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Shrinkage Estimation of Dynamic Panel Data Models with Interactive Fixed Effects

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Abstract

We consider the problem of determining the number of factors and selecting the proper regressors in linear dynamic panel data models with interactive fixed effects. Based on the preliminary estimates of the slope parameters and factors a la Bai and Ng (2009) and Moon and Weidner (2010a), we propose a method for simultaneous selection of regressors and factors and estimation through the method of adaptive group Lasso (least absolute shrinkage and selection operator). We show that with probability approaching one, our method can correctly select all relevant regressors and factors and shrink the coefficients of irrelevant regressors and redundant factors to zero. Further, we demonstrate that our shrinkage estimators of the nonzero slope parameters exhibit some oracle property. We conduct Monte Carlo simulations to demonstrate the superb finite-sample performance of the proposed method. We apply our method to study the determinants of economic growth and find that in addition to three common unobserved factors selected by our method, government consumption share has negative effects, whereas investment share and lagged economic growth have positive effects on economic growth.

JEL Classification: C13, C23, C51

Key Words: Adaptive Lasso; Dynamic panel; Factor selection; Group Lasso; Interactive fixed effects; Oracle property; Selection consistency

1 Introduction

We consider a panel data model with interactive fixed effects as proposed and studied in Pesaran (2006), Bai (2009), Moon and Weidner (2010a, 2010b), Pesaran and Tosetti (2011), Greenaway-McGregor et al. (2012), Su and Jin (2012), Su et al. (2013), among others. This model has been widely applied in empirical research, as it allows more flexible modeling of heterogeneity than traditional fixed effects.
models and provides an effective way to model cross section dependence that is widely present in macro and financial data. To use this model, we need to determine the number of factors in the multi-factor error component and select the proper regressors to be included in the model. This paper provides a novel automated estimation method that combines both estimation of parameters of interest and selection of the number of factors and regressors.

Specifically, we consider the following interactive fixed-effects panel data model

\[
Y_{it} = \beta^0 X_{it} + \lambda^0_i F^0_t + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T, \tag{1.1}
\]

where \(X_{it}\) is a \(K \times 1\) vector of regressors, \(\beta^0\) the corresponding vector of slope coefficients, \(\lambda^0_i\) an \(R_0 \times 1\) vector of unknown factor loadings, \(F^0_t\) an \(R_0 \times 1\) vector of unknown common factors, and \(\varepsilon_{it}\) the idiosyncratic error term. Here the factor structure \(\lambda^0_i F^0_t\) is referred to as interactive fixed effects in Bai (2009) and Moon and Weidner (2010a, 2010b) as one allows both \(\lambda^0_i\) and \(F^0_t\) to be correlated with elements of \(X_{it}\) and \(\lambda^0_i F^0_t + \varepsilon_{it}\) is called the multi-factor error structure in Pesaran (2006). We are interested in estimating \(\beta^0, \lambda^0_i\) and \(F^0_t\). It has been argued that the factor structure can capture more flexible heterogeneity across individuals and over time than the traditional fixed-effects model. The latter takes the form

\[
Y_{it} = \beta^0 X_{it} + \alpha^0_i + \xi^0_i + \varepsilon_{it}
\]

and can be thought of as a special case of the interactive fixed-effects panel data model by letting \(F^0_t = (1, \xi^0_i)\) and \(\lambda^0_i = (\alpha^0_i, 1)'\), where \(\alpha^0_i\) and \(\xi^0_i\) are individual-specific and time-specific fixed effects, respectively. When \(X_{it}\) is absent in (1.1), the model becomes the pure factor model studied in Bai and Ng (2002) and Bai (2003), among others.

Given the correct number \(R_0\) of factors and the proper regressors \(X_{it}\), several estimation methods have been proposed in the literature. For example, Pesaran (2006) proposes common correlated effects (CCE) estimators; Bai (2009) and Moon and Weidner (2010a, 2010b) provide estimators based on Gaussian quasi-maximum likelihood estimation (QMLE) and the principal component analysis (PCA). To apply the latter methods, we must first determine the number of factors and appropriate regressors to be included in the model. Nevertheless, in practice, we do not have a priori knowledge about the true number of factors in almost all cases. Also there may be a large number of potential regressors, some of which may be irrelevant. Thus it is desirable to use a parsimonious model by choosing a subset of regressors. The common procedure is to perform some model selection in the first step and then conduct estimation based on the selected regressors and the chosen number of factors. To select regressors, a wide range of methods can be adopted. For example, one can apply the Bayesian information criterion (BIC) or some cross-validation methods. To determine the number of factors, one can apply the information criteria proposed in Bai and Ng (2002) or the testing procedure introduced in Onatski (2009), Kapetanios (2010), or Ahn and Horenstein (2013). Bai and Ng (2006, 2007) provide some empirical examples of the determination of number of factors in economic applications. Hallin and Liška (2007) study the determination of the number of factors in general dynamic factor models. Nevertheless, there are some problems associated with these approaches. First, in terms of determining the number of factors, when either the cross section dimension \(N\) or the time dimension \(T\) is small, or the magnitude of \(N\) is not comparable to that of \(T\) (e.g., \(N = 100, T = 10\), or the other way around), the finite sample performances of these selection methods tend to be very poor; see, e.g., the simulation in Bai and Ng (2002, p.207). Onatski’s (2009) test has low power when \(N\) is small or the explanatory power of the factors is low. Second, these approaches suffer from the problem of post-selection inference, as it is difficult to establish inference theory for post-selection estimators. See Leeb and Pötscher (2005).
In this paper, we explore a different approach. We use shrinkage techniques to combine the estimation with the selection of the number of factors and regressors in a single step. Following Bai (2009) or Moon and Weidner (2010a, 2010b), we can set a maximum number of factors ($R$, say) and obtain the preliminary estimates of the slope parameters and factors in (1.1). Then we consider a penalized least squares (PLS) regression of $Y_{it}$ on $X_{it}$ and the estimated factors via the adaptive (group) Lasso. We include two penalty terms in the PLS, one for the selection of regressors in $X_{it}$ via adaptive Lasso and the other for the selection of the exact number of factors via adaptive group Lasso. Despite the use of estimated factors that have slow convergence rates, we show that our new method can consistently determine the number of factors, consistently select all relevant regressors, and shrink the estimates of the coefficients of irrelevant regressors and redundant factors to zero with probability approaching 1 (w.p.a.1). We also demonstrate the oracle property of our method. That is, our estimator of the non-zero regression coefficients is asymptotically equivalent to the least squares estimator based on the factor-augmented regression where both the true number of factors and the set of relevant regressors are known. The bias-corrected version of our shrinkage estimator of the non-zero regression coefficients is asymptotically equivalent to Moon and Weidner’s (2010b) bias-corrected QML estimator in the case where all regressors are relevant (i.e., there is no selection of regressors). In the presence of irrelevant regressors, the variance-covariance matrix for our shrinkage estimator of the non-zero coefficients is smaller than that of Moon and Weidner’s QML estimator. In addition, we emphasize that even though Moon and Weidner (2010) show that the limiting distribution of the QML estimator is independent of the number of factors used in the estimation as long as the number of factors does not fall below the true number of factors, we find that in finite samples the inclusion of redundant factors can result in significant loss of efficiency (see section 4.3 for detail). For this reason, it is very important to include the correct number of factors in the model especially when the cross section or time dimension is not very large. Our shrinkage method effectively selects all relevant regressors and factor estimates and get rid of irrelevant regressors or redundant factor estimates.

There is a large statistics literature on the shrinkage type of estimation methods. See, for example, Tibshirani (1996) for the origin of Lasso, Knight and Fu (2000) for the first systematic study of the asymptotic properties of Lasso-type estimators, and Fan and Li (2001) for SCAD (smoothly clipped absolute deviation) estimators. Zou (2006) establishes the oracle property of adaptive Lasso; Yuan and Lin (2006) propose the method of group Lasso; Wang and Leng (2008) and Wei and Huang (2010) study the properties of adaptive group Lasso; Huang et al. (2008) study Bridge estimators in sparse high dimensional regression models. Recently there have been an increasing number of applications of the shrinkage techniques in the econometrics literature. For example, Caner (2009) and Fan and Liao (2011) consider covariate selection in the GMM estimation. Belloni et al. (2013) and García (2011) consider selection of instruments in the GMM framework. Liao (2013) provides a shrinkage GMM method for moment selection and Cheng and Liao (2012) consider the selection of valid and relevant moments via penalized GMM. Liao and Phillips (2012) apply adaptive shrinkage techniques to cointegrated systems. Kock (2013) considers Bridge estimators of static linear panel data models with random or fixed effects. Caner and Knight (2013) apply Bridge estimators to differentiate a unit root from a stationary alternative. Caner (2011) proposes a Bridge estimator for pure factor models and shows the selection consistency. This paper adds to the literature by applying the shrinkage idea to panel data models with factor structures and considering generated regressors. The method proposed in this paper has a wide range of applications. For example, it can be used to
estimate a structural panel model that allows a more flexible form of heterogeneity. A specific example is to study cross-country economic growth. Let $Y_{it}$ be the economic growth for country $i$ in period $t$ and $X_{it}$ be a large number of potential observable causes of economic growth, such as physical capital investment, consumption, population growth, government consumption, and lagged economic growth, among others. Economic growth may also be caused by many unobservable common factors $F_{i0}$. It is of great interest to know which observable causes are important to determine economic growth and the number of common unobserved factors that affect all countries’ economic growth. Our new method is directly applicable to this important economic question. Another example of application is to forecast asset returns, as factor models are often used to model asset returns. Specifically, let $Y_{it}$ be the excess returns on asset $i$ in period $t$ and $X_{it}$ be observable factors such as Fama-French factors (small market capitalization and book-to-market ratio), divided yields, dividend payout ratio and consumption gap, among others. The asset returns may also be affected by an unknown number of common unobserved factors. Our method automatically selects the important observable factors and unobservable common factors. Thus it provides a powerful tool to predict future asset returns.

The paper is organized as follows. Section 2 introduces our adaptive group Lasso estimators. Section 3 analyzes their asymptotic properties. In section 4 we report the Monte Carlo simulation results for our method and compare it with the methods of Bai and Ng (2002) and Onatski (2009). In section 5 we apply our method to study the determinants of economic growth in the framework of dynamic panel data models with interactive fixed effects, and find that in addition to three common unobserved factors selected by our method, government consumption share has negative effects, whereas investment share and lagged economic growth have positive effects on economic growth. Final remarks are contained in Section 6. The proofs of all theorems are delegated to the appendix. Additional proofs for the technical lemmas are provided in the online supplementary material.

NOTATION. For an $m \times n$ real matrix $A$, we denote its transpose as $A'$, its Frobenius norm as $\|A\| \equiv \|A\|_F \equiv \sqrt{\text{tr}(AA')}^{1/2}$, its spectral norm as $\|A\|_{\text{sp}} \equiv \sqrt{\mu_1(A'A)}$ and its Moore-Penrose generalized inverse as $A^+$, where $\equiv$ means “is defined as” and $\mu_s(\cdot)$ denotes the $s$th largest eigenvalue of a real symmetric matrix by counting eigenvalues of multiplicity multiple times. Note that the two norms are equal when $A$ is a vector. We will frequently use the submultiplicative property of these norms and the fact that $\|A\|_{\text{sp}} \leq \|A\| \leq \|A\|_{\text{sp}} \text{rank}(A)^{1/2}$. We also use $\mu_{\max}(B)$ and $\mu_{\min}(B)$ to denote the largest and smallest eigenvalues of a symmetric matrix $B$, respectively. We use $B > 0$ to denote that $B$ is positive definite. Let $P_A \equiv A(A'A)^+A'$ and $M_A \equiv I_m - P_A$, where $I_m$ denotes an $m \times m$ identity matrix. The operator $\overset{p}{\to}$ denotes convergence in probability, $\overset{d}{\to}$ convergence in distribution, and plim probability limit. We use $(N, T) \to \infty$ to denote that $N$ and $T$ pass to infinity jointly.

2 Penalized Estimation of Panel Data Models with Interactive Fixed Effects

In this section, we consider penalized least squares (PLS) estimation of panel data models with interactive fixed effects where the number of unobservable factors is unknown and some observable regressors may be irrelevant.
2.1 Panel Data Models with Interactive Fixed Effects

Consider the panel data model with interactive fixed effects

$$Y_{it} = \beta^0 X_{it} + \lambda^0_i F^0_t + \varepsilon_{it}, \ i = 1, \ldots, N, \ t = 1, \ldots, T,$$

(2.1)

where $X_{it}$ is a $K \times 1$ vector of regressors that may contain lagged dependent variables, $\beta^0 \equiv (\beta^0_1, \ldots, \beta^0_K)'$ is a $K \times 1$ vector of unknown slope coefficients, $\lambda^0_i$ is an $R_0 \times 1$ vector of factor loadings, and $F^0_t$ is an $R_0 \times 1$ vector of common factors, $\varepsilon_{it}$ is the idiosyncratic error term, and $\beta^0, \lambda^0_i, F^0_t$ and $\varepsilon_{it}$ are unobserved. Here $\{\lambda^0_i\}$ and $\{F^0_t\}$ may be correlated with $\{X_{it}\}$. We consider estimation and inference on $\beta^0$ when the true number of factors $R_0$ is unknown and some variables in $X_{it}$ may be irrelevant.

