2.001 [.553](.553)	1.991 [.551](.551)	2.004 [.554](.554)
	5.0	5.750 [1.78](1.61)	5.033 [1.41](1.41)	5.010 [1.41](1.41)	5.043 [1.42](1.41)
100	0.5	0.535 [.105](.099)	0.502 [.093](.093)	0.502 [.093](.093)	0.503 [.093](.093)
	0.8	0.855 [.168](.159)	0.802 [.149](.149)	0.801 [.149](.149)	0.803 [.149](.149)
	1.0	1.068 [.209](.197)	1.002 [.185](.185)	1.001 [.185](.185)	1.003 [.185](.185)
	2.0	2.135 [.417](.395)	2.003 [.371](.371)	2.000 [.370](.370)	2.004 [.371](.371)
	5.0	5.316 [1.02](.969)	4.986 [.909](.909)	4.980 [.909](.909)	4.989 [.909](.909)

DOI: http://dx.doi.org/10.1080/00949655

Table 8. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 2$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.607 [.201](.170)	0.495 [.139](.139)	0.486 [.138](.137)	0.502 [.141](.141)
	0.8	0.971 [.321](.271)	0.792 [.222](.221)	0.779 [.219](.218)	0.803 [.224](.224)
	1.0	1.224 [.407](.340)	0.998 [.278](.278)	0.981 [.274](.273)	1.013 [.282](.281)
	2.0	2.429 [.807](.684)	1.981 [.558](.558)	1.947 [.551](.549)	2.009 [.565](.565)
	5.0	6.066 [1.99](1.68)	4.946 [1.37](1.37)	4.862 [1.36](1.35)	5.018 [1.39](1.39)
20	0.5	0.549 [.111](.100)	0.500 [.091](.091)	0.498 [.090](.090)	0.502 [.091](.091)
	0.8	0.874 [.172](.155)	0.796 [.142](.142)	0.794 [.141](.141)	0.800 [.142](.142)
	1.0	1.099 [.222](.199)	1.001 [.181](.181)	0.997 [.180](.180)	1.005 [.182](.182)
	2.0	2.198 [.447](.401)	2.002 [.365](.365)	1.995 [.364](.364)	2.011 [.366](.366)
	5.0	5.491 [1.11](.990)	5.001 [.903](.903)	4.983 [.900](.900)	5.023 [.906](.906)
50	0.5	0.517 [.059](.057)	0.499 [.055](.055)	0.499 [.055](.055)	0.500 [.055](.055)
	0.8	0.828 [.094](.090)	0.799 [.087](.087)	0.799 [.087](.087)	0.800 [.087](.087)
	1.0	1.036 [.119](.113)	1.000 [.109](.109)	0.999 [.109](.109)	1.001 [.109](.109)
	2.0	2.070 [.238](.227)	1.998 [.219](.219)	1.997 [.219](.219)	2.000 [.220](.220)
	5.0	5.173 [.591](.566)	4.992 [.546](.546)	4.989 [.546](.546)	4.998 [.546](.546)
100	0.5	0.509 [.040](.039)	0.500 [.038](.038)	0.500 [.038](.038)	0.501 [.038](.038)
	0.8	0.815 [.064](.062)	0.801 [.061](.061)	0.801 [.061](.061)	0.801 [.061](.061)
	1.0	1.018 [.079](.077)	1.001 [.076](.076)	1.001 [.076](.076)	1.001 [.076](.076)
	2.0	2.035 [.160](.156)	1.999 [.153](.153)	1.999 [.153](.153)	2.001 [.153](.153)
	5.0	5.082 [.392](.384)	4.993 [.377](.377)	4.992 [.377](.377)	4.996 [.377](.377)
		$p = 0.5$			
10	0.5	0.685 [.319](.260)	0.491 [.186](.186)	0.469 [.180](.178)	0.505 [.191](.191)
	0.8	1.101 [.513](.416)	0.789 [.299](.298)	0.754 [.289](.285)	0.811 [.306](.306)
	1.0	1.370 [.632](.513)	0.981 [.369](.368)	0.937 [.358](.352)	1.009 [.378](.378)
	2.0	2.725 [1.24](1.01)	1.952 [.724](.722)	1.865 [.704](.691)	2.008 [.742](.742)
	5.0	6.877 [3.21](2.60)	4.927 [1.87](1.86)	4.708 [1.81](1.78)	5.067 [1.92](1.92)
20	0.5	0.575 [.154](.134)	0.497 [.116](.116)	0.493 [.115](.115)	0.501 [.116](.116)
	0.8	0.917 [.240](.209)	0.793 [.181](.181)	0.785 [.180](.179)	0.798 [.182](.182)
	1.0	1.155 [.308](.267)	0.998 [.231](.231)	0.989 [.229](.229)	1.005 [.232](.232)
	2.0	2.302 [.616](.537)	1.989 [.465](.464)	1.971 [.461](.460)	2.002 [.467](.467)
	5.0	5.764 [1.53](1.33)	4.979 [1.15](1.15)	4.935 [1.14](1.14)	5.014 [1.15](1.15)
50	0.5	0.529 [.078](.073)	0.501 [.069](.069)	0.500 [.069](.069)	0.502 [.069](.069)
	0.8	0.845 [.123](.114)	0.801 [.109](.109)	0.800 [.108](.108)	0.802 [.109](.109)
	1.0	1.054 [.153](.143)	0.998 [.136](.136)	0.997 [.136](.136)	0.999 [.136](.136)
	2.0	2.111 [.310](.290)	1.999 [.274](.274)	1.996 [.274](.274)	2.002 [.275](.275)
	5.0	5.292 [.774](.717)	5.011 [.680](.679)	5.005 [.679](.679)	5.018 [.680](.680)
100	0.5	0.513 [.050](.048)	0.500 [.047](.047)	0.499 [.047](.047)	0.500 [.047](.047)
	0.8	0.821 [.081](.078)	0.800 [.076](.076)	0.799 [.076](.076)	0.800 [.076](.076)
	1.0	1.027 [.101](.097)	1.000 [.094](.094)	1.000 [.094](.094)	1.001 [.094](.094)
	2.0	2.057 [.200](.192)	2.002 [.187](.187)	2.002 [.187](.187)	2.003 [.187](.187)
	5.0	5.121 [.497](.482)	4.986 [.470](.470)	4.984 [.470](.470)	4.989 [.470](.469)
		$p = 0.3$			
10	0.5	0.952 [.767](.619)	0.453 [.300](.296)	0.358 [.275](.235)	0.506 [.343](.343)
	0.8	1.519 [1.25](1.02)	0.723 [.495](.489)	0.571 [.452](.389)	0.807 [.546](.546)
	1.0	1.934 [1.60](1.30)	0.921 [.625](.620)	0.726 [.562](.491)	1.028 [.736](.735)
	2.0	3.762 [2.99](2.42)	1.790 [1.17](1.15)	1.413 [1.09](.915)	1.998 [1.33](1.33)
	5.0	9.395 [7.44](6.00)	4.470 [2.90](2.85)	3.526 [2.69](2.26)	4.989 [3.24](3.24)
20	0.5	0.653 [.267](.219)	0.492 [.165](.165)	0.477 [.162](.160)	0.502 [.168](.168)
	0.8	1.038 [.422](.348)	0.782 [.264](.263)	0.757 [.258](.255)	0.797 [.268](.268)
	1.0	1.304 [.536](.441)	0.983 [.333](.333)	0.951 [.326](.322)	1.002 [.339](.339)
	2.0	2.608 [1.06](.873)	1.965 [.658](.658)	1.902 [.644](.636)	2.003 [.670](.670)
	5.0	6.563 [2.71](2.21)	4.946 [1.67](1.67)	4.786 [1.63](1.62)	5.040 [1.70](1.70)
50	0.5	0.550 [.115](.103)	0.498 [.094](.094)	0.496 [.093](.093)	0.499 [.094](.094)
	0.8	0.883 [.183](.164)	0.799 [.148](.148)	0.795 [.148](.147)	0.801 [.148](.148)
	1.0	1.104 [.235](.211)	0.999 [.191](.191)	0.995 [.190](.190)	1.002 [.191](.191)
	2.0	2.206 [.459](.410)	1.997 [.372](.372)	1.989 [.370](.370)	2.003 [.372](.372)
	5.0	5.500 [1.15](1.04)	4.978 [.938](.937)	4.956 [.935](.934)	4.993 [.940](.940)
100	0.5	0.524 [.071](.067)	0.500 [.064](.064)	0.499 [.064](.064)	0.500 [.064](.064)
	0.8	0.839 [.115](.108)	0.799 [.103](.103)	0.799 [.103](.103)	0.800 [.103](.103)
	1.0	1.049 [.143](.135)	1.000 [.128](.128)	0.999 [.128](.128)	1.001 [.128](.128)
	2.0	2.101 [.287](.269)	2.002 [.256](.256)	2.000 [.256](.256)	2.004 [.256](.256)
	5.0	5.246 [.715](.671)	5.000 [.641](.641)	4.995 [.640](.640)	5.004 [.641](.641)

DOI: http://dx.doi.org/10.1080/00949655

Table 9. Empirical mean [rmse](se) of MLE-type estimators of β, Type-II data, $k = 8$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE}}$
		$p = 0.7$			
10	0.5	0.576 [.107](.075)	0.494 [.065](.065)	0.491 [.065](.064)	0.501 [.065](.065)
	0.8	0.920 [.169](.120)	0.788 [.104](.103)	0.783 [.104](.102)	0.800 [.104](.104)
	1.0	1.151 [.213](.150)	0.987 [.130](.129)	0.980 [.130](.128)	1.002 [.131](.131)
	2.0	2.299 [.423](.299)	1.971 [.258](.256)	1.958 [.258](.255)	2.001 [.260](.260)
	5.0	5.759 [1.07](.755)	4.937 [.651](.647)	4.904 [.650](.643)	5.012 [.657](.657)
20	0.5	0.535 [.058](.047)	0.499 [.044](.044)	0.498 [.044](.044)	0.501 [.044](.044)
	0.8	0.854 [.092](.074)	0.796 [.069](.069)	0.795 [.069](.069)	0.799 [.069](.069)
	1.0	1.069 [.117](.094)	0.997 [.088](.088)	0.996 [.087](.087)	1.001 [.088](.088)
	2.0	2.133 [.230](.188)	1.989 [.176](.176)	1.986 [.176](.175)	1.996 [.176](.176)
	5.0	5.338 [.576](.467)	4.978 [.436](.436)	4.971 [.436](.435)	4.997 [.437](.437)
50	0.5	0.514 [.031](.028)	0.500 [.027](.027)	0.500 [.027](.027)	0.501 [.027](.027)
	0.8	0.822 [.050](.045)	0.800 [.044](.044)	0.800 [.043](.043)	0.801 [.044](.044)
	1.0	1.027 [.062](.055)	1.000 [.054](.054)	1.000 [.054](.054)	1.001 [.054](.054)
	2.0	2.053 [.124](.112)	2.000 [.109](.109)	1.999 [.109](.109)	2.001 [.109](.109)
	5.0	5.134 [.309](.278)	5.001 [.271](.271)	5.000 [.271](.271)	5.004 [.271](.271)
100	0.5	0.506 [.020](.019)	0.500 [.019](.019)	0.500 [.019](.019)	0.500 [.019](.019)
	0.8	0.810 [.033](.031)	0.800 [.031](.031)	0.800 [.031](.031)	0.800 [.031](.031)
	1.0	1.013 [.040](.038)	1.000 [.038](.038)	1.000 [.038](.038)	1.000 [.038](.038)
	2.0	2.027 [.081](.077)	2.001 [.076](.076)	2.001 [.076](.076)	2.001 [.076](.076)
	5.0	5.065 [.203](.193)	4.999 [.191](.191)	4.999 [.191](.191)	5.000 [.191](.191)
		$p = 0.5$			
10	0.5	0.622 [.161](.105)	0.481 [.084](.082)	0.472 [.085](.080)	0.499 [.085](.085)
	0.8	0.998 [.261](.170)	0.772 [.134](.132)	0.758 [.136](.129)	0.801 [.136](.136)
	1.0	1.249 [.327](.212)	0.966 [.168](.164)	0.949 [.169](.161)	1.002 [.170](.170)
	2.0	2.500 [.654](.422)	1.934 [.334](.327)	1.899 [.337](.321)	2.006 [.338](.338)
	5.0	6.242 [1.63](1.06)	4.829 [.835](.818)	4.740 [.844](.803)	5.007 [.846](.846)
20	0.5	0.554 [.082](.061)	0.496 [.055](.055)	0.494 [.055](.055)	0.500 [.055](.055)
	0.8	0.886 [.130](.098)	0.793 [.088](.088)	0.791 [.088](.088)	0.800 [.088](.088)
	1.0	1.110 [.166](.123)	0.994 [.111](.111)	0.991 [.111](.110)	1.002 [.111](.111)
	2.0	2.218 [.327](.245)	1.985 [.220](.219)	1.978 [.219](.218)	2.001 [.221](.221)
	5.0	5.541 [.820](.617)	4.960 [.555](.553)	4.943 [.554](.551)	5.000 [.557](.557)
50	0.5	0.521 [.041](.035)	0.500 [.034](.034)	0.499 [.034](.034)	0.500 [.034](.034)
	0.8	0.833 [.065](.056)	0.800 [.054](.054)	0.799 [.054](.053)	0.801 [.054](.054)
	1.0	1.040 [.080](.069)	0.998 [.067](.067)	0.998 [.067](.067)	0.999 [.067](.067)
	2.0	2.082 [.163](.141)	1.998 [.135](.135)	1.997 [.135](.135)	2.001 [.135](.135)
	5.0	5.203 [.405](.351)	4.995 [.337](.337)	4.993 [.337](.337)	5.001 [.337](.337)
100	0.5	0.510 [.026](.024)	0.500 [.023](.023)	0.500 [.023](.023)	0.500 [.023](.023)
	0.8	0.816 [.042](.039)	0.800 [.038](.038)	0.799 [.038](.038)	0.800 [.038](.038)
	1.0	1.020 [.052](.048)	1.000 [.047](.047)	1.000 [.047](.047)	1.000 [.047](.047)
	2.0	2.040 [.103](.095)	2.000 [.093](.093)	2.000 [.093](.093)	2.001 [.093](.093)
	5.0	5.098 [.260](.241)	4.997 [.237](.237)	4.997 [.237](.237)	4.999 [.237](.237)
		$p = 0.3$			
10	0.5	0.770 [.333](.195)	0.431 [.130](.110)	0.389 [.149](.099)	0.498 [.126](.126)
	0.8	1.239 [.542](.318)	0.692 [.208](.178)	0.625 [.238](.161)	0.802 [.206](.206)
	1.0	1.551 [.680](.398)	0.867 [.260](.223)	0.783 [.296](.202)	1.004 [.258](.258)
	2.0	3.094 [1.36](.799)	1.730 [.523](.448)	1.562 [.597](.405)	2.002 [.517](.517)
	5.0	7.734 [3.36](1.96)	4.325 [1.29](1.10)	3.906 [1.48](.992)	5.005 [1.27](1.27)
20	0.5	0.608 [.144](.095)	0.489 [.078](.077)	0.482 [.078](.076)	0.502 [.079](.079)
	0.8	0.972 [.229](.151)	0.781 [.123](.122)	0.771 [.124](.120)	0.802 [.125](.125)
	1.0	1.215 [.285](.187)	0.976 [.152](.151)	0.963 [.153](.149)	1.003 [.155](.154)
	2.0	2.421 [.564](.376)	1.946 [.307](.302)	1.920 [.309](.298)	1.999 [.310](.310)
	5.0	6.056 [1.41](.934)	4.869 [.764](.753)	4.804 [.768](.743)	5.001 [.772](.772)
50	0.5	0.538 [.063](.050)	0.499 [.046](.046)	0.498 [.046](.046)	0.501 [.047](.047)
	0.8	0.860 [.099](.079)	0.798 [.073](.073)	0.796 [.073](.073)	0.801 [.073](.073)
	1.0	1.074 [.123](.098)	0.996 [.091](.091)	0.994 [.091](.091)	1.000 [.091](.091)
	2.0	2.150 [.248](.197)	1.993 [.183](.183)	1.990 [.183](.183)	2.001 [.183](.183)
	5.0	5.380 [.622](.492)	4.989 [.457](.457)	4.981 [.457](.456)	5.007 [.458](.458)
100	0.5	0.518 [.037](.032)	0.499 [.031](.031)	0.499 [.031](.031)	0.500 [.031](.031)
	0.8	0.828 [.060](.053)	0.799 [.051](.051)	0.799 [.051](.051)	0.800 [.051](.051)
	1.0	1.036 [.075](.066)	0.999 [.064](.064)	0.999 [.064](.064)	1.000 [.064](.064)
	2.0	2.070 [.147](.129)	1.997 [.125](.125)	1.996 [.125](.125)	1.998 [.125](.125)
	5.0	5.180 [.375](.329)	4.997 [.318](.318)	4.995 [.318](.318)	5.001 [.318](.318)