To proceed, let $X_{it,k}$ denote the $k$th element of $X_{it}$ for $k = 1, \ldots, K$. Define

$$Y_i \equiv (Y_{i1}, \ldots, Y_{iT})', \ X_i \equiv (X_{i1}, \ldots, X_{iT})', \ \varepsilon_i \equiv (\varepsilon_{i1}, \ldots, \varepsilon_{iT})',
$$

$$F^0 \equiv (F^0_1, \ldots, F^0_T)', \ \lambda^0 \equiv (\lambda^0_1, \ldots, \lambda^0_K)', \ X_{i,k} \equiv (X_{i1,k}, \ldots, X_{iT,k})',
$$

$$Y \equiv (Y_1, \ldots, Y_N)', \ X_k \equiv (X_{1,k}, \ldots, X_{N,k})', \ \text{and } \varepsilon \equiv (\varepsilon_1, \ldots, \varepsilon_N)'.$$  

Apparently $Y$, $X_k$, and $\varepsilon$ are all $N \times T$ matrices. Then we can write the model (2.1) in matrix form

$$Y = \sum_{k=1}^K \beta^0_k X_k + \lambda^0 F^0 + \varepsilon.$$  

(2.2)

Without loss of generality (Wlog), we assume that only the first $K_0$ elements of $X_{it}$ have nonzero slope coefficients, and write $X_{it} = (X_{it(1)}, X_{it(2)})'$, where $X_{it(1)}$ and $X_{it(2)}$ are $K_0 \times 1$ and $(K - K_0) \times 1$ vectors, respectively, and the true coefficients of $X_{it(1)}$ are nonzero while those of $X_{it(2)}$ are zero. Accordingly, we decompose $\beta^0$ as $\beta^0 \equiv (\beta^0_{(1)}, \beta^0_{(2)})' = (\beta^0_{(1)}, 0)'$.

Following Bai (2009) and Moon and Weidner (2010a, 2010b), we consider the Gaussian QMLE $(\hat{\beta}, \hat{\lambda}, \hat{F})$ of $(\beta^0, \lambda^0, F^0)$ which is given by

$$
\left( \hat{\beta}, \hat{\lambda}, \hat{F} \right) = \arg \min_{(\beta, \lambda, F)} \mathcal{L}^0_{NT}(\beta, \lambda, F),
$$

(2.3)

where

$$
\mathcal{L}^0_{NT}(\beta, \lambda, F) \equiv \frac{1}{NT} \text{tr} \left[ (Y - \sum_{k=1}^K \beta_k X_k - \lambda F')' \left( Y - \sum_{k=1}^K \beta_k X_k - \lambda F' \right) \right],
$$

(2.4)

$\beta \equiv (\beta_1, \ldots, \beta_K)'$ is a $K \times 1$ vector, $F \equiv (F_1, \ldots, F_T)'$ is a $T \times R$ matrix, and $\lambda \equiv (\lambda_1, \ldots, \lambda_N)'$ is an $N \times R$ matrix. Note that $R$ may be different from $R_0$. One can first obtain the profile-likelihood estimate $\hat{\beta}$ and then the estimate $(\hat{\lambda}, \hat{F})$ via the PCA method under the identification restrictions: $F'F/T = I_R$ and $X'\lambda$ is a diagonal matrix. Namely, $(\hat{\lambda}, \hat{F})$ solves

$$
\left[ \frac{1}{NT} \sum_{i=1}^N \left( Y_i - X_i \hat{\beta} \right) \left( Y_i - X_i \hat{\beta} \right)' \right] \hat{F} = \hat{F} V_{NT} \text{ and } \hat{\lambda} = T^{-1} \left( Y - \sum_{k=1}^K \hat{\beta}_k X_k \right) \hat{F}
$$

(2.5)

where $V_{NT}$ is a diagonal matrix consisting of the $R$ largest eigenvalues of the above matrix in the square bracket, arranged in descending order. Moon and Weidner (2010a) show that as long as $R \geq R_0$, the limiting distribution of the QMLE for $\beta$ is independent of $R$, the number of unobserved factors used in
the estimation. Throughout the paper, we assume that $R \geq R_0$ and use $\hat{\beta}^c = (\hat{\beta}_1^c, \ldots, \hat{\beta}_K^c)'$ to denote the bias-corrected version of $\hat{\beta}$ based on the formula in Moon and Weidner (2010b). After obtaining $\hat{\beta}^c$, we obtain the final estimate $(\hat{\lambda}, \hat{F})$ via (2.5) with $\hat{\beta}$ replaced by $\hat{\beta}^c$.

Note that $R$ is usually different from $R_0$ and one cannot expect $\hat{F}$ to be a consistent estimator of $F^0$ or a rotational version of $F^0$. Below we assume that $R \geq R_0$. If we define $\hat{Y} = Y - \sum_{k=1}^K \hat{\beta}^c_k X_k$, $\hat{F} = (NT)^{-1} \hat{Y}' \hat{Y} \hat{F}$ and $H = H_{NT} = (N^{-1} \lambda_0^0 \lambda_0^0)' (T^{-1} F^0 \hat{F})$, then we can follow Bai and Ng (2002) and show that under certain regularity conditions, $\sqrt{T} \left\| \hat{F} - F^0 H \right\|^2 = O_P(\delta_{NT}^2)$ and $\left\| \hat{F}_t - H^T P_t^0 \right\|^2 = O_P(\delta_{NT}^{-2})$ for each $t$, where $\hat{F}_t$ denotes the $t$th row of $\hat{F}$ and $\delta_{NT} = \min(\sqrt{N}, \sqrt{T})$. In addition, we show in Appendix A that $H = H_{NT}$ converges in probability to a sparse matrix

$$H^0 = [H^0_{(1)}, \mathbf{0}_{R_0 \times (R - R_0)}],$$

where $H^0_{(1)}$ is an $R_0 \times R_0$ full rank matrix and $\mathbf{0}_{a \times b}$ denotes an $a \times b$ matrix of zeros. As a result, $\lambda^* = H^0 \lambda^*$ also exhibits a sparse structure asymptotically. Let $\hat{\Sigma}_F = \hat{F}' \hat{F}/T$ and $\tau_1, \ldots, \tau_R$ denote the eigenvalues of $\hat{\Sigma}_F$ arranged in descending order. We show in Appendix A that $\tau_1, \ldots, \tau_{R_0}$ converge to some finite positive numbers whereas $\tau_{R_0+1}, \ldots, \tau_R$ converge to zero at the rate of $\delta_{NT}$. This means that $\hat{\Sigma}_F$ suggests the sparsity nature of $\lambda^*$.

The above discussions motivate the PLS estimation of $(\beta^0, \lambda^*)$ where $\lambda^* = (\lambda_1^*, \ldots, \lambda_N^*)'$. 

### 2.2 Penalized Least Squares Estimation of $(\beta^0, \lambda^*)$

Using the above definitions of $H$ and $\lambda^*$, we can rewrite (2.1) as

$$Y_{it} = \beta_0^0 X_{it} + \lambda_0^0 H^T F_{it} + \epsilon_{it} = \beta_0^0 X_{it} + \lambda_0^* H^T F_{it} + \epsilon_{it}. \quad (2.6)$$

Motivated by the literature on adaptive group Lasso (see, Yuan and Li (2006), Zou (2006), Huang et al. (2008)), we propose to estimate $(\beta^0, \lambda^*)$ by minimizing the following PLS criterion function

$$Q_\gamma(\beta, \lambda) = \mathcal{L}_{NT}(\beta, \lambda) + \gamma_{1NT} \sum_{k=1}^K \frac{1}{|\beta_k|} |\beta_k| + \frac{\gamma_{2NT}}{\sqrt{N}} \sum_{r=1}^R \frac{1}{\tau_r} \|\lambda_r\|, \quad (2.7)$$

where $\mathcal{L}_{NT}(\beta, \lambda) = \mathcal{L}_{NT}^0(\beta, \lambda, \hat{F})$, $\lambda_r$ denotes the $r$th column of $\lambda$, $\gamma = (\gamma_{1NT}, \gamma_{2NT})$ is a vector of tuning parameters, and $\kappa_1, \kappa_2 > 0$ are usually taken as either 1 or 2 in the literature. Note that we have replaced the unobserved factor vector $H^T F^0$ by its estimate $\hat{F}_t$ in (2.7). (2.7) contains two penalty terms: $\gamma_{1NT}$ for the regression coefficients $\beta_k$’s and $\gamma_{2NT}$ for the loading vectors $\lambda_r$’s. Note that we explicitly account for the fact that $\|\lambda_r\| = O(\sqrt{N})$ for relevant factors by dividing the second penalty term $\gamma_{2NT}$ by $\sqrt{N}$.

Note that the objective function in (2.7) is convex in $(\beta, \lambda)$ so that the global minimizer of $Q_\gamma$ can be found easily for any given tuning parameter $\gamma$. Let $(\hat{\lambda}, \hat{\lambda}) = (\hat{\lambda}(\gamma), \hat{\lambda}(\gamma))$ denote the solution to the above minimization problem. We frequently suppress the dependence of $(\hat{\lambda}, \hat{\lambda})$ on $\gamma$ as long as no confusion arises. Below we will propose a data-driven method to choose $\gamma$. 

---

1 Noting that the $R_0 \times R$ matrix $H$ is right invertible, by Proposition 6.1.5 in Bernstein (2005, p.225) we have $H^+ = H' (HH')^{-1}$, which further implies that $HH^+ = I_{R_0}$. 

---
3 Asymptotic Properties

In this section we study the asymptotic properties of the proposed adaptive group Lasso estimator \((\hat{\beta}, \hat{\lambda})\).

3.1 Estimation Consistency

Let \(W_{NT} = \frac{1}{NT} \sum_{i=1}^{N} \tilde{X}_i' M_{F0} \tilde{X}_i\) where \(\tilde{X}_i = X_i - \frac{1}{T} \sum_{j=1}^{N} \lambda_i^0 (N^{-1} \lambda_i^0 \lambda_i^0)^{-1} \lambda_j^0 X_j\). Let \(C\) denote a generic finite positive constant that may vary across lines. We make the following assumptions.

**Assumption A.1**

\(\det(\tilde{X}_i' M_{F0} \tilde{X}_i) = 0\) and \(E(\varepsilon_{it}) = 0\) and \(E(\varepsilon_{it}^2) \leq C\).

Let \(\tilde{X}_i = X_i - \frac{1}{T} \sum_{j=1}^{N} \lambda_i^0 (N^{-1} \lambda_i^0 \lambda_i^0)^{-1} \lambda_j^0 X_j\). Let \(C\) denote a generic finite positive constant that may vary across lines. We make the following assumptions.

**Assumption A.2**

Let \(\tilde{X}_i = X_i - \frac{1}{T} \sum_{j=1}^{N} \lambda_i^0 (N^{-1} \lambda_i^0 \lambda_i^0)^{-1} \lambda_j^0 X_j\). Let \(C\) denote a generic finite positive constant that may vary across lines. We make the following assumptions.

**Assumption A.3**

(i) As \((N, T) \rightarrow \infty\), \(T = O(N^{3/2})\) and \(N/T^2 \rightarrow 0\).

(ii) As \((N, T) \rightarrow \infty\), \(T^{1/2} \gamma_{1,NT} \rightarrow 0\), and \(T^{1/2} \gamma_{2,NT} \rightarrow 0\).

A.1(ii) is a high-level assumption. Primitive conditions can be found in Moon and Weidner (2010a, 2010b) which ensure the \(\sqrt{NT}\)-consistency of a bias-corrected preliminary estimate. It is possible to relax this assumption to allow for a non-bias-corrected estimator of \(\beta\), in which case the rate of consistency would be \(\delta_{NT}\) and one would still be able to obtain the main results in this paper without essential changes. A.1(ii)-(iv) mainly impose standard moment conditions on \(F_i\), \(\lambda_i\), and \(X_i\); see, e.g., Bai and Ng (2002) and Bai (2003, 2009). Note that Bai and Ng (2002) assume only the fourth moment for \(F_i\) but require that \(\lambda_i^0\) be uniformly bounded. Moon and Weidner (2010a) demonstrate that A.1(i) can be satisfied for various error processes. A.1(iv) requires weak exogeneity of the regressor \(X_i\). A.1(iv) can be satisfied under various primitive conditions and it implies that \(\|\lambda_i^0 \varepsilon_i\| = O_P(N^{1/2}T^{1/2})\) by Chebyshev inequality, which further implies that \(\|\lambda_i^0 \varepsilon_i\| = O_P(NT^{1/2})\) under Assumptions A.1(ii)-(iii) by standard matrix operations.

A.2 is adopted from Bai and Ng (2002) and Bai (2009). It allows for weak forms of both cross sectional dependence and serial dependence in the error processes. A.3(i) requires that \(T\) should not grow too fast in comparison with \(N\) and vice versa, which is different from the minimum requirement \((T/N^2 \rightarrow 0 \text{ and } N/T^2 \rightarrow 0)\) in Bai (2003) for \(\sqrt{N}\) and \(\sqrt{T}\)-consistent estimation of factors and factor loadings, respectively. Nevertheless, it allows \(N\) and \(T\) to pass to infinity at either the same or different rates and either one can pass to infinity faster than the other. We need \(T = O(N^{3/2})\) to ensure that the smallest \((R - R_0)\) eigenvalues of \(\hat{\Sigma}_F\) converge to zero at the \(\sqrt{N}\)-rate. A.3(ii) is a condition that
ensures a preliminary $\sqrt{T}$-rate of consistency of our shrinkage estimator $\hat{\beta}$ (see Theorem 3.1 below) and it essentially says that the two penalty terms cannot be too large.

The following theorem establishes the consistency of the shrinkage estimator $(\hat{\beta}, \hat{\lambda})$.