DOI: http://dx.doi.org/10.1080/00949655

Table 10. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 1$

n	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE-I}}$	$\hat{\beta}_{\text{MMLE-II}}$
		$p = 0.7$				
10	0.5	0.577 [.200](.185)	0.490 [.157]	0.495 [.159]	0.519 [.162](.161)	0.490 [.163](.163)
	0.8	0.918 [.316](.293)	0.781 [.249]	0.788 [.251]	0.823 [.256](.255)	0.783 [.257](.257)
	1.0	1.156 [.392](.360)	0.982 [.306]	0.991 [.309]	1.031 [.313](.312)	0.991 [.315](.314)
	2.0	2.338 [.782](.705)	1.981 [.599]	2.000 [.604]	2.044 [.611](.609)	2.028 [.615](.614)
	5.0	5.865 [1.97](1.77)	4.970 [1.51]	5.016 [1.52]	5.099 [1.54](1.53)	5.099 [1.54](1.53)
20	0.5	0.532 [.114](.110)	0.495 [.102]	0.496 [.102]	0.511 [.104](.103)	0.493 [.104](.103)
	0.8	0.853 [.179](.171)	0.792 [.159]	0.794 [.159]	0.817 [.162](.161)	0.791 [.161](.161)
	1.0	1.066 [.222](.212)	0.990 [.197]	0.992 [.198]	1.019 [.200](.199)	0.991 [.200](.200)
	2.0	2.147 [.431](.406)	1.991 [.377]	1.995 [.378]	2.023 [.379](.378)	2.009 [.381](.381)
	5.0	5.372 [1.11](1.05)	4.980 [.972]	4.992 [.974]	5.032 [.980](.980)	5.032 [.980](.980)
50	0.5	0.511 [.064](.063)	0.497 [.061]	0.498 [.061]	0.504 [.062](.062)	0.496 [.061](.061)
	0.8	0.819 [.102](.100)	0.796 [.097]	0.796 [.097]	0.807 [.099](.098)	0.795 [.098](.098)
	1.0	1.025 [.126](.124)	0.997 [.120]	0.997 [.120]	1.011 [.122](.122)	0.997 [.121](.121)
	2.0	2.055 [.244](.238)	1.997 [.231]	1.998 [.231]	2.015 [.231](.231)	2.003 [.232](.232)
	5.0	5.133 [.607](.592)	4.987 [.576]	4.989 [.577]	5.006 [.577](.577)	5.006 [.577](.577)
100	0.5	0.506 [.045](.044)	0.499 [.044]	0.499 [.044]	0.503 [.044](.044)	0.499 [.044](.044)
	0.8	0.809 [.070](.069)	0.798 [.068]	0.798 [.068]	0.804 [.069](.069)	0.798 [.068](.068)
	1.0	1.011 [.085](.084)	0.997 [.083]	0.997 [.083]	1.005 [.084](.084)	0.997 [.083](.083)
	2.0	2.029 [.166](.164)	2.001 [.161]	2.001 [.162]	2.012 [.162](.162)	2.004 [.162](.162)
	5.0	5.072 [.411](.405)	5.000 [.400]	5.001 [.400]	5.010 [.400](.400)	5.010 [.400](.400)
		$p = 0.5$				
10	0.5	0.575 [.204](.190)	0.489 [.162]	0.495 [.163]	0.524 [.171](.169)	0.478 [.167](.166)
	0.8	0.922 [.331](.308)	0.783 [.262]	0.794 [.264]	0.843 [.280](.277)	0.761 [.269](.266)
	1.0	1.152 [.417](.389)	0.977 [.329]	0.992 [.332]	1.055 [.357](.353)	0.953 [.339](.336)
	2.0	2.337 [.843](.773)	1.977 [.652]	2.003 [.660]	2.124 [.711](.700)	1.977 [.666](.666)
	5.0	5.852 [1.97](1.78)	4.950 [1.50]	5.005 [1.52]	5.230 [1.62](1.60)	5.038 [1.54](1.54)
20	0.5	0.533 [.119](.114)	0.495 [.106]	0.497 [.106]	0.513 [.111](.110)	0.488 [.108](.107)
	0.8	0.852 [.195](.188)	0.791 [.175]	0.794 [.175]	0.821 [.182](.181)	0.779 [.177](.176)
	1.0	1.064 [.240](.231)	0.988 [.215]	0.991 [.215]	1.026 [.224](.223)	0.974 [.218](.216)
	2.0	2.147 [.468](.444)	1.990 [.412]	1.996 [.413]	2.069 [.433](.428)	1.987 [.416](.416)
	5.0	5.383 [1.12](1.05)	4.986 [.976]	5.001 [.979]	5.145 [1.02](1.01)	5.021 [.984](.983)
50	0.5	0.511 [.068](.067)	0.498 [.065]	0.498 [.065]	0.504 [.066](.066)	0.494 [.066](.065)
	0.8	0.821 [.110](.108)	0.798 [.105]	0.799 [.105]	0.809 [.107](.106)	0.793 [.105](.105)
	1.0	1.024 [.135](.133)	0.996 [.130]	0.996 [.130]	1.010 [.132](.132)	0.990 [.130](.130)
	2.0	2.054 [.262](.257)	1.995 [.249]	1.996 [.250]	2.027 [.255](.254)	1.994 [.250](.250)
	5.0	5.161 [.628](.607)	5.013 [.590]	5.015 [.591]	5.090 [.606](.599)	5.025 [.592](.592)
100	0.5	0.506 [.047](.047)	0.500 [.046]	0.500 [.046]	0.503 [.047](.046)	0.498 [.046](.046)
	0.8	0.810 [.076](.075)	0.799 [.074]	0.799 [.074]	0.804 [.075](.075)	0.796 [.074](.074)
	1.0	1.012 [.095](.094)	0.998 [.093]	0.998 [.093]	1.005 [.093](.093)	0.995 [.093](.093)
	2.0	2.027 [.178](.176)	1.998 [.173]	1.999 [.173]	2.014 [.175](.175)	1.998 [.174](.174)
	5.0	5.079 [.422](.414)	5.006 [.409]	5.007 [.409]	5.047 [.414](.412)	5.013 [.409](.409)
		$p = 0.3$				
10	0.5	0.574 [.213](.200)	0.488 [.170]	0.496 [.172]	0.526 [.183](.181)	0.467 [.175](.171)
	0.8	0.925 [.366](.344)	0.782 [.290]	0.800 [.295]	0.853 [.322](.317)	0.735 [.298](.291)
	1.0	1.164 [.462](.433)	0.980 [.362]	1.004 [.369]	1.074 [.408](.401)	0.920 [.371](.363)
	2.0	2.355 [.953](.884)	1.962 [.721]	2.009 [.737]	2.185 [.849](.829)	1.894 [.748](.741)
	5.0	6.046 [2.41](2.17)	4.973 [1.71]	5.077 [1.76]	5.637 [2.15](2.05)	5.034 [1.79](1.79)
20	0.5	0.533 [.125](.121)	0.496 [.112]	0.497 [.113]	0.514 [.117](.116)	0.484 [.113](.112)
	0.8	0.855 [.209](.201)	0.792 [.187]	0.796 [.187]	0.823 [.196](.194)	0.769 [.189](.186)
	1.0	1.075 [.269](.258)	0.995 [.239]	1.000 [.240]	1.035 [.252](.250)	0.966 [.241](.239)
	2.0	2.152 [.527](.504)	1.983 [.463]	1.993 [.465]	2.079 [.496](.490)	1.951 [.470](.467)
	5.0	5.449 [1.27](1.19)	4.998 [1.08]	5.021 [1.09]	5.283 [1.19](1.15)	5.021 [1.10](1.10)
50	0.5	0.511 [.071](.070)	0.497 [.068]	0.498 [.068]	0.504 [.069](.069)	0.493 [.069](.068)
	0.8	0.820 [.120](.119)	0.797 [.115]	0.798 [.115]	0.808 [.117](.117)	0.788 [.116](.115)
	1.0	1.028 [.151](.149)	0.998 [.144]	0.999 [.144]	1.012 [.147](.147)	0.986 [.145](.144)
	2.0	2.058 [.290](.284)	1.995 [.275]	1.996 [.276]	2.030 [.283](.281)	1.982 [.277](.277)
	5.0	5.165 [.676](.656)	5.000 [.634]	5.004 [.634]	5.103 [.657](.649)	5.008 [.637](.637)
100	0.5	0.505 [.049](.049)	0.498 [.048]	0.498 [.048]	0.501 [.049](.049)	0.495 [.048](.048)
	0.8	0.809 [.082](.082)	0.798 [.081]	0.798 [.081]	0.803 [.081](.081)	0.793 [.081](.081)
	1.0	1.013 [.104](.103)	0.999 [.101]	0.999 [.101]	1.006 [.102](.102)	0.993 [.102](.101)
	2.0	2.031 [.201](.199)	2.001 [.196]	2.001 [.196]	2.018 [.199](.198)	1.994 [.196](.196)
	5.0	5.085 [.451](.443)	5.005 [.436]	5.006 [.436]	5.055 [.444](.440)	5.009 [.436](.436)

Note: for $\hat{\beta}^{bc2}_{\text{MLE}}$ and $\hat{\beta}^{bc3}_{\text{MLE}}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.

DOI: http://dx.doi.org/10.1080/00949655

Table 11. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 2$

n_i	β	$\hat{\beta}_{\text{MLE}}$	$\hat{\beta}^{bc2}_{\text{MLE}}$	$\hat{\beta}^{bc3}_{\text{MLE}}$	$\hat{\beta}_{\text{MMLE-I}}$	$\hat{\beta}_{\text{MMLE-II}}$
		$p = 0.7$				
10	0.5	0.552 [.130](.119)	0.495 [.107]	0.494 [.106]	0.510 [.109](.109)	0.489 [.109](.109)
	0.8	0.881 [.201](.184)	0.790 [.165]	0.788 [.165]	0.814 [.168](.168)	0.784 [.168](.167)
	1.0	1.104 [.252](.229)	0.989 [.205]	0.987 [.205]	1.017 [.208](.207)	0.986 [.209](.208)
	2.0	2.222 [.492](.440)	1.987 [.394]	1.983 [.394]	2.017 [.395](.395)	2.001 [.397](.397)
	5.0	5.587 [1.26](1.12)	4.993 [1.00]	4.982 [.999]	5.042 [1.01](1.01)	5.042 [1.01](1.01)
20	0.5	0.522 [.078](.075)	0.497 [.071]	0.497 [.071]	0.504 [.072](.072)	0.493 [.072](.071)
	0.8	0.837 [.123](.117)	0.796 [.111]	0.796 [.111]	0.808 [.113](.113)	0.792 [.112](.112)
	1.0	1.047 [.151](.143)	0.996 [.136]	0.995 [.136]	1.011 [.138](.138)	0.992 [.137](.137)
	2.0	2.105 [.298](.278)	1.999 [.264]	1.998 [.264]	2.016 [.265](.264)	2.004 [.265](.265)
	5.0	5.251 [.729](.685)	4.985 [.651]	4.983 [.651]	5.003 [.653](.653)	5.003 [.653](.653)
50	0.5	0.509 [.046](.045)	0.499 [.044]	0.499 [.044]	0.502 [.044](.044)	0.497 [.044](.044)
	0.8	0.815 [.072](.071)	0.799 [.069]	0.799 [.069]	0.804 [.070](.070)	0.797 [.070](.069)
	1.0	1.019 [.089](.087)	1.000 [.085]	1.000 [.085]	1.006 [.086](.086)	0.998 [.085](.085)
	2.0	2.039 [.170](.166)	1.999 [.162]	1.999 [.162]	2.009 [.163](.163)	2.000 [.163](.163)
	5.0	5.103 [.419](.406)	5.002 [.399]	5.002 [.399]	5.009 [.399](.399)	5.009 [.399](.399)
100	0.5	0.504 [.031](.031)	0.499 [.031]	0.499 [.031]	0.500 [.031](.031)	0.498 [.031](.031)
	0.8	0.807 [.050](.049)	0.799 [.049]	0.799 [.049]	0.801 [.049](.049)	0.798 [.049](.049)
	1.0	1.009 [.060](.060)	1.000 [.059]	1.000 [.059]	1.003 [.059](.059)	0.999 [.059](.059)
	2.0	2.019 [.116](.115)	1.999 [.114]	1.999 [.114]	2.006 [.114](.114)	2.000 [.114](.114)
	5.0	5.047 [.285](.282)	4.997 [.279]	4.997 [.279]	5.000 [.279](.279)	5.000 [.279](.279)
		$p = 0.5$				
10	0.5	0.546 [.129](.121)	0.491 [.108]	0.490 [.108]	0.506 [.112](.112)	0.477 [.112](.109)
	0.8	0.877 [.211](.196)	0.788 [.176]	0.787 [.176]	0.813 [.183](.183)	0.763 [.181](.177)
	1.0	1.098 [.262](.243)	0.985 [.218]	0.984 [.218]	1.019 [.227](.226)	0.957 [.224](.219)
	2.0	2.222 [.521](.472)	1.988 [.422]	1.985 [.421]	2.064 [.443](.438)	1.967 [.426](.424)
	5.0	5.585 [1.28](1.14)	4.990 [1.02]	4.980 [1.02]	5.149 [1.07](1.06)	5.006 [1.03](1.03)
20	0.5	0.522 [.082](.079)	0.497 [.075]	0.497 [.075]	0.504 [.076](.076)	0.490 [.076](.075)
	0.8	0.834 [.131](.127)	0.794 [.121]	0.794 [.121]	0.804 [.123](.123)	0.780 [.122](.121)
	1.0	1.046 [.163](.156)	0.995 [.149]	0.995 [.149]	1.009 [.152](.151)	0.980 [.150](.149)
	2.0	2.099 [.312](.296)	1.994 [.281]	1.994 [.281]	2.029 [.289](.287)	1.982 [.283](.282)