**Theorem 3.1** Suppose Assumptions A.1, A.2, and A.3(i)-(ii) hold. Then

$$
\left\| \hat{\beta} - \beta^0 \right\| = O_P \left( T^{-1/2} \right), \text{ and } N^{-1} \left\| \hat{\lambda} H' - \lambda^0 \right\|^2 = \frac{1}{N} \sum_{i=1}^{N} \left\| H \hat{\lambda}_i - \lambda_i^0 \right\|^2 = O_P \left( T^{-1} \right).
$$

**Remark 1.** Theorem 3.1 establishes the $\sqrt{T}$-rate of consistency for $\hat{\beta}$ and the $T$-rate of consistency for the cross sectional average of the squared deviations between the estimated factor loadings (with rotation) and the true factor loadings. The latter is the best rate of consistency one can obtain. It is worth mentioning that the second part of the result in the above theorem in general does not imply

$$
\frac{1}{N} \left\| \hat{\lambda} - \lambda^0 H' \right\|^2 = O_P \left( T^{-1} \right) \text{ unless } R = R_0.
$$

To see why, notice that

$$
\frac{1}{N} \left\| \hat{\lambda} - \lambda^0 H' \right\|^2 = \frac{1}{N} \left\| \hat{\lambda} (I_R - H'H') + (\hat{\lambda} H' - \lambda^0) H' \right\|^2 \\
\leq \frac{2}{N} \left\| \hat{\lambda} (I_R - H'H') \right\|^2 + \frac{2}{N} \left\| (\hat{\lambda} H' - \lambda^0) H' \right\|^2.
$$

Even though Theorem 3.1 implies that the second term is $O_P \left( T^{-1} \right)$, the first term does not vanish asymptotically as $H'H' \neq I_R$ for any $R > R_0$. Nevertheless, by the triangle inequality, the fact that $H$ has full row rank asymptotically, and Assumption A.1(iii), we can readily show that $N^{-1} \left\| \hat{\lambda} \right\|^2 = O_P \left( 1 \right)$.

### 3.2 Selection Consistency

To study the selection consistency, we write $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)'$, $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2)$, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are column vectors of dimensions $K_0$ and $K - K_0$, respectively, and $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are $N \times R_0$ and $N \times (R - R_0)$ matrices, respectively.

To state the next result, we augment Assumption A.3 with one further condition.

**Assumption A.3 (iii)** As $(N, T) \to \infty$, $(NT)^{0.1/2} T^{1/2} \gamma_{1NT} \to \infty$, and $N^{\alpha_3/2} T^{1/2} \gamma_{2NT} \to \infty$.

Clearly, A.3(iii) requires that the two penalty terms should not be too small. The next theorem establishes the selection consistency of our adaptive group Lasso procedure.

**Theorem 3.2** Suppose Assumptions A.1, A.2 and A.3(i)-(iii) hold. Then

$$
P \left( \left\| \hat{\beta}_2 \right\| = 0 \text{ and } \left\| \hat{\lambda}_2 \right\| = 0 \right) \to 1 \text{ as } (N, T) \to \infty.
$$

**Remark 2.** Theorem 3.2 says that with w.p.a.1 all the zero elements in $\beta^0$ and all the factor loadings of the redundant factor estimates must be estimated to be exactly zero. On the other hand, by Theorem 3.1, we know that the estimates of the nonzero elements in $\beta^0$ and the factor loadings of the non-redundant factor estimates must be consistent. This implies that w.p.a.1, all the relevant regressors and estimated factors must be identified by nonzero coefficients and nonzero factor loadings, respectively. Put together, Theorems 3.1 and 3.2 imply that the adaptive group Lasso has the ability to identify the true regression model with the correct number of factors consistently.
3.3 Oracle Property

Decompose \( \hat{F} = (\hat{F}_1, \hat{F}_2) \) where \( \hat{F}_1 \) and \( \hat{F}_2 \) are \( T \times R_0 \) and \( T \times (R - R_0) \) submatrices, respectively. Analogously, let \( \hat{F} = (\hat{F}_1, \hat{F}_2) \) and \( H = (H_1, H_2) \), where \( \hat{F}_i \equiv (NT)^{-1} \hat{Y}^\prime \hat{Y} \hat{F}_i \) and \( H_i \equiv H_i^{NT} = (N^{-1} \lambda_0^0 \lambda_0^0) (T^{-1} \hat{F}_i \hat{F}_i \hat{F}_i) \) for \( i = 1, 2 \). Let \( F_i^* \equiv F_i H_i^0 \lambda_i^* \equiv \lambda_0^0 H_i^0 \). Let \( F_i^* \) and \( \lambda_i^* \) denote the \( i \)th rows of \( F_i^* \) and \( \lambda_i^* \), respectively. Define \( \hat{F}_i \) and \( \hat{\lambda}_i \) analogously. Further, write \( X_i = (X_{i1}, X_{i2}) \) where \( X_{i1} \) and \( X_{i2} \) are \( T \times K_0 \) and \( T \times (K - K_0) \) submatrices of \( X_i \), respectively. Let \( \hat{D}_{F\theta} \equiv (NT)^{-1} \sum_{i=1}^N X_i \theta M_{F\theta} X_i \) and \( C^{NT}_{\theta} \equiv (NT)^{-2} \sum_{i=1}^N \sum_{j=1}^N \epsilon_i \theta \epsilon_j \hat{F}_i \hat{F}_i \lambda_i^* X_i \lambda_j^* M_{F\theta} X_j \).

Let \( D \equiv \sigma (F_0, \lambda_0^0) \), the sigma-field generated by \( (F_0, \lambda_0^0) \), and \( E_D (A) \equiv E (A | D) \). Define

\[
\begin{align*}
B_{1NT} &\equiv (NT)^{-5/2} \sum_{i=1}^N X_{i1} F_i^* \left( F_i^* (F_i^*) \right)^{-1} F_i^* F_0 \lambda_0^0 \epsilon_i \epsilon_i^0 \lambda_0^0 F_0 \hat{F}_i \hat{\lambda}_i^* , \\
B_{2NT} &\equiv (NT)^{-3/2} \sum_{i=1}^N X_{i1} M_{F\theta} \epsilon_i \epsilon_i^0 \hat{\lambda}_i^* , \\
B_{3NT} &\equiv N^{-1} T^{-2} \sum_{i=1}^N \sum_{j=1}^N X_{i1} F_i^* \left( F_i^* (F_i^*) \right)^{-1} F_i^* F_0 \epsilon_i \epsilon_j^0 \epsilon_i^0 \epsilon_j^0 , \\
B_{4NT} &\equiv (NT)^{-1/2} \sum_{i=1}^N [X_{i1} - E_D (X_{i1})]' P_{F\theta} \epsilon_i .
\end{align*}
\]

Under Assumptions A.1-A.2, we can readily verify that \( B_{1NT} = O_P (\sqrt{T/N}) \), \( B_{2NT} = O_P (1 + \sqrt{N/T}) \), \( B_{3NT} = O_P (\sqrt{T/N}) \), and \( B_{4NT} = O_P (\sqrt{N/T}) \).

To study the oracle property of \( \hat{\beta}_1 \) and \( \hat{\lambda}_1 \), we add the following assumptions.

**Assumption A.4**

(i) There exists a \( K_0 \times K_0 \) matrix \( D_{F\theta} > 0 \) such that \( \hat{D}_{F\theta} = D_{F\theta} + o_P (1) \).

(ii) There exists a \( K \times K_0 \) matrix \( C_0 \) such that \( C_{NT} = C_0 + o_P (1) \).

(iii) \( \| F \theta \epsilon_i X \lambda_0^0 \|^2 = O_P \left( N^2 T^2 (N + T) \right) \).

(iv) \( \| \lambda_0^0 \epsilon_i \|^2 = O_P \left( (N + T) \right) \).

**Assumption A.5**

(i) Let \( \tilde{X}_{i1} \equiv X_{i1} - P_{F\theta} E_D (X_{i1}) - M_{F\theta} \tilde{X}_{1NT} + [X_i - E_D (X_i) - M_{F\theta} \tilde{X}_{2NT}] W_0^{-1} C_0 \), where \( \tilde{X}_{1NT} = \frac{1}{N} \sum_{i=1}^N \lambda_0^0 (T^{-1} F_0 \hat{F}_i) \lambda_i^* (T^{-1} F_0 \hat{F}_i) \lambda_i^* (E_D (X_i) - X_i) \) and \( \tilde{X}_{2NT} = \frac{1}{N} \sum_{i=1}^N \lambda_0^0 (N^{-1} \lambda_0^0 \lambda_0^0) (T^{-1} F_0 \hat{F}_i) \lambda_i^* (E_D (X_i) - X_i) \).

There exists \( \Theta > 0 \) such that \( \frac{1}{N} \sum_{i=1}^N \tilde{Z}_{i}^2 \tilde{Z}_{i} \tilde{Z}_{i} \tilde{Z}_{i} \tilde{Z}_{i} \leq N (0, \Theta) \).

(ii) There exists \( \Theta_{i1} F_{i1}^* > 0 \) such that \( \frac{1}{\sqrt{T}} \sum_{i=1}^T F_{i1}(\epsilon_i) \tilde{\epsilon}_{it} \tilde{\epsilon}_{it} \leq N (0, \Theta_{i1} F_{i1}^* ) \).

**Assumption A.6**

As \( (N, T) \to \infty \), \( (NT)^{1/2} \gamma_{1NT} \to 0 \) and \( (NT)^{1/2} \gamma_{2NT} \to 0 \).

Assumptions A.4(i)-(ii) are weak as one can readily show that both \( \hat{D}_{F\theta} \) and \( C_{NT} \) are \( O_P (1) \). The positive definiteness of \( D_{F\theta} \) is ensured by Assumption A.1(viii). \( C_0 \) is generally not zero in A.A(ii), but it can be zero under fairly restrictive conditions on the data generating processes for \( \{ X_{it}, \epsilon_{it}, F_{i1}, X_{it} \} \). See Greenaway-McGregory et al. (2012, GHS hereafter) and the discussion in Remark 4 below. A.4(iii)-(iv) are high level assumptions. A.5 parallels Assumption E in Bai (2009) which is also a high level assumption. Note that both cross sectional and serial dependence and heteroskedasticity are allowed in the error terms. We verify these assumptions in the supplementary appendix by allowing lagged dependent variables in \( X_{it} \). A.6 is needed to obtain the oracle property for our adaptive group Lasso estimator.

The following theorem establishes the asymptotic distributions of both \( \hat{\beta}_1 \) and \( \hat{\lambda}_1 \).
asymptotic properties of such a one-step PLS estimator. For this reason, we compare our estimator to (

Remark 4.

Despite the oracle property of , it possesses four bias terms that have to be corrected

2 Bai and Liao (2013) propose a one-step shrinkage estimator for a pure factor model where the error terms are het-

Remark 3. To understand the results in Theorem 3.3, we consider an oracle who knows the exact

number of factors and exact regressors that should be included in the panel regression model. In this case, one can consider the estimation of both the finite dimensional slope coefficients and the factors and factor loadings via the Gaussian QMLE method of Bai (2009). This one-step oracle estimator is asymptotically efficient under Gaussian errors and some other conditions. Ideally, one can consider a one-step SCAD or Bridge-type PLS regression where the penalty terms on both and some identification restrictions on and are required, it is very challenging to study the asymptotic properties of such a one-step PLS estimator.

For this reason, we compare our estimator with an alternative two-step estimator that is obtained by a second step augmented regression with estimated factors obtained using Bai's (2009) PCA-based QMLE method from a first step estimation; see, e.g., Kapetanios and Pesaran (2007) and GHS. This two-step augmented estimator is only as efficient as the one-step QMLE under some restrictive assumptions and has more bias terms to be corrected otherwise. But after the bias correction, it is asymptotically equivalent to the bias-corrected one-step QMLE estimator. See also Remark 7 below.

Specifically, let and denote the least squares (LS) estimates of and in the following augmented panel regression

where is the new error term that takes into account the estimation error from the first stage estimation. Then we can readily show that

where

\[ \hat{\beta}_{(1)} = \bar{D}_{F_{(1)}}^{-1} \frac{1}{NT} \sum_{i=1}^{N} X'_{i(1)} M_{F_{(1)}} Y_{i(1)}, \quad \hat{\lambda}_{i(1)} = \Sigma_{F_{(1)}}^{-1} \frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{i(1)} (Y_{it} - X'_{i(1)} \hat{\beta}_{(1)}), \]

Theorem 3.3 Suppose Assumptions A.1-A.6 hold. Then

(i) \( \sqrt{NT} (\hat{\beta}_{(1)} - \beta_{0(1)}) \rightarrow_{D} N(0, D_{F_{(1)}}^{-1} \Sigma_{F_{(1)}}^{-1} D_{F_{(1)}}) \),

(ii) \( \sqrt{T} (\hat{\lambda}_{i(1)} - H_{(1)}^+ \lambda_{i}^0) \rightarrow_{D} N(0, \Sigma_{F_{(1)}}^{-1} \Theta_{i(1)} \Sigma_{F_{(1)}}^{-1}) \),

where \( \Xi_{NT} = \bar{D}_{F_{(1)}}^{-1} (\Xi_{1NT} - \Xi_{2NT} - \Xi_{3NT} - \Xi_{4NT}) \), \( \Sigma_{F_{(1)}} = H_{(1)}^0 \Sigma_{F} H_{(1)}^0 \), and \( H_{(1)}^0 \) is the probability limit of \( H_{(1)} \).

\[ Y_{it} = \beta_{0(1)}^0 X_{it(1)} + \lambda_{i}^0 \tilde{F}_{i(1)} + \epsilon_{it}, \quad (3.1) \]

where is the first stage estimation. Then we can readily show that

\[ \hat{\beta}_{(1)} = \bar{D}_{F_{(1)}}^{-1} \frac{1}{NT} \sum_{i=1}^{N} X'_{i(1)} M_{F_{(1)}} Y_{i(1)}, \quad \hat{\lambda}_{i(1)} = \Sigma_{F_{(1)}}^{-1} \frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{i(1)} (Y_{it} - X'_{i(1)} \hat{\beta}_{(1)}), \]

where \( \bar{D}_{F_{(1)}} \equiv (NT)^{-1} \sum_{i=1}^{N} X'_{i(1)} M_{F_{(1)}} X_{i(1)} \) and \( \Sigma_{F_{(1)}} \equiv \frac{1}{T} \sum_{t=1}^{T} \tilde{F}_{i(1)} \tilde{F}_{i(1)}' \). As demonstrated in the proof of Theorem 3.3, Assumption A.6 is essential to ensure that \( (\hat{\beta}_{(1)}, \hat{\lambda}_{i(1)}) \) is asymptotically equivalent to \( (\beta_{(1)}, \lambda_{i(1)}) \) in the sense that they share the same first order asymptotic distribution. For this reason, we say that our estimator \( (\hat{\beta}_{(1)}, \hat{\lambda}_{i(1)}) \) is as oracle efficient as a two-step augmented estimator by knowing the exact number of factors and regressors.