	5.0	5.271 [.764](.715)	5.004 [.678]	5.002 [.678]	5.091 [.697](.691)	5.007 [.680](.680)
50	0.5	0.508 [.048](.047)	0.499 [.046]	0.499 [.046]	0.501 [.046](.046)	0.495 [.046](.046)
	0.8	0.814 [.077](.076)	0.799 [.074]	0.799 [.074]	0.802 [.075](.075)	0.793 [.075](.074)
	1.0	1.018 [.097](.096)	0.999 [.094]	0.999 [.094]	1.004 [.095](.094)	0.993 [.094](.094)
	2.0	2.040 [.181](.177)	2.000 [.174]	2.000 [.174]	2.013 [.175](.175)	1.995 [.174](.174)
	5.0	5.102 [.431](.419)	5.001 [.411]	5.001 [.411]	5.038 [.415](.413)	5.002 [.411](.411)
100	0.5	0.504 [.033](.033)	0.499 [.032]	0.499 [.032]	0.500 [.032](.032)	0.497 [.032](.032)
	0.8	0.807 [.053](.053)	0.799 [.052]	0.799 [.052]	0.801 [.052](.052)	0.796 [.052](.052)
	1.0	1.009 [.066](.065)	1.000 [.065]	1.000 [.065]	1.002 [.065](.065)	0.996 [.065](.065)
	2.0	2.019 [.123](.122)	1.999 [.120]	1.999 [.120]	2.006 [.121](.121)	1.997 [.120](.120)
	5.0	5.054 [.294](.289)	5.004 [.287]	5.004 [.287]	5.023 [.288](.287)	5.005 [.286](.286)
		$p = 0.3$				
10	0.5	0.546 [.133](.125)	0.491 [.112]	0.490 [.112]	0.506 [.116](.116)	0.470 [.116](.112)
	0.8	0.881 [.231](.216)	0.792 [.193]	0.792 [.193]	0.816 [.202](.201)	0.748 [.198](.191)
	1.0	1.100 [.292](.274)	0.987 [.245]	0.988 [.244]	1.019 [.256](.256)	0.931 [.252](.242)
	2.0	2.232 [.594](.547)	1.988 [.482]	1.988 [.482]	2.078 [.520](.514)	1.912 [.494](.486)
	5.0	5.695 [1.48](1.31)	5.017 [1.14]	5.010 [1.14]	5.339 [1.28](1.23)	5.001 [1.15](1.15)
20	0.5	0.521 [.084](.082)	0.497 [.078]	0.497 [.078]	0.502 [.079](.079)	0.486 [.079](.078)
	0.8	0.837 [.142](.137)	0.797 [.131]	0.797 [.131]	0.806 [.133](.133)	0.774 [.132](.130)
	1.0	1.044 [.178](.173)	0.994 [.164]	0.994 [.164]	1.006 [.167](.167)	0.965 [.167](.163)
	2.0	2.098 [.347](.333)	1.990 [.315]	1.990 [.315]	2.027 [.325](.323)	1.952 [.319](.315)
	5.0	5.307 [.844](.787)	5.009 [.738]	5.009 [.738]	5.147 [.779](.765)	4.999 [.742](.742)
50	0.5	0.509 [.050](.049)	0.500 [.048]	0.500 [.048]	0.502 [.048](.048)	0.495 [.048](.048)
	0.8	0.813 [.083](.082)	0.798 [.081]	0.798 [.081]	0.801 [.081](.081)	0.789 [.081](.080)
	1.0	1.017 [.104](.103)	0.998 [.101]	0.998 [.101]	1.002 [.102](.102)	0.986 [.102](.101)
	2.0	2.039 [.203](.200)	1.998 [.195]	1.998 [.195]	2.011 [.198](.197)	1.983 [.196](.195)
	5.0	5.110 [.465](.452)	4.999 [.442]	4.999 [.442]	5.049 [.450](.447)	4.994 [.442](.442)
100	0.5	0.504 [.035](.034)	0.500 [.034]	0.500 [.034]	0.501 [.034](.034)	0.497 [.034](.034)
	0.8	0.807 [.057](.057)	0.800 [.056]	0.800 [.056]	0.801 [.056](.056)	0.795 [.056](.056)
	1.0	1.008 [.073](.072)	0.999 [.072]	0.999 [.072]	1.001 [.072](.072)	0.993 [.072](.071)
	2.0	2.022 [.140](.138)	2.002 [.137]	2.002 [.137]	2.008 [.138](.137)	1.994 [.137](.137)
	5.0	5.059 [.319](.313)	5.005 [.310]	5.005 [.310]	5.029 [.313](.311)	5.002 [.310](.310)

Note: for $\hat{\beta}^{bc2}_{\text{MLE}}$ and $\hat{\beta}^{bc3}_{\text{MLE}}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.

DOI: http://dx.doi.org/10.1080/00949655

Table 12. Empirical mean [rmse](se) of MLE-type estimators of β, random censoring, $k = 8$

n_i	β	$\hat{\beta}_{\rm MLE}$	$\hat{\beta}^{bc2}_{\rm MLE}$	$\hat{\beta}^{bc3}_{\rm MLE}$	$\hat{\beta}_{\rm MMLE-I}$	$\hat{\beta}_{\rm MMLE-II}$
		$p = 0.7$				
10	0.5	0.534 [.065](.055)	0.497 [.051]	0.497 [.051]	0.497 [.052](.052)	0.488 [.053](.052)
	0.8	0.856 [.103](.087)	0.797 [.081]	0.795 [.081]	0.797 [.081](.081)	0.785 [.082](.081)
	1.0	1.070 [.128](.107)	0.996 [.099]	0.995 [.099]	0.998 [.100](.100)	0.985 [.101](.099)
	2.0	2.158 [.261](.207)	2.005 [.193]	2.001 [.192]	2.011 [.192](.192)	1.998 [.192](.192)
	5.0	5.401 [.653](.515)	5.015 [.479]	5.005 [.478]	5.008 [.477](.477)	5.008 [.477](.477)
20	0.5	0.515 [.039](.036)	0.499 [.035]	0.499 [.035]	0.498 [.035](.035)	0.494 [.036](.035)
	0.8	0.824 [.062](.057)	0.798 [.055]	0.798 [.055]	0.797 [.055](.055)	0.791 [.056](.055)
	1.0	1.032 [.077](.070)	0.998 [.068]	0.998 [.068]	0.998 [.068](.068)	0.991 [.069](.068)
	2.0	2.072 [.152](.134)	2.002 [.129]	2.002 [.129]	2.005 [.129](.129)	1.997 [.129](.129)
	5.0	5.187 [.383](.334)	5.012 [.323]	5.010 [.323]	5.004 [.322](.322)	5.004 [.322](.322)
50	0.5	0.506 [.023](.022)	0.500 [.022]	0.500 [.022]	0.499 [.022](.022)	0.497 [.022](.022)
	0.8	0.809 [.035](.034)	0.799 [.034]	0.799 [.034]	0.798 [.034](.034)	0.796 [.034](.034)
	1.0	1.012 [.044](.043)	1.000 [.042]	0.999 [.042]	0.999 [.042](.042)	0.996 [.042](.042)
	2.0	2.028 [.087](.082)	2.001 [.081]	2.001 [.081]	2.002 [.081](.081)	1.999 [.081](.081)
	5.0	5.074 [.215](.202)	5.007 [.200]	5.007 [.200]	5.003 [.199](.199)	5.003 [.199](.199)
100	0.5	0.503 [.016](.015)	0.500 [.015]	0.500 [.015]	0.499 [.015](.015)	0.499 [.015](.015)
	0.8	0.805 [.025](.024)	0.800 [.024]	0.800 [.024]	0.800 [.024](.024)	0.799 [.024](.024)
	1.0	1.006 [.030](.030)	1.000 [.029]	1.000 [.029]	0.999 [.029](.029)	0.998 [.029](.029)
	2.0	2.014 [.059](.057)	2.001 [.057]	2.001 [.057]	2.002 [.057](.057)	2.000 [.057](.057)
	5.0	5.034 [.143](.139)	5.001 [.139]	5.001 [.139]	4.999 [.139](.139)	4.999 [.139](.139)
		$p = 0.5$				
10	0.5	0.530 [.064](.057)	0.496 [.053]	0.495 [.053]	0.492 [.054](.053)	0.480 [.057](.053)
	0.8	0.850 [.106](.093)	0.795 [.087]	0.794 [.087]	0.789 [.088](.087)	0.768 [.092](.086)
	1.0	1.063 [.132](.116)	0.994 [.108]	0.992 [.108]	0.988 [.109](.109)	0.962 [.114](.108)
	2.0	2.148 [.265](.219)	2.001 [.204]	1.997 [.203]	2.004 [.205](.205)	1.966 [.206](.203)
	5.0	5.402 [.663](.528)	5.019 [.491]	5.010 [.490]	5.048 [.496](.494)	4.984 [.489](.489)
20	0.5	0.514 [.041](.038)	0.499 [.037]	0.499 [.037]	0.496 [.037](.037)	0.490 [.038](.037)
	0.8	0.823 [.065](.061)	0.798 [.059]	0.798 [.059]	0.794 [.060](.059)	0.784 [.061](.059)
	1.0	1.030 [.082](.076)	0.998 [.074]	0.998 [.074]	0.994 [.074](.074)	0.981 [.076](.074)
	2.0	2.071 [.162](.145)	2.004 [.141]	2.004 [.141]	2.003 [.141](.141)	1.985 [.141](.140)