Remark 4. Despite the oracle property of \( \hat{\beta}_{(1)} \), it possesses four bias terms that have to be corrected in practice in order to ensure its \( \sqrt{NT} \)-consistency and zero-mean asymptotic normality. Interestingly,
under a different set of assumptions, GHS establish formal asymptotics for the factor-augmented panel regressions. They show that the replacement of the unobservable factor $F_1 \tilde{1}$ by the PCA estimate $\hat{F}_1(1)$ in (3.1) does not affect the limiting distribution of the LS estimates of $\beta^0_{1(1)}$ under four key conditions: (i) $T/N \to 0$ and $N/T^3 \to 0$, (ii) there is no dynamic lagged dependent variable in the regression, (iii) $X_i$ also possesses a factor structure: $X_i = F^X \lambda^X_i + V_i$, and the estimated factors associated with $X_i$ are also included into the augmented regression, and (iv) the exact number of factors and the exact regressors that should be included in the model are known. Note that we relax all the four assumptions in this paper. We relax condition (iv) by considering the shrinkage estimation. Under condition (i), both $B_{1NT}$ and $B_{3NT}$ are $o_P(1)$. Under condition (ii), $F^X$ is a submatrix of $F^0$ so that $F^X'M_0 = 0$ and the factor component of $X_{i(1)}$ does not contribute to $B_{2NT}$; under GHS’s conditions on $V_i, \varepsilon_i, F^0_1$ and $\lambda^0_1$ (see their Assumptions A(v)-(viii)), the error component of $X_{i(1)}$ does not contribute to $B_{2NT}$ either. That is, $B_{2NT}$ is asymptotically negligible under their conditions. To understand the sources of asymptotic bias and variance of our estimator, we consider the following expansion used in the proof of Theorem 3.3:

\[
\sqrt{NT} \left( \beta_{1(1)} - \beta^0_{1(1)} \right) = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) M_{F^0} \varepsilon_i + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) M_{F^1} (F^1_i - \hat{F}_1) \lambda_{i(1)}^0 + o_P(1)
\]

\[
\equiv S_{1NT} + S_{2NT} + S_{3NT} + o_P(1), \quad (3.2)
\]

$s_{1NT}$ is present even if one observes $F^1_1$ (in which case $\hat{F}_1(1)$ is replaced by $\hat{F}^0_1$). We show that $S_{2NT}$ contributes to both the asymptotic bias and variance whereas $S_{3NT}$ only contributes to the asymptotic bias:

\[
S_{2NT} = \frac{1}{\sqrt{NT}} \left( B_{1NT} - B_{2NT} - V_{1NT} + V_{2NT} \right) + o_P(1), \quad \text{and} \quad S_{3NT} = -\frac{1}{\sqrt{NT}} \left( B_{3NT} + o_P(1) \right),
\]

where $V_{1NT} \equiv (NT)^{-1/2} \sum_{i=1}^{N} X'_{i(1)} M_{F^0} \varepsilon_i$, $V_{2NT} \equiv C'_{NT} \sqrt{NT} (\beta - \beta^0)$, and $X'_{i(1)} = \frac{1}{N} \sum_{j=1}^{N} X_{ij} [T^{-1} F^0_{1(1)}/\hat{F}(1)] \lambda^0_{i(1)} X'_j(1)$. In general, the parameter estimation error plays an important role. Nevertheless, under GHS’s key condition (ii) in conjunction with some other regularity conditions specified in their Assumption A, one can show that $V_{1NT} = o_P(1)$ and $C_{NT} = o_P(1)$ (and hence $C_0 = 0$ in our Assumption A.4(ii) and $V_{2NT} = o_P(1)$ under our Assumption A.1(ii)). In this case, only $S_{1NT}$ contributes to the asymptotic variance of their augmented estimator of $\beta^0_{1(1)}$ and both $S_{2NT}$ and $S_{3NT}$ are asymptotically negligible under GHS’s key conditions (i), (iii) and (iv). If their key condition (ii) is also satisfied, one can show that $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)} M_{F^0} \varepsilon_i$ converges to a zero-mean normal distribution; otherwise, one has to consider bias-correction as in Moon and Weidner (2010a).

**Remark.** The presence of $B_{1NT}$ and the complicated structure of $Z_i$ in Assumption A.5(i) are mainly due to the allowance of lagged dependent variables because $X_{it}$ can be correlated with $\varepsilon_{is}$ for $t > s$. In this case, $S_{1NT}$ is not centered around zero asymptotically, whereas both $V_{1NT}$ and $V_{2NT}$ are centered.

---

3Condition (i) is explicitly mentioned in GHS. Lagged dependent variables are ruled out by the second part of Assumption B in their paper. The first part of (iii) is explicitly assumed in their equation (3) and the second part is implicitly assumed because the factors in their equation (6) include the maximal common factor set of the observable variables $(Y_i, X_i)$. (iv) is also implicitly assumed in their paper.

4In the absence of lagged dependent variables, one can simply combine $S_{1NT}$ with $D_{F(1)}^{-1} (-V_{1NT} + V_{2NT})$ to obtain the asymptotic distribution without GHS’s key conditions (i), (iii), and (iv).
around 0 asymptotically.\(^5\) We have to decompose \(S_{1NT}\) into an asymptotic bias term (which is associated with \(B_{4NT}\)) and an asymptotic variance term (which enters \(\hat{Z}_i\) via \(X_{i(1)} - F_{00}E_D(X_{i(1)})\)):

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)}M_{F0}\varepsilon_i = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \left[ X'_{i(1)}\varepsilon_i - E_D(X'_{i(1)})P_{F0}\varepsilon_i \right] - B_{4NT}.
\]

We can find primitive conditions to ensure that the first term in the last expression converges to a zero mean normal distribution, the conditional expectation \(B_{4NT}\) of the second term given \(D\) contributes to the asymptotic bias which can be corrected, and \(B_{4NT} - \tilde{B}_{4NT}\) is asymptotically negligible. For further details, see the proofs of Theorem 3.3 and Corollary 3.4 in the appendix and the supplementary appendix.

**Remark 6.** Now we consider some special cases where the formulae for the asymptotic bias and variance terms can be simplified.

1. If all regressors are strictly exogenous as in Pesaran (2006), Bai (2009), and GHS, then one can set \(B_{4NT} = 0\) and \(\hat{Z}_i = M_{F0}X_{i(1)} - X_{1NT} + (X_i - X_{2NT})W_0^{-1}C_0\) in Assumption A.5(i), where \(X_{1NT} = \frac{1}{N}\sum_{j=1}^{N} X_i^0(T^{-1}F_00\hat{F}_j(1))X_j(1)\) and \(X_{2NT} = \frac{1}{N}\sum_{j=1}^{N} X_i^0(N^{-1}(N^0)\lambda^0)^{-1}X_j\).

In short, there is no need to consider conditioning on the “exogenous” set of factors and factor loadings.

2. If in addition, \(X_i\) also follows a factor structure as in GHS, then there is no need to correct \(B_{2NT}\) and \(B_{4NT}\) under the conditions specified in GHS, and one can reset \(\hat{Z}_i = M_{F0}X_{i(1)}\) in Assumption A.5(i).

3. If in addition, \(T/N \rightarrow 0\), there is no need to correct any bias term.

Note that we present Theorem 3.3 under a set of fairly general and high level assumptions. To estimate the asymptotic bias and variance, one generally needs to add more specific assumptions as in Bai (2009). In the supplementary appendix, we specify a set of assumptions (Assumptions B.1-B.2) that ensure all the high level conditions specified in Assumptions A.1(vi)-(vii), A.2(ii)-(iii), A.4(iii)-(iv) and A.5(i)-(ii) are satisfied. Note that Assumption B.1(i) relies on the key notion of conditional strong mixing that has recently been introduced by Prakasa Rao (2009) and Roussas (2008) and applied to the econometrics literature by Su and Chen (2013). With the notion of conditional strong mixing, there is no need to decompose the predetermined variable into a strictly exogenous component plus a weakly exogenous component as in Moon and Weidner (2010b, Assumption 5(iii)). Assumptions B.1-B.2 are also used to establish the consistency of the asymptotic bias and variance estimates.

In particular, under the martingale difference sequence (m.d.s.) condition in Assumption B.2, we have \(\Theta = \lim_{(N,T) \rightarrow \infty} \Theta_{NT}\) and \(\Theta_{F_{(1)}}, \Theta_{NT,F_{(1)}}\) where

\[
\Theta_{NT} \equiv (NT)^{-1} \left\{ \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it}^2 \hat{Z}_{it} \hat{Z}_{it}' , \Theta_{NT,F_{(1)}} \equiv \frac{1}{T} \sum_{t=1}^{T} E\left[ F_{it(1)}^* F_{it(1)}^2 \varepsilon_{it}^2 \right],
\]

and \(\hat{Z}_{it}\) denotes the \(t\)th row of \(\hat{Z}_i\). One can consistently estimate \(\Theta_{F_{(1)}}, \Theta_{NT,F_{(1)}}\) by \(\hat{\Theta}_{F_{(1)}}, \hat{\Theta}_{NT,F_{(1)}}\) where \(\hat{\varepsilon}_{it} = Y_{it} - X'_{it}^{0} \hat{\beta}_{(1)} - \hat{\lambda}_{it}^{0} \hat{F}_{i(1)}\). Below we focus on inferential theory for \(\beta_{(1)}^{0}\).

---

\(^5\)\(\nu_{1NT}\) is centered around 0 asymptotically because \(X_{1NT}\) is defined as a weighted average of \(X_{j(1)}\) which asymptotically smooths out the endogenous component of \(X_{jt}\); \(\nu_{2NT}\) is also asymptotically centered around 0 because of the adoption of a bias-corrected estimate \(\tilde{\beta}_c\) in its definition.
Let $\hat{\Psi}_{NT} \equiv \text{diag}(\hat{\psi}_{1T}, \ldots, \hat{\psi}_{NT})$ and $\hat{\Phi}_{NT} \equiv \text{diag}(\hat{\phi}_{1N}, \ldots, \hat{\phi}_{TN})$ where $\hat{\psi}_{iT} = T^{-1} \sum_{t=1}^{T} \hat{\varepsilon}_{it}^2$ and $\hat{\phi}_{iN} = N^{-1} \sum_{i=1}^{N} \hat{\varepsilon}_{it}^2$. Let $\hat{Z}_{i} = X_{i}(1) - P_{F(i)} X_{i} - M_{F(i)} \hat{X}_{1NT} + [X_{i} - P_{F(i)} X_{i} - M_{F(i)} \hat{X}_{2NT}] \hat{W}_{NT} \hat{C}_{NT}$, where $\hat{X}_{1NT} = \frac{1}{N} \sum_{j=1}^{N} \lambda_{i(j)}^{'} (T^{-1} \hat{F}(i) \hat{F}(i)) \hat{X}_{j(j)}$, and $\hat{X}_{2NT} = \frac{1}{N} \sum_{j=1}^{N} \lambda_{i(j)}^{'} (N^{-1} \hat{\lambda}_{i(j)}^{'} \hat{\lambda}_{i(j)})^{-1} \lambda_{i(j)}^{'} X_{j}$, $\hat{C}_{NT} = (NT)^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{\lambda}_{i(j)}^{'} (T^{-1} \hat{F}(i) \hat{F}(i)) \hat{\lambda}_{j(j)} \hat{X}_{i(j)} M_{F(i)} X_{j}$, $\hat{W}_{NT} = \frac{1}{NT} \sum_{t=1}^{T} \hat{X}_{t} M_{F(i)} \hat{X}_{t}$, and $\hat{\lambda}_{i} = X_{i} - \hat{X}_{2NT}$. Note that we can write the $k$th elements of $B_{3NT}$ and $\tilde{B}_{4NT} \equiv E D (B_{4NT})$ respectively as

$$
B_{3NT,k} = (NT)^{-1/2} \text{tr} \left[ (F_{1(i)} F_{1(i)})^{-1} \hat{F}_{1(i)} \hat{F}_{1(i),c}^{'} \hat{\varepsilon}^{'} \hat{X}_{k} \hat{F}_{1(i)} \right] 
\text{and} 
\tilde{B}_{4NT,k} = (NT)^{-1/2} \text{tr} \left[ P_{F(i)} (\hat{\varepsilon}^{'} \hat{X}_{k}) \right].
$$

We propose to estimate the bias and variance terms as follows:

$$
\hat{B}_{1NT} \equiv N^{-5/2} T^{-3/2} \sum_{i=1}^{N} X_{i(i)}^{'} \hat{F}_{1(i)} (\hat{F}_{1(i)} \hat{F}_{1(i)})^{-1} \hat{F}_{1(i)} \hat{F}_{1(i)}^{'} \hat{\Psi}_{NT} \hat{\lambda}_{i(1)} \hat{\lambda}_{i(1)}^{'} \hat{\Phi}_{NT},
$$

$$
\hat{B}_{2NT} \equiv N^{-1/2} T^{-3/2} \sum_{i=1}^{N} X_{i(i)}^{'} M_{F(i)} \hat{\Phi}_{NT} \hat{F}_{1(i)} \hat{\lambda}_{i(1)}^{'} ,
$$

$$
\hat{B}_{3NT,k} \equiv (NT)^{-1} \text{tr} \left[ (\hat{F}_{1(i)} \hat{F}_{1(i)})^{-1} \hat{F}_{1(i)} \hat{F}_{1(i)}^{'} \hat{\Psi}_{NT} \hat{\lambda}_{i(1)} \hat{\lambda}_{i(1)}^{'} \hat{X}_{k} \hat{F}_{1(i)} \right] \text{for} \ k = 1, \ldots, K_{0},
$$

$$
\hat{B}_{4NT,k} \equiv \frac{1}{\sqrt{NT}} \text{tr} \left[ P_{F(i)} (\hat{\varepsilon}^{'} \hat{X}_{k})^{\text{trunc}} \right] \text{for} \ k = 1, \ldots, K_{0},
$$

$$
\hat{\Theta}_{NT} \equiv \frac{1}{N T} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{\varepsilon}_{it}^{2} \hat{Z}_{it} \hat{Z}_{it}^{'},
$$

where $A^{\text{trunc}} = \sum_{t=1}^{T-M} \sum_{s=t}^{t+M} A_{ts}^{*}$ for any $T \times T$ matrix $A = (A_{ts})$ and $A_{ts}^{*}$ is a $T \times T$ matrix with $(t, s)$th element given by $A_{ts}$ and zeros elsewhere, and $\hat{Z}_{it}^{'}$ denotes the $t$th row of $\hat{Z}_{i}$. Let $\tilde{B}_{3NT} \equiv (\tilde{B}_{3NT,1}, \ldots, \tilde{B}_{3NT,K_{0}})^{'}$ and $\tilde{B}_{4NT} \equiv (\tilde{B}_{4NT,1}, \ldots, \tilde{B}_{4NT,K_{0}})^{'}$. We define the bias-corrected adaptive group Lasso estimator of $\beta_{(1)}^{0}$ as

$$
\tilde{\beta}_{(1)}^{c} = \hat{\beta}_{(1)} - (NT)^{-1/2} \tilde{D}_{F(i)}^{-1} (\hat{B}_{1NT} - \hat{B}_{2NT} - \hat{B}_{3NT} - \hat{B}_{4NT}).
$$

The following corollary establishes the asymptotic distribution of $\tilde{\beta}_{(1)}^{c}$.

**Corollary 3.4** Suppose Assumptions A.1(i), (v), (viii), A.3, A.4(i)-(ii), A.6, and B.1-B.2 hold. Then $\sqrt{NT}(\tilde{\beta}_{(1)}^{c} - \beta_{(1)}^{0}) \Rightarrow N (0, D_{F(0)}^{-1} \Theta D_{F(0)}^{-1})$.

**Remark 7.** The proof of the above corollary is quite involved and we delegate it to the supplementary appendix. If only strictly exogenous regressors are present in the model, following Remark 6, we can set $\hat{B}_{4NT} = 0$ and redefine $\hat{Z}_{i} \equiv M_{F(i)} [X_{i(i)} - \hat{X}_{i1NT} + (X_{i} - \hat{X}_{i2NT}) \hat{W}_{NT}^{-1} \hat{C}_{NT}]$ to be used in the variance estimation. When other conditions are also satisfied, both the bias and variance estimates can be further simplified with obvious modifications according to Remark 6. It is worth mentioning that our bias-corrected estimator is asymptotically equivalent to Moon and Weidner’s (2010b) bias-corrected estimator in the case where all regressors are relevant (i.e., there is no selection of regressors). In the presence of irrelevant regressors, the variance-covariance matrix for our shrinkage estimator of the non-zero coefficients is smaller than that of Moon and Weidner’s estimator.

**Remark 8.** Belloni and Chernozhukov (2013) study post-model selection estimators which apply ordinary least squares to the model selected by first-step penalized estimators and show that the post Lasso
estimators perform at least as well as Lasso in terms of the rate of convergence and have the advantage of having a smaller bias. After we apply our adaptive group Lasso procedure, we can re-estimate the panel data model based on the selected regressors and number of factors and the QMLE method of Bai (2009) or Moon and Weidner (2010a, 2010b). We will compare the performance of these post-Lasso estimators with the Lasso estimators through simulations.

### 3.4 Choosing the Tuning Parameter $\gamma$

Let $S_\beta(\gamma)$ and $S_\lambda(\gamma)$ denote the index set of nonzero elements in $\hat{\beta}(\gamma)$ and nonzero columns in $\hat{\lambda}(\gamma)$, respectively. Let $S(\gamma) = S_\beta(\gamma) \times S_\lambda(\gamma)$. Let $|S|$ denote the cardinality of the index set $S$. We propose to select the tuning parameter $\gamma = (\gamma_1, \gamma_2)$ by minimizing the following information criterion:

$$IC(\gamma) = \hat{\sigma}^2(\gamma) + \rho_{1NT} |S_\beta(\gamma)| + \rho_{2NT} |S_\lambda(\gamma)|,$$

where $\hat{\sigma}^2(\gamma) = \mathcal{L}_{NT}(\hat{\beta}(\gamma), \hat{\lambda}(\gamma))$. Similar information criteria are proposed by Wang et al. (2007) and Liao (2013) for shrinkage estimation in different contexts.

Let $S_{F,\lambda} = \{1, ..., K\}$ and $S_{T,\beta} = \{1, ..., K_0\}$ denote the index sets for the full set of covariates and the (true) set of relevant covariates in $X_{it}$, respectively. Similarly, $S_{F,\lambda} = \{1, ..., R\}$ and $S_{T,\lambda} = \{1, ..., R_0\}$ denote the index sets for the full set of factors and the (true) set of relevant factors in $F_t$, respectively. Let $S = (S_\beta, S_\lambda)$ be an arbitrary index set with $S_\beta = \{j_1, ..., j_{K_0}\} \subset S_{F,\beta}$ and $S_\lambda = \{j_1, ..., j_{R_0}\} \subset S_{F,\lambda}$ where $0 \leq K^* \leq K$ and $0 \leq R^* \leq R$. Consider a candidate model with regressor index $S_\beta$ and factor index $S_\lambda$. Then any candidate model with either $S_\beta \nsubseteq S_{T,\beta}$ or $S_\lambda \nsubseteq S_{T,\lambda}$ is referred to as an under-fitted model in the sense that it misses at least one important covariate or factor. Similarly, any candidate model with $S_\beta \supset S_{T,\beta}$, $S_\lambda \supset S_{T,\lambda}$ and either $S_\beta \nsubseteq S_{T,\beta}$ or $S_\lambda \nsubseteq S_{T,\lambda}$ (i.e., $|S_\beta| + |S_\lambda| > |S_{T,\beta}| + |S_{T,\lambda}|$) is referred as an over-fitted model in the sense that it contains not only all relevant covariates and factors but also at least one irrelevant covariate or factor.

Denote $\Omega_1 = [0, \gamma_{1\max}]$ and $\Omega_2 = [0, \gamma_{2\max}]$, two bounded intervals in $\mathbb{R}^+$, where the potential tuning parameters $\gamma_{1NT}$ and $\gamma_{2NT}$ take values, respectively. Here we suppress the dependence of $\Omega_1$, $\Omega_2$, $\gamma_{1\max}$ and $\gamma_{2\max}$ on $(N, T)$. We divide $\Omega = \Omega_1 \times \Omega_2$ into three subsets $\Omega_0$, $\Omega_-$ and $\Omega_+$ as follows

$$\begin{align*}
\Omega_0 &= \{ \gamma \in \Omega : S_\beta(\gamma) = S_{T,\beta} \quad \text{and} \quad S_\lambda(\gamma) = S_{T,\lambda} \}, \\
\Omega_- &= \{ \gamma \in \Omega : S_\beta(\gamma) \nsubseteq S_{T,\beta} \quad \text{or} \quad S_\lambda(\gamma) \nsubseteq S_{T,\lambda} \}, \\
\Omega_+ &= \{ \gamma \in \Omega : S_\beta(\gamma) \supset S_{T,\beta} \quad \text{and} \quad S_\lambda(\gamma) \supset S_{T,\lambda}, \quad |S_\beta| + |S_\lambda| > |S_{T,\beta}| + |S_{T,\lambda}| \}.
\end{align*}$$

Clearly, $\Omega_0$, $\Omega_-$ and $\Omega_+$ denote the three subsets of $\Omega$ in which the true, under- and over-fitted models can be produced.

For any $S = S_\beta \times S_\lambda$ with $S_\beta = \{j_1, ..., j_{|S_\beta|}\} \subset S_{F,\beta}$ and $S_\lambda = \{l_1, ..., l_{|S_\lambda|}\} \subset S_{F,\lambda}$, we use $\beta_{S_\beta} = (\beta_{j_1}, ..., \beta_{j_{|S_\beta|}})'$ to denote an $|S_\beta| \times 1$ subvector of $\beta$ and $\lambda_{S_\lambda} = (\lambda_{l_1}, ..., \lambda_{l_{|S_\lambda|}})'$ to denote an $N \times |S_\lambda|$ submatrix of $\lambda$. Similarly, $X_{it,S_\beta}$ and $\hat{F}_{t,S_\lambda}$ denote the $|S_\beta| \times 1$ subvector of $X_{it}$ and $|S_\lambda| \times 1$ subvector of $F_t$ according to $S$. Let $\hat{\beta}_{S_\beta}$ and $\hat{\lambda}_{S_\lambda}$ denote the ordinary least squares (OLS) estimators of $\beta_{S_\beta}$ and $\lambda_{S_\lambda}$, respectively, by regressing $Y_{it}$ on $X_{it,S_\beta}$ and $\hat{F}_{t,S_\lambda}$. Define

$$\hat{\sigma}^2_S = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left( Y_{it} - \hat{\beta}_{S_\beta} X_{it,S_\beta} - \hat{\lambda}_{S_\lambda} \hat{F}_{t,S_\lambda} \right)^2,$$

(3.4)
where $\hat{\lambda}_{i,S_i}$ denotes the $i$th row of $\hat{\lambda}_{S_i}$. Let $S_T = S_{T,\beta} \times S_{T,\lambda}$. One can readily show that $\hat{\sigma}^2_{S_T} \overset{P}{\to} \sigma^2_{S_T} \equiv \lim_{(N,T) \to \infty} \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T E(\varepsilon^2_{it})$ under Assumptions A.1-A.2.

To proceed, we add the following two assumptions.

**Assumption A.7** For any $S = S_\beta \times S_\lambda$ with either $S_\beta \not\subset S_{T,\beta}$ or $S_\lambda \not\subset S_{T,\lambda}$, there exists $\sigma^2_S$ such that $\hat{\sigma}^2_S \overset{P}{\to} \sigma^2_S > \sigma^2_{S_T}$.

**Assumption A.8** As $(N,T) \to \infty$, $\rho_{1NT} \to 0$, $\rho_{2NT} \to 0$, $\rho_{1NT}\delta_{NT}^2 \to \infty$, and $\rho_{2NT}\delta_{NT}^2 \to \infty$.

Assumption A.7 is intuitively clear. It requires that all under-fitted models yield asymptotic mean square errors that are larger than $\sigma^2_{S_T}$, which is delivered by the true model. A.8 reflects the usual conditions for the consistency of model selection. The penalty coefficients $\rho_{1NT}$ and $\rho_{2NT}$ cannot shrink to zero either too fast or too slowly.

Let $\gamma^0_{NT} = (\gamma^0_{1NT}, \gamma^0_{2NT})$ where $\gamma^0_{1NT}$ and $\gamma^0_{2NT}$ satisfy the conditions on $\gamma_1$ and $\gamma_2$, respectively in Assumptions A.3(ii)-(iii).

**Theorem 3.5** Suppose that Assumptions A.1, A.2, A.3(i), A.7 and A.8 hold. Then

$$P \left( \inf_{\gamma \in \Omega_{1,T}, \Omega_{1,t}} IC(\gamma) > IC(\gamma^0_{NT}) \right) \to 1 \text{ as } (N,T) \to \infty.$$ 

**Remark 9.** Note that we do not impose Assumptions A.3(ii)-(iii) and A.4-A.6 in the above theorem. Theorem 3.5 implies that the $\gamma$’s that yield the over- or under-selected sets of regressors or number of factors fail to minimize the information criterion w.p.a.1. Consequently, the minimizer of $IC(\gamma)$ can only be the one that meets Assumptions A.3(ii)-(iii) so that both estimation and selection consistency can be achieved.

## 4 Monte Carlo Simulations

In this section, we conduct Monte Carlo simulations to evaluate the finite sample performance of our proposed adaptive group Lasso (agLasso) method.

### 4.1 Data Generating Processes

We consider the following data generating processes (DGPs):

**DGP 1:** $Y_{it} = \lambda_{i1}^0 + \lambda_{i2}^0 + \theta_{i1} + \varepsilon_{it}$.

**DGP 2:** $Y_{it} = \beta_{i1}^0X_{i1,t} + \beta_{i2}^0X_{i2,t} + \lambda_{i1}^0F_{i1,t} + \lambda_{i2}^0F_{i2,t} + \theta_{i1} + \varepsilon_{it}$, where $(\beta_{i1}^0, \beta_{i2}^0) = (1, 1)$.

**DGP 3:** $Y_{it} = \beta_{i1}^0X_{i1,t} + \beta_{i2}^0Y_{i1,t-1} + \lambda_{i1}^0F_{i1,t} + \lambda_{i2}^0F_{i2,t} + \theta_{i1} + \varepsilon_{it}$, where $(\beta_{i1}^0, \beta_{i2}^0) = (1, 0.25)$.

**DGP 4:** $Y_{it} = \beta_{i1}^0X_{i1,t} + \beta_{i2}^0X_{i2,t} + \beta_{i3}^0X_{i3,t} + \beta_{i4}^0X_{i4,t} + \beta_{i5}^0X_{i5,t} + \lambda_{i1}^0F_{i1,t} + \lambda_{i2}^0F_{i2,t} + \theta_{i1} + \varepsilon_{it}$, where $(\beta_{i1}^0, \beta_{i2}^0, \beta_{i3}^0, \beta_{i4}^0, \beta_{i5}^0) = (1, 1, 0, 0, 0)$.