	5.0	5.188 [.393](.345)	5.014 [.334]	5.012 [.334]	5.023 [.335](.334)	4.993 [.333](.333)
50	0.5	0.506 [.024](.023)	0.500 [.023]	0.500 [.023]	0.498 [.023](.023)	0.496 [.023](.023)
	0.8	0.809 [.039](.038)	0.800 [.037]	0.800 [.037]	0.797 [.037](.037)	0.793 [.038](.037)
	1.0	1.012 [.048](.047)	1.000 [.046]	1.000 [.046]	0.998 [.046](.046)	0.993 [.047](.046)
	2.0	2.026 [.091](.087)	2.001 [.086]	2.001 [.086]	2.000 [.086](.086)	1.993 [.086](.086)
	5.0	5.070 [.219](.207)	5.005 [.205]	5.004 [.205]	5.007 [.205](.205)	4.996 [.204](.204)
100	0.5	0.503 [.017](.016)	0.500 [.016]	0.500 [.016]	0.499 [.016](.016)	0.498 [.016](.016)
	0.8	0.804 [.026](.026)	0.800 [.026]	0.800 [.026]	0.799 [.026](.026)	0.797 [.026](.026)
	1.0	1.006 [.033](.033)	1.000 [.032]	1.000 [.032]	0.999 [.032](.032)	0.997 [.032](.032)
	2.0	2.013 [.063](.062)	2.001 [.062]	2.001 [.062]	2.000 [.062](.062)	1.997 [.062](.061)
	5.0	5.036 [.150](.145)	5.004 [.145]	5.004 [.145]	5.005 [.145](.144)	4.999 [.144](.144)
		$p = 0.3$				
10	0.5	0.529 [.066](.059)	0.496 [.055]	0.495 [.055]	0.490 [.056](.055)	0.475 [.060](.055)
	0.8	0.847 [.109](.099)	0.794 [.092]	0.793 [.092]	0.784 [.093](.092)	0.752 [.102](.090)
	1.0	1.060 [.137](.124)	0.994 [.115]	0.992 [.115]	0.981 [.117](.115)	0.939 [.129](.113)
	2.0	2.138 [.281](.244)	1.993 [.226]	1.989 [.225]	1.989 [.230](.229)	1.911 [.241](.224)
	5.0	5.444 [.731](.580)	5.027 [.531]	5.015 [.529]	5.103 [.556](.547)	4.954 [.532](.530)
20	0.5	0.514 [.041](.039)	0.499 [.038]	0.499 [.038]	0.495 [.038](.038)	0.487 [.040](.038)
	0.8	0.823 [.070](.066)	0.799 [.064]	0.799 [.064]	0.792 [.064](.064)	0.777 [.067](.063)
	1.0	1.029 [.088](.083)	0.998 [.081]	0.998 [.080]	0.991 [.081](.080)	0.970 [.085](.080)
	2.0	2.067 [.176](.162)	2.001 [.156]	2.000 [.156]	1.997 [.158](.158)	1.960 [.161](.156)
	5.0	5.206 [.427](.374)	5.018 [.359]	5.016 [.359]	5.049 [.366](.363)	4.981 [.359](.358)
50	0.5	0.505 [.025](.024)	0.500 [.024]	0.500 [.024]	0.498 [.024](.024)	0.495 [.024](.024)
	0.8	0.809 [.041](.040)	0.800 [.040]	0.800 [.040]	0.797 [.040](.040)	0.791 [.041](.040)
	1.0	1.011 [.052](.051)	0.999 [.050]	0.999 [.050]	0.996 [.050](.050)	0.988 [.051](.050)
	2.0	2.024 [.100](.098)	1.999 [.096]	1.999 [.096]	1.996 [.097](.096)	1.982 [.098](.096)
	5.0	5.081 [.235](.220)	5.010 [.217]	5.009 [.217]	5.021 [.219](.218)	4.995 [.217](.217)
100	0.5	0.503 [.017](.017)	0.500 [.017]	0.500 [.017]	0.499 [.017](.017)	0.497 [.017](.017)
	0.8	0.804 [.029](.028)	0.800 [.028]	0.800 [.028]	0.798 [.028](.028)	0.795 [.028](.028)
	1.0	1.005 [.036](.036)	1.000 [.035]	1.000 [.035]	0.998 [.035](.035)	0.994 [.036](.035)
	2.0	2.012 [.068](.067)	1.999 [.067]	1.999 [.067]	1.998 [.067](.067)	1.991 [.067](.067)
	5.0	5.038 [.159](.154)	5.003 [.153]	5.003 [.153]	5.008 [.153](.153)	4.996 [.153](.153)

Note: for $\hat{\beta}^{bc2}_{\rm MLE}$ and $\hat{\beta}^{bc3}_{\rm MLE}$, the empirical sds are almost identical to the rmses and hence are not reported to conserve space.

DOI: http://dx.doi.org/10.1080/00949655

Table 13. Empirical coverage probability (length) of confidence intervals for β_0, complete data, $k = 8$

		$1-\gamma=0.90$		$1-\gamma=0.95$		$1-\gamma=0.99$	
n_i	β_0	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$	$\hat{\beta}_{\mathrm{MLE}}$	$\hat{\beta}^{bc2}_{\mathrm{MLE}}$
10	0.5	0.8131 (0.1586)	0.9272 (0.1734)	0.8943 (0.1890)	0.9684 (0.2066)	0.9734 (0.2484)	0.9940 (0.2715)
	0.8	0.8055 (0.2542)	0.9271 (0.2780)	0.8867 (0.3029)	0.9675 (0.3313)	0.9707 (0.3981)	0.9936 (0.4353)
	1.0	0.8037 (0.3178)	0.9263 (0.3474)	0.8857 (0.3787)	0.9680 (0.4140)	0.9684 (0.4977)	0.9931 (0.5441)
	2.0	0.8086 (0.6350)	0.9328 (0.6941)	0.8909 (0.7567)	0.9698 (0.8271)	0.9744 (0.9944)	0.9947 (1.0870)
	5.0	0.8054 (1.5882)	0.9288 (1.7376)	0.8906 (1.8924)	0.9672 (2.0705)	0.9726 (2.4871)	0.9954 (2.7211)
20	0.5	0.8611 (0.1064)	0.9173 (0.1113)	0.9256 (0.1268)	0.9579 (0.1326)	0.9832 (0.1667)	0.9933 (0.1743)
	0.8	0.8563 (0.1704)	0.9185 (0.1782)	0.9219 (0.2030)	0.9609 (0.2124)	0.9826 (0.2668)	0.9941 (0.2791)
	1.0	0.8569 (0.2130)	0.9137 (0.2226)	0.9219 (0.2538)	0.9572 (0.2652)	0.9812 (0.3335)	0.9912 (0.3486)
	2.0	0.8501 (0.4261)	0.9148 (0.4455)	0.9174 (0.5077)	0.9607 (0.5309)	0.9832 (0.6673)	0.9928 (0.6977)
	5.0	0.8587 (1.0645)	0.9205 (1.1134)	0.9195 (1.2684)	0.9609 (1.3266)	0.9805 (1.6670)	0.9924 (1.7435)
50	0.5	0.8833 (0.0653)	0.9070 (0.0664)	0.9405 (0.0778)	0.9556 (0.0791)	0.9876 (0.1023)	0.9903 (0.1040)
	0.8	0.8839 (0.1046)	0.9089 (0.1064)	0.9402 (0.1247)	0.9570 (0.1268)	0.9884 (0.1638)	0.9929 (0.1666)
	1.0	0.8830 (0.1307)	0.9065 (0.1329)	0.9388 (0.1558)	0.9543 (0.1584)	0.9874 (0.2047)	0.9912 (0.2082)
	2.0	0.8861 (0.2614)	0.9074 (0.2659)	0.9407 (0.3115)	0.9545 (0.3169)	0.9843 (0.4094)	0.9899 (0.4164)
	5.0	0.8835 (0.6536)	0.9070 (0.6652)	0.9399 (0.7789)	0.9530 (0.7927)	0.9880 (1.0236)	0.9903 (1.0417)
100	0.5	0.8863 (0.0458)	0.8989 (0.0461)	0.9407 (0.0545)	0.9454 (0.0550)	0.9879 (0.0717)	0.9890 (0.0723)
	0.8	0.8911 (0.0732)	0.9045 (0.0738)	0.9455 (0.0873)	0.9513 (0.0879)	0.9887 (0.1147)	0.9902 (0.1156)
	1.0	0.8896 (0.0915)	0.8995 (0.0922)	0.9420 (0.1091)	0.9528 (0.1099)	0.9884 (0.1434)	0.9889 (0.1445)
	2.0	0.8849 (0.1831)	0.9011 (0.1846)	0.9442 (0.2182)	0.9540 (0.2200)	0.9888 (0.2868)	0.9909 (0.2891)
	5.0	0.8897 (0.4577)	0.9016 (0.4614)	0.9433 (0.5454)	0.9498 (0.5498)	0.9883 (0.7168)	0.9899 (0.7226)