**DGP 5:** $Y_{it} = \beta_{i1}^0X_{i1,t} + \beta_{i2}^0Y_{i1,t-1} + \beta_{i3}^0X_{i2,t} + \beta_{i4}^0X_{i4,t} + \beta_{i5}^0Y_{i5,t-2} + \lambda_{i1}^0F_{i1,t} + \lambda_{i2}^0F_{i2,t} + \theta_{i1} + \varepsilon_{it}$, where $(\beta_{i1}^0, \beta_{i2}^0, \beta_{i3}^0, \beta_{i4}^0, \beta_{i5}^0) = (1, 0.25, 0, 0, 0)$.

In all the five DGPs, $\lambda_{i1}^0$, $\lambda_{i2}^0$, and $\varepsilon_{it}$ are independent $N(0,1)$ random variables. In DGPs 1, 2 and 4, $F_{i1,t}$ and $F_{i2,t}$ are independently and standard normally distributed. In DGPs 3 and 5, we consider an AR(1) structure for the factors: $F_{i1,t} = 0.5F_{i1,t-1} + \zeta_{i1,t}$ and $F_{i2,t} = 0.5F_{i2,t-1} + \zeta_{i2,t}$, where $(\zeta_{i1,t}, \zeta_{i2,t})$ are independent $N(0,1)$ random variables. $X_{it,k} = 0.25(\lambda_{i1}^0F_{i1} + \lambda_{i2}^0F_{i2}) + \xi_{it,k}$, where $\xi_{it,k}$ are IID $N(0,1)$.
across both \(i\) and \(t\) for \(k = 1, \ldots, 5\). We use \(\theta\) to control for the signal-to-noise (SN) ratio, which is defined as \(\text{Var}(\beta^0 X_{it} + \lambda_i^0 F_{it}^0) / \text{Var}(\theta \varepsilon_{it})\). For each DGP, we choose two \(\theta\)'s such that the SN ratio equals either 2 or 1.

DGP 1 is a pure factor structure without any regressor \(X_{it}\). DGPs 2 and 3 are static and dynamic panel structures with interactive fixed effects, respectively. DGP 4 is identical to DGP 2 except that in DGP 4 we include three more irrelevant regressors: \(X_{it,3}, X_{it,4}\) and \(X_{it,5}\). Hence, in DGP 4, we consider both the selection of the regressors and determination of the number of factors, while in DGP 2 we only consider the latter. DGP 5 is identical to DGP 3, except that DGP 5 includes three irrelevant regressors: \(X_{it,2}, X_{it,3}\) and \(Y_{it-2}\). Thus, we select both the regressors and number of factors in DGP 5.

The true number of factors is 2 in all the above five DGPs. In our simulations, we assume that we do not know the true number of factors. We consider different combinations of \((N, T) : (20, 20), (40, 40), (20, 60), (60, 20)\) and \((60, 60)\). The number of replications is 250.

4.2 Implementation

One of the important steps in our method is to choose the tuning parameters \(\gamma_{1NT}\) and \(\gamma_{2NT}\). Following our theoretical arguments above, we use the information criterion in (3.3). Let \(s_i^2\) denote the sample variance of \(Y_{it}\). For DGPs 1-3 where we only choose the number of factors, we set \(\rho_{1NT} = 0\) and \(\rho_{2NT} = s_i^2 \ln(\delta_{NT}) / (N \min(N, T))\). For DGPs 4-5, we set \(\rho_{1NT} = 0.05s_i^2 \ln(\delta_{NT}) / \min(N, T)\) and \(\rho_{2NT} = s_i^2 \ln(\delta_{NT}) / (N \min(N, T))\). In DGPs 1-3, we only select factors, hence we let \(\gamma_{1NT} = 0\) and choose \(\gamma_{2NT}\) from the set: \(\{C_\gamma s_i^2 \gamma_1^{c2}(NT)^{-1/2}(\ln(NT))^{-1}\}\), where \(C_\gamma\) are 50 constants that increase geometrically from 0.01 to 25, i.e., \(C_\gamma = 0.01, 0.014, \ldots, 18.045, 25\). For DGPs 4-5, we let the candidate set of \((\gamma_{1NT}, \gamma_{2NT})\) be the Cartesian product: \(\{C_\gamma s_i^2 (NT)^{-1/2}(\ln(NT))^{-1}\} \times \{C_\gamma s_i^2 \gamma_1^{c2}(NT)^{-1/2}(\ln(NT))^{-1}\}\), where \(C_\gamma\) are 25 constants that increase geometrically from 0.01 to 25. We set \(\kappa_1 = \kappa_2 = 2\) in all cases.

We compare our agLasso method with the methods of determining the number of factors proposed in Bai and Ng (2002) and Onatski (2009). Their methods only apply to pure factor structures without regressors. Thus, we have to modify their methods to account for the presence of regressors. Specifically, we apply their methods to the factor component: \(\lambda_i^0 F_{it}^0 + \varepsilon_{it}\), which can be consistently estimated by \(\hat{Y}_{it} = Y_{it} - \tilde{\beta}^e X_{it}\), where \(\tilde{\beta}^e\) is Moon and Weidner’s (2010b) bias-corrected estimator of \(\beta\) using the largest number of factors \(R\). We briefly describe these methods here. Bai and Ng (2002, p.201) consider the following information criteria to select the number of factors:

\[
PC_1(r) = V(r) + r \cdot V(R) \cdot \left( \frac{N + T}{NT} \right) \cdot \ln \left( \frac{NT}{N + T} \right),
\]

\[
PC_2(r) = V(r) + r \cdot V(R) \cdot \left( \frac{N + T}{NT} \right) \cdot \ln \left( \min(N, T) \right),
\]

\[
IC_1(r) = \ln(V(r)) + r \cdot \left( \frac{N + T}{NT} \right) \cdot \ln \left( \frac{NT}{N + T} \right),
\]

\[
IC_2(r) = \ln(V(r)) + r \cdot \left( \frac{N + T}{NT} \right) \cdot \ln \left( \min(N, T) \right),
\]

where \(V(r) = (NT)^{-1} \sum_{i=1}^N \tilde{\varepsilon}_{i,r} \tilde{\varepsilon}_{i,r}\), and \(\tilde{\varepsilon}_{i,r}\) is the \(T \times 1\) residual vector when \(r\) factors are included in the model.

Onatski (2009) develops a test to test the null hypothesis that the true number of factors \(R_0 = r\), against the alternative \(r < R_0 \leq R\). The test can be used to determine the number of factors. Specifically,
we start by testing $H_0 : R_0 = 0$ versus $H_1 : 0 < R_0 \leq R$. If $H_0$ is not rejected, then we conclude $R_0 = 0$. Otherwise, we continue to test $H_0 : R_0 = 1$ versus $H_1 : 1 < R_0 \leq R$. We repeat the procedure until $H_0 : R_0 = r$ is not rejected and conclude $R_0 = r$. The test is based on the largest eigenvalue of the smoothed periodogram estimate of the spectral density matrix of data and the details are described in Section 4 in Onatski (2009, p.1455).

4.3 Effects of the Number of Factors on the Estimation of $\beta$’s

Before we compare various methods, we first examine the effects of the number of factors included in the model on the performance of the estimators of $\beta$’s. Table 1 presents the mean squared errors (MSEs) of Moon and Weidner’s (2010b) bias-corrected estimators of $\beta_1$ and $\beta_2$ with different numbers of factors $r$: $r = 0, 1, 2, 4, 6$ and 8.\textsuperscript{5} It is easy to see that when $r = 2$ (the true number of factors), the MSEs are the smallest. In general, the number of factors has substantial effects on the MSEs, especially when $N$ or $T$ is small. For example, when $N = 20$ and $T = 20$ and the SN ratio is 1 in DGP 5, the MSEs of the estimate of $\beta_1$ with $r = 0$ and $r = 8$ are 10.8 and 7.5 times as large as those with $r = 2$, respectively; the MSEs of the estimate of $\beta_2$ with $r = 0$ and $r = 8$ are 4.6 and 69.5 times as large as those with $r = 2$, respectively. The simulations suggest that the finite sample performance of the estimates of $\beta$’s crucially depends on the number of included unobservable factors, especially when $N$ or $T$ is small.

4.4 Illustration of the Main Ideas

The main innovation of this paper is to use agLasso to determine the number of factors. There are three key ideas underlying our approach. First, the smallest $R_0$ eigenvalues of $\hat{\Sigma}_F \equiv \hat{F}'\hat{F}/T$ converge to zero, while the largest $R_0$ eigenvalues converge to some positive numbers, which ensures the adaptive nature of our approach. Second, the penalty term $\lambda_{2NT}$ controls the number of factors selected. The larger the penalty term $\lambda_{2NT}$ is, the fewer factors are selected. Third, the information criterion chooses an appropriate penalty term $\lambda_{2NT}$. Below, we set $R = 8$ and use the simple DGP 1 with the SN ratio being 1 to illustrate these three main ideas. Note that in DGP 1, there is no regressor so that we only consider the selection of the number of factors.

Plots (a) and (b) in Figure 1 show the medians of the eight eigenvalues of $\hat{\Sigma}_F$ over the 250 replications for $(N, T) = (20, 20)$ and $(40, 40)$, respectively. It is clear that the two largest eigenvalues are greater than zero, while the six smallest eigenvalues are all close to zero. Plots (c) and (d) show the effects of the penalty term $\lambda_{2NT} = C_\gamma s_1^2 T^{-1/2} \sqrt{N^2T^{-1}\ln(NT)}$ on the selected number of factors for $(N, T) = (20, 20)$ and $(40, 40)$, respectively. To make the picture clearer, we choose a wide range of $C_\gamma$ values: 250 points that increase geometrically from 0.001 to 25. We can see that when $C_\gamma$ (and thus $\lambda_{2NT}$) increases, i.e., the penalty becomes larger, a smaller number of factors are selected. We also note that for this DGP, there is quite a wide range of $C_\gamma$ values (and thus $\lambda_{2NT}$) that correctly select the number of factors, especially for $(N, T) = (40, 40)$. Plots (e) and (f) show how our information criterion (IC) changes with respect to $C_\gamma$ for $(N, T) = (20, 20)$ and $(40, 40)$, respectively. In general, the minimizer of IC falls in the range of $C_\gamma$ that correctly selects the number of factors.

\textsuperscript{5}The results for $r = 3, 5$, and 7 are available upon request. In DGPs 4 and 5, all the five regressors are included in the models and the estimation results for $\beta_3, \beta_4$ and $\beta_5$ are also available upon request.
4.5 Simulation Results

The simulation results are summarized in Tables 2-4. Table 2 reports the proportions of the replications in which the number of factors is correctly determined out of total 250 replications. Our agLasso method is among the best performers in general. For example, for DGP 2, when \( N = 20, T = 20 \) and the SN ratio is 1, our agLasso method selects the true number of factors with a correct rate of 53%, while PC1, PC2, IC1, IC2, and Onatski with 0%, 0%, 14%, 8%, and 1%, respectively. In general, when \( N \) and \( T \) increase, the performances of all the methods improve.

Table 3 shows the proportions of the replications in which the estimates of the \( \beta \)'s are shrunk to zero out of total 250 replications for DGPs 4-5. Note that the methods proposed by Bai and Ng (2002) and Onatski (2009) cannot select regressors. Thus we only report the results using our agLasso method. For DGP 4, the relevant regressors \( X_{it,1} \) and \( X_{it,2} \) are always selected, while a large proportion (e.g., above 90% for \( N = 60 \) and \( T = 60 \)) of the estimated coefficients of the irrelevant regressors \( (\beta_3, \beta_4 \) and \( \beta_5 \)) are shrunk to zero. For DGP 5, when \( T \) is large, the relevant regressors are always selected, though a small proportion of relevant regressors are not selected when \( T \) is small. For the irrelevant regressors, a large proportion of estimated \( \beta_3 \)'s and \( \beta_4 \)'s are shrunk to zero. However, a large proportion of estimated \( \beta_5 \)'s are shrunk to zero only when \( T \) is large.

Table 4 reports the MSEs of estimated \( \beta_1 \)'s and \( \beta_2 \)'s for DGPs 2-5 using different methods.\(^7\) In addition to our agLasso estimators, we also report the performances of the bias-corrected agLasso (BC-agLasso) estimator introduced in Section 3.3 and the post-adaptive group Lasso (post-agLasso) estimators, which is Moon and Weidner’s (2010b) bias-corrected estimators using the number of factors and regressors selected by our agLasso method. For most of the cases, our agLasso, BC-agLasso, post-agLasso estimators achieve smaller MSEs than other methods when \( N \) and \( T \) are small. For example, for \( \beta_1 \) in DGP 2 with the SN ratio being 1 and \( (N, T) = (20, 20) \), the 100×MSEs for our agLasso, BC-agLasso and post-agLasso estimators are 4.08, 4.11 and 3.84 respectively, while those for PC1, PC2, IC1, IC2 and Onatski (2009) are 9.98, 9.40, 7.94, 11.18 and 17.53 respectively. When \( N \) and \( T \) are large, all the methods perform similarly well. In general, the post-agLasso estimator performs best among the three agLasso-type of estimators, whereas the agLasso and BC-agLasso estimators perform similarly.\(^8\)

5 Empirical Application

In this section, we apply our method to study the determinants of economic growth. There is a large literature on the empirical studies of economic growth. For example, Barro (1991) and Sala-i-Martin et al. (2004) investigate this question using cross-sectional data. For panel data, Islam (1995) employs country fixed effect models and Moral-Benito (2012) uses a Bayesian model averaging approach. Durlauf et al. (2005, DJT) provide a comprehensive literature review. To the best of our knowledge, none of

\(^7\)For DGPs 4-5, our agLasso method both selects the regressors and determines the number of factors, while Bai and Ng’s (2002) information criteria (PC1, PC2, IC1, and IC2) and Onatski’s (2009) testing method are capable of the latter only. So all five regressors are included for their methods. The estimation results for \( \beta_3, \beta_4, \) and \( \beta_5 \) are available upon request.

\(^8\)For the BC-agLasso estimators, we ignore the fact that the exogenous regressors share the same factor structure as the dependent variable in which case there is no need to correct some of the bias terms (\( B_{2NT} \) in particular). See Remark 6.
the existing studies allows interactive fixed effects. However, it is plausible that the economic growth is determined not only by observable regressors but also some common unobservable shocks or factors. Thus the panel data model with interactive fixed effects provides much more flexibility in this context. Nevertheless, in practice, we do not have a priori knowledge about the number of unobservable factors that should be included in the model. In addition, there are a large number of potential observable variables that may determine economic growth and economic theory does not provide much guidance for the selection of them. For example, DJT survey 145 possible determinants of economic growth and point out that “approximately as many growth determinants have been proposed as there are countries for which data are available. It is hard to believe that all these determinants are central...”. Thus, it is important to determine the number of factors and select the relevant regressors in this context.

5.1 Data and Implementation

Let \( Y_{it} \) be the growth rate of real GDP per capita for country \( i \) in year \( t \). \( X_{it} \) includes 9 variables as listed in Table 5. The sample covers 108 countries for the period 1970-2005, i.e., \( N = 108 \) and \( T = 36 \). Here we need a balanced panel, and the selection of the 108 countries is completely determined by the availability of data. The data sources are the Penn World Table (Penn Table) and World Bank World Development Indicators (WDI).

We include maximum 8 factors in the model. As in the simulations, we choose \( (\gamma_{1NT}, \gamma_{2NT}) \) from the set \( \left\{ C_i s_i^2 \frac{1}{\sqrt{NT \ln(NT)}} \right\} \times \left\{ C_j s_j^2 \frac{1}{\sqrt{NT \ln(NT)}} \right\} \), where \( C_i \) are 100 constants that increase geometrically from 0.01 to 25. Both \( \kappa_1 \) and \( \kappa_2 \) are equal to 2. The information criterion is the same as that in the simulations.

5.2 Estimation Results

5.2.1 Estimation without regressors

We first consider a pure factor structure without including any regressors. Our agLasso method chooses 3 factors. The eigenvalues \( (\tau_1, \ldots, \tau_8) \) used in our agLasso are shown in Figure 2(a). Bai and Ng’s (2002) PC1, PC2, IC1, IC2 choose 6, 5, 4, and 3 factors, respectively and Onatski’s (2009) testing method chooses 1 factor.

5.2.2 Linear estimation

We consider a linear specification that uses the 9 variables listed in Table 5 and the first 3 lags of \( Y_{it} \) as regressors. The first half of Table 6 shows the estimation results for the different numbers of factors, \( r = 0, 3, 5 \) and 8. The estimates of the coefficients vary substantially with different numbers of factors. For example, the coefficient of consumption is negative and significant when \( r \) is smaller than 4 and becomes insignificant when \( r \) is greater than 4. However, the coefficients of government consumption share, investment share, and the first lag of economic growth are significant for most of the numbers of factors.

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9 The only exception is Su et al. (2012), who apply a specification test of panel models with fixed effects to economic growth data. However, they do not provide estimation results.
10 The list of the 108 countries is available upon request.
11 The results for the number of factors \( r = 1, 2, 4, 6 \) and 7 are available upon request.
Our agLasso method chooses 3 factors, while PC₁, PC₂, IC₁, IC₂, and Onatski choose 6, 6, 4, 3, and 1 factors, respectively. The eigenvalues used in our agLasso are shown in Figure 2(b). The estimation results are presented in Table 6. Our agLasso selects five regressors: population growth, government consumption share, investment share, and the first and second lags of economic growth. Among them, government consumption share, investment share, and the first lag of economic growth are significant. The government consumption share has a negative effect on economic growth, while the investment share and lagged economic growth have positive effects.

5.2.3 Nonlinear estimation

In this subsection, we examine the nonlinear effects of the regressors. Our agLasso method selects five regressors in the linear specification. Thus, we include the squared and interaction terms of those five selected regressors in addition to the 12 regressors in the linear specification. The total number of regressors included is 27. In this case, our agLasso selects 3 factors again, while PC₁, PC₂, IC₁, IC₂, and Onatski choose 6, 6, 4, 4, and 1 factors, respectively. The eigenvalues used in our agLasso are shown in Figure 2(c). Among the 27 regressors, our agLasso method selects 9 regressors. Table 7 presents the estimation results for the 9 selected regressors. Based on the post-agLasso method, we find that consumption share, investment share, and the interaction term of government consumption share and investment share are significant at the 1% level, while the first lag of economic growth is significant at the 10% level. The signs of those significant regressors are in general consistent with those in the linear specification. The government consumption share has a negative effect through its interaction term with investment share, while the lagged economic growth has positive effects. The effect of investment share is \(0.219 - 0.008 \times \text{Gov}\), which is positive for most values of government consumption share in the sample. However, population growth becomes insignificant, and consumption share becomes significant with a negative sign.

To further examine the nonlinear effect, we consider a “high dimensional” model by including the linear, squared, and interaction terms of all the original 12 regressor (i.e., the 9 variables listed in Table 5 and the first 3 lags of the economic growth). The total number of regressors is 90 in this case. Our agLasso selects 3 factors again, while PC₁, PC₂, IC₁, IC₂, and Onatski choose 6, 6, 4, 4, and 1 factors, respectively. Among the 90 regressors, our agLasso method selects 11 regressors as shown in Table 8. The first half of Table 8 also reports the estimation results for the 11 regressors using the model that includes all the 90 regressors with different numbers of factors.\(^{12}\) Note that almost all the regressors are insignificant when all 90 regressors are included. This is not surprising, as the standard errors can be easily inflated when a large number of regressors are included. Nevertheless, our agLasso is effective in selecting the relevant regressors. Based on the post-agLasso estimation results, consumption share, government consumption share, investment share, the first lag of economic growth and the interaction term of fertility rates and the lagged economic growth are significant. Among them, consumption share and government consumption share have negative effects, while investment share and lagged economic growth have positive effects on economic growth.\(^{13}\)

To summarize, we find that in general there are 3 unobservable factors that determine economic growth. Among the observable regressors, considering both linear and nonlinear specifications, we find

\(^{12}\) The estimation results for the other 79 regressors are available upon request.

\(^{13}\) The effect of the first lag of economic growth is 0.321 – 0.049×Fert, which is positive in general.
that government consumption share has a negative effect, while investment share and lagged economic growth have positive effects on economic growth. This finding is largely consistent with the existing empirical literature on economic growth (see, e.g., DJT, Appendix 2).

6 Conclusion

We propose a novel adaptive group Lasso procedure for simultaneous selection of factors and relevant regressors and estimation in dynamic panel models with interactive fixed effects. We show that our method consistently determines the number of factors and selects relevant regressors. Our estimators of the slope parameters in the models achieve an oracle property. Our simulations suggest that our new method performs well in finite samples. We apply our method to study the determinants of economic growth and find that government consumption share has negative effects, whereas investment share and lagged economic growth have positive effects on economic growth.

There are several interesting topics for further research. First, the numbers of relevant regressors \( K_0 \) and factors \( R_0 \) are fixed in this paper. It is desirable to extend our method to allow \( K_0, R_0 \), or both, to increase with both \( N \) and \( T \). The divergence of \( K_0 \) to infinity is particularly useful for the nonparametric sieve estimation of dynamic panel models with interactive fixed effects (see, e.g., Su and Zhang (2013)), as the number of terms in sieve regressions increases with both \( N \) and \( T \). The divergence of \( R_0 \) to infinity is important in approximate factor models. Second, as an alternative to the adaptive group Lasso used in this paper, the SCAD method of Fan and Li (2001) can also be employed and it will be interesting to compare our method with that based on the SCAD method. Third, it is also possible to allow endogeneity in panel data models with interactive fixed effects, in which case various important issues would arise, including how to extend the usual instrumental variable (IV) estimation to the current framework, how to determine the set of instruments, and how to select the number of factors and relevant regressors. Endogeneity naturally arises in dynamic panel data models with measurement error (e.g., Lee et al., 2012) and in various macro and micro panel data models. Fourth, instead of considering variable and factor selection in augmented panel regression models, an alternative is to consider SCAD- or Bridge-based penalized PCA in one step. We are exploring some of these topics in ongoing works.
A Some Technical Lemmas

Recall \( \bar{Y} = Y - \sum_{k=1}^{K} \beta_k^* X_k \), \( \bar{F} = (NT)^{-1} \bar{Y}' \bar{Y} \bar{F} \), \( H = H_{NT} = (N^{-1}X'X)^0 (T^{-1}F^0 F) \), and \( T^{-1} \bar{F}' \bar{F} = I_R \). Recall \( \bar{F} = (\hat{F}_1(1), \hat{F}_2(2)) \), \( \hat{F} = (\hat{F}_1(1), \hat{F}_2(2)) \) and \( H = (H_{1}, H_{2}) \), where, e.g., \( \hat{F}_1(1) \) and \( \hat{F}_2(2) \) are \( T \times R \) and \( T \times (R - R_0) \) submatrices, respectively. Noting that \( T^{-1} \| \hat{F} \|^2 = \text{tr}(T^{-1} \hat{F}' \hat{F}) = R \), \( \| \hat{F} \| = O_P(T^{-1/2}) \).

Write \( V_{NT} = \text{diag}(V_{NT,11}, V_{NT,22}) \) where \( V_{NT,11} \) and \( V_{NT,22} \) are \( R_0 \times R_0 \) and \( (R - R_0) \times (R - R_0) \) submatrices of \( V_{NT} \), respectively. We first state some technical lemmas whose proofs are given in the supplementary appendix.

Lemma A.1 Suppose Assumption A.1 holds. Then

(i) \( T^{-1} \bar{F}' (NT)^{-1} \bar{Y}' \bar{Y} \bar{F} = V_{NT} \overset{p}{\rightarrow} V \),
(ii) \( T^{-1} \bar{F}' F_0 (N^{-1}X_0'X_0) (T^{-1}F_0' F_0) P \overset{P}{\rightarrow} V \),
(iii) \( T^{-1} F_0' H_1(1)^{P} \Delta_1 \) and \( T^{-1} F_0' H_2(2)^{P} \rightarrow 0 \),
(iv) \( H_{1} \overset{P}{\rightarrow} \Sigma_{1} \Delta_{1} \) and \( H_{2} \overset{P}{\rightarrow} 0 \),

where \( V_{NT} \) is an \( R \times R \) diagonal matrix consisting of the \( R \) largest eigenvalues of \( (NT)^{-1} \bar{Y}' \bar{Y} \), and \( V = \begin{bmatrix} V_{11} & 0 \\ 0 & 0 \end{bmatrix} \) with \( V_{11} \) being an \( R_0 \times R_0 \) matrix consisting of the \( R_0 \) eigenvalues of \( \Sigma^0 \Sigma_{P0} \), both arranged in descending order; \( \Delta_1 \) is an \( R_0 \times R_0 \) full rank matrix.

Lemma A.2 Suppose Assumptions A.1, A.2 and A.3(i) hold. Let \( \tau_1, ..., \tau_R \) denote the eigenvalues of \( T^{-1} H' F_0' F_0 H \) in descending order. Then

(i) \( T^{-1} \| \hat{F} - F_0 H \|^2 = O_P(\delta^{-2}_{NT}) \),
(ii) \( T^{-1} \| \hat{F}_1(1) - F_0 H_{1,11} \|^2 = O_P(\delta^{-2}_{NT}) \) and \( \| H_{2}(2) \| = O_P(\delta^{-1}_{NT}) \),
(iii) \( T^{-1} (\hat{F}_1(1) - F_0 H_{1,11})' F_0 = O_P(\delta^{-2}_{NT}) \) and \( T^{-1} (\hat{F}_2(2) - F_0 H_{2,22})' F_0 = O_P(N^{-1/2}) \),
(iv) \( T^{-1} (\hat{F}_1(1) - F_0 H_{1,11})' \hat{F}_1(1) = O_P(\delta^{-2}_{NT}) \), \( T^{-1} (\hat{F}_2(2) - F_0 H_{2,22})' \hat{F}_2(2) = O_P(N^{-1/2}) \),
(v) \( T^{-1} (\hat{F}_1(1) - H_{1,11})' F_0' F_0 H_{1,11} = O_P(\delta^{-2}_{NT}) \) and \( T^{-1} (\hat{F}_1(1) - H_{1,11})' \hat{F}_1(1) = O_P(N^{-1/2}) \),
(vi) \( \| P_{\hat{F}_1(1)} - P_{F_1(1)} \| = O_P(\delta^{-1}_{NT}) \),
(vii) \( \tau_{l} - \tau_{l} = O_P(\delta^{-2}_{NT}) \) for \( l = 1, 2, ..., R_0 \) and \( \tau_{l} = O_P(N^{-1/2}) \) for \( l = R_0 + 1,...,R_0 \), where \( \tau_{l} \), ..., \( \tau_{R} \) denote the \( R \) eigenvalues of \( T^{-1} \hat{F}' \hat{F} \) arranged in descending order.

Lemma A.3 Suppose Assumptions A.1, A.2, and A.3(i) hold. Let \( F_1(1)^{i} = F_0 H_{1,11} \) and \( \lambda_1^{i} = \lambda^0 H_{1,11}^{i} \).

Let \( \lambda_1^{(i)} \) denote the \( i \)-th row of \( \lambda_1^{i} \). Then

(i) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(P_{\hat{F}_1(1)} - P_{F_1(1)})(\hat{F}_1(1) - F_1(1))\lambda_1^{i} = \mathbb{B}_{1 NT} + O_P(1) \),
(ii) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(\hat{F}_1(1) - F_1(1))' \hat{F}_1(1)^{-1} F_1^{*}(1)^{i} \epsilon_i = O_P(1 + (T/N)^{1/2}) \),
(iii) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(\hat{F}_1(1) - F_1(1))^{-1} (\hat{F}_1(1) - F_1(1))' \epsilon_i = \mathbb{B}_{3 NT} + O_P(\delta^{-1}_{NT} \Delta_1) \),
(iv) \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(\hat{F}_1(1) - F_1(1))' \epsilon_i = O_P(\delta^{-2}_{NT}) \) for \( i = 1,...,N \),

where \( \mathbb{B}_{1 NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(F_1^i)'(F_1^i)^{-1} \hat{F}_1(1)^{i} F_0^0 \lambda^0 \epsilon \epsilon F_1(1)^j \lambda_0^{i} \), \( \mathbb{B}_{3 NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(F_1^i)'(F_1^i)^{-1} \hat{F}_1(1)^{i} F_0^0 \lambda^0 \epsilon \epsilon F_1(1)^j \lambda_0^{i} \), \( b_{3 NT} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i'(1)(F_1^i)'(F_1^i)^{-1} \hat{F}_1(1)^{i} F_0^0 \lambda^0 \epsilon \epsilon F_1(1)^j \lambda_0^{i} = O_P(1) \).
B Proof of Theorem 3.1

Recall that $\lambda^*_s = H^+ \lambda^0$ and $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_N)' = \lambda^0 H^+$. Let $\beta = \beta^0 + T^{-1/2} b$ and $\lambda = \lambda^* + T^{-1/2} u$ where $b = (b_1, \ldots, b_K)'$ and $u = (u_1, \ldots, u_K)$ are matrices of dimensions $K \times 1$ and $N \times R$, respectively. Apparently, we use $u_r$ to denote the $r$th column of $u$ for $r = 1, 2, \ldots, R$. Let $u'_i$ denote the $i$th row of $u : u = (u_1, \ldots, u_N)'$. By (2.7),

$$T \left[ Q_\gamma (\beta, \lambda) - Q_\gamma (\beta^0, \lambda^*) \right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \left( Y_{it} - \beta^0 X_{it} - \lambda^*_s \hat{F}_i - T^{-1/2} b' X_{it} - T^{-1/2} u'_i \hat{F}_i \right)^2 - \left( Y_{it} - \beta^0 X_{it} - \lambda^*_s \hat{F}_i \right)^2 \right]$$

$$+ T_{\gamma 1 NT} \sum_{k=1}^{K} \frac{1}{|\beta_k^*|} \left( |\beta_k^0 + T^{-1/2} b_k| - |\beta_k^0| \right) + \frac{T_{\gamma 2 NT}}{\sqrt{N}} \sum_{r=1}^{R} \frac{1}{|\tau_r^*|^{1/2}} \left( \| \lambda^*_r + T^{-1/2} u_r \| - \| \lambda^*_r \| \right)$$

$$= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left( b' X_{it} + u'_i H' F_i^0 \right)^2 - \frac{2}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} \left( b' X_{it} + u'_i H' F_i^0 \right)^2$$

$$+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \left( b' X_{it} + u'_i \hat{F}_i \right)^2 - \left( b' X_{it} + u'_i H' F_i^0 \right)^2 \right]$$

$$- \frac{2}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \varepsilon_{it} \left( b' X_{it} + u'_i \hat{F}_i \right) - \varepsilon_{it} \left( b' X_{it} + u'_i H' F_i^0 \right) \right]$$

$$+ T_{\gamma 1 NT} \sum_{k=1}^{K} \frac{1}{|\beta_k^*|} \left( |\beta_k^0 + T^{-1/2} b_k| - |\beta_k^0| \right) + \frac{T_{\gamma 2 NT}}{\sqrt{N}} \sum_{r=1}^{R} \frac{1}{|\tau_r^*|^{1/2}} \left( \| \lambda^*_r + T^{-1/2} u_r \| - \| \lambda^*_r \| \right)$$

$$\equiv A_1 (b, u) - 2A_2 (b, u) + A_3 (b, u) - 2A_4 (b, u) + A_5 (b) + A_6 (u),$$

where $\varepsilon_{it} = Y_{it} - \beta^0 X_{it} - \lambda^*_s \hat{F}_i$. We want to show that $A_1$ and $A_2$ are $O_P (1)$ and $A_3$ and $A_4$ are $O_P (1)$, both uniformly in $(b, u)$ such that $\| b \| < L$ and $\| u \| < LN^{1/2}$ for any fixed positive constant $L$.

In addition, we will show that $A_5 (b)$ and $A_6 (u)$ are uniformly bounded from below by $O_P (1)$.

First, $A_2 (b, u) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} X_{it} + \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} F_i^0 H u_i = A_21 (b) + A_22 (u)$. Noting that $\| b \| < L$ and $(NT)^{-1/2} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} X_{it} = O_P (1)$ by Assumption A.1(vi), $A_2 (b, u) = O_P (N^{-1/2})$ uniformly in $b$. By the Cauchy-Schwarz inequality, Assumption A.1(vii), and the fact that $N^{-1/2} \| H u' \| < \| H \| L$, $|A_22 (u)| = \frac{1}{NT^{1/2}} \| \varepsilon F H u' \| < \frac{1}{\sqrt{N}} \| \varepsilon F \| \| H u' \| = O_P (1)$ uniformly in $u$. Thus $A_2 (b, u) = \frac{1}{NT^{1/2}} \| \varepsilon F H u' \| + O_P (1) = O_P (1)$ uniformly in $(b, u)$. Next, by the fact that $a_1^2 - a_2^2 = (a_1 - a_2)^2 + 2(a_1 - a_2) a_2$,

$$A_3 (b, u) = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} \left( F_i - H' F_i^0 \right)' \left( F_i - H' F_i^0 \right)' u_i + \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} \left( F_i - H' F_i^0 \right)' F_i^0 H' u_i$$

$$+ \frac{2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} b' X_{it} \left( F_i - H' F_i^0 \right)' u_i$$

$$\equiv A_{31} (u) + 2A_{32} (u) + 2A_{33} (b, u),$$

By the submultiplicative property of the Frobenius norm, the triangle inequality, and Lemmas A.2(i) and (iii), $|A_{31} (u)| \leq \frac{1}{N} \| \tilde{F} - F^0 H \| \| H \| \| u \|^2 = O_P \left( \frac{\delta_{NT}^2}{\sqrt{N}} \right), \text{ and } |A_{32} (u)| \leq \frac{1}{N} \| (\tilde{F} - F^0 H) F^0 H \| \| u \|^2 = O_P (\delta_{NT}^2), \text{ and } |A_{33} (b, u)| \leq \| b \| T^{-1/2} \| \tilde{F} - F^0 H \| \| H \| \sum_{i=1}^{N} \| X_i \| \| u_i \| = O_P (\delta_{NT}^{-1}).$ So $A_3 (b, u) = O_P (1)$ uniformly in $(b, u)$.
Let $\tilde{\varepsilon}_i \equiv (\tilde{\varepsilon}_{i1}, ..., \tilde{\varepsilon}_{iT})'$. In view of the fact that $\tilde{\varepsilon}_{it} - \varepsilon_{it} = -\lambda_i^* (\hat{F}_i - H'F^0_i)$, we have $\tilde{\varepsilon}_i - \varepsilon_i = -(\hat{F} - F^0 H)\lambda_i^*$ and

$$A_4(b, u) = b' \frac{1}{NT^{1/2}} \sum_{i=1}^{N} X'_i (\hat{F}_i - F^0 H)\lambda_i^* - \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \lambda_i^* (\hat{F} - F^0 H)\hat{u}_i + \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \varepsilon'_i (\hat{F} - F^0 H)\hat{u}_i$$

$$= -A_41(b) - A_{42}(u) + A_{43}(u), \text{ say.}$$

Using the fact that $\lambda_i^* = H^+ \lambda_i^0 = H' (HH')^{-1} \lambda_i^0$ and $H = (H_{(1)}, H_{(2)})$, we have

$$A_{41}(b) = b' \frac{1}{NT^{1/2}} \sum_{i=1}^{N} X'_i (\hat{F}_{(1)} - F^0 H_{(1)})H_{(1)}(HH')^{-1} \lambda_i^0$$

$$+ b' \frac{1}{NT^{1/2}} \sum_{i=1}^{N} X'_i (\hat{F}_{(2)} - F^0 H_{(2)})H_{(2)}(HH')^{-1} \lambda_i^0$$

$$= A_{41,1}(b) + A_{41,2}(b), \text{ say.}$$

Following the proof of Lemma A.3(iii) in Bai (2009), we can show that $\frac{1}{NT^{1/2}} \sum_{i=1}^{N} X'_i (\hat{F}_{(1)} - F^0 H_{(1)})H_{(1)}(HH')^{-1} \lambda_i^0 \times \lambda_i^0 = O_P(\delta_{N_T}^{-2})$. It follows that $A_{41,1}(b) = O_P\left(T^{1/2}\delta_{N_T}^{-2}\right)$ uniformly in $b$. For $A_{41,2}(b)$, we have by the triangle and Cauchy-Schwarz inequalities, and Lemmas A.2(ii)-(iii),

$$|A_{41,2}(b)| \leq \|b\| \frac{1}{NT^{1/2}} \left\| \hat{F}_{(2)} - F^0 H_{(2)} \right\| \left\{ \sum_{i=1}^{N} \|X_i\|^2 \sum_{i=1}^{N} \|\lambda_i^0\|^2 \right\}^{1/2} \|H_{(2)}\| \left\| (HH')^{-1} \right\|$$

$$= \frac{1}{NT^{1/2}} O_P\left(T^{1/2}\delta_{N_T}^{-1}\right) O_P\left(NT^{1/2}\right) O_P\left(\delta_{N_T}^{-1}\right) O_P(1) = O_P\left(T^{1/2}\delta_{N_T}^{-2}\right)$$

uniformly in $b$. Then $A_{41}(b) = O_P\left(T^{1/2}\delta_{N_T}^{-2}\right)$ uniformly in $b$. Note that $A_{42}(u) = \frac{1}{NT^{1/2}} \left[ \lambda^* (\hat{F} - F^0 H) F^0 H u' \right] + \frac{1}{NT^{1/2}} \left[ \lambda^* (\hat{F} - F^0 H) (\hat{F} - F^0 H) u' \right] \equiv A_{42,1}(u) + A_{42,2}(u), \text{ say.}$ Using $\lambda^* = \lambda^0 H^+$, we further decompose $A_{42,1}(u)$ as follows $A_{42,1}(u) = \frac{1}{NT^{1/2}} \left[ \lambda^0 (HH')^{-1} H_{(1)}(\hat{F}_i - F^0 H_{(1)})F^0 H u' \right] + \frac{1}{NT^{1/2}} \left[ \lambda^0 (HH')^{-1} H_{(2)}(\hat{F}_i - F^0 H_{(2)})F^0 H u' \right] \equiv A_{42,1a}(u) + A_{42,1b}(u), \text{ say.}$ By Cauchy-Schwarz inequality and Lemma A.2(iii), $|A_{42,1a}(u)| \leq \frac{1}{NT^{1/2}} \left\| \lambda_i^0 \right\| \|H_{(1)}\| \left\| (HH')^{-1} H_{(1)} \right\| \left\| \hat{F}_i - F^0 H_{(1)} \right\| \left\| F^0 H \right\| \left\| u' \right\| \leq \frac{1}{NT^{1/2}} O_P\left(N^{1/2}\right) O_P\left(T\delta_{N_T}^2\right) O_P\left(1 \right) = O_P\left(T^{1/2}\delta_{N_T}^{-2}\right)$ uniformly in $u$. Similarly, by Lemma A.2(iii), we can show that $A_{42,1b}(u) = O_P\left(T^{1/2}\delta_{N_T}^{-2} N^{-1/2}\right)$ uniformly in $u$. In addition, by Lemma A.2(i), we can readily show that $A_{42,2}(u) = O_P\left(\delta_{N_T}^{-1}\right)$ uniformly in $u$. It follows that $A_{42}(u) = O_P\left(1\right)$ uniformly in $u$. By the fact that $|\text{tr}(A_1)| \leq \text{rank}(A_1) \|A_1\|_{sp}$, $\|A_1 A_2\|_{sp} \leq \|A_1\|_{sp} \|A_2\|_{sp}$ and $\|A_1\|_{sp} \leq \|A_1\|_{sp}$, and Lemma A.2(i),

$$|A_{43}(u)| = \frac{\delta_{NT}}{NT} \|\varepsilon_{(2)}(\hat{F} - F^0 H)\| \leq \frac{R\delta_{NT}}{NT} \|\varepsilon_{(2)}\| \|\hat{F} - F^0 H\| \|u\| \leq \delta_{NT} N^{-1/2} T^{1/2} O_P\left(N^{1/2}\right) O_P\left(T^{1/2}\delta_{N_T}^{-1}\right) O_P\left(N^{1/2}\right) = O_P\left(\delta_{N_T}^{-1}\right)$$

uniformly in $u$. Consequently, $A_4(b, u) = O_P\left(1\right)$ uniformly in $(b, u)$.

Under Assumption A.3(ii) we can apply the triangle inequality to show that $A_5(b) \geq T\gamma_{1NT} \sum_{k=1}^{K_0} \frac{1}{|\rho_k|} \times (|\beta_{k}^{\delta_0} + T^{-1/2} b_k| - |\beta_{k}^{\delta_0}|) = O_P(T^{1/2}\gamma_{1NT}) = O_P\left(1\right)$ uniformly in $b$ and similarly $A_6(u) \geq O_P\left(1\right)$ uni-
formally in \( u \). It follows that \( T \left[ Q_\gamma (\beta, \lambda) - Q_\gamma (\beta^0, \lambda^*) \right] \geq \Pi_{NT} (b, u) + o_P (1) \) uniformly in \( (b, u) \), where

\[
\Pi_{NT} (b, u) = b' Ab + \frac{1}{\sqrt{N}} \text{tr} \left( \frac{u H' F_0 F_0 H u'}{\sqrt{T}} \right) + 2 - \frac{1}{\sqrt{N}} \sum_{i=1}^{T} \text{tr} \left( \frac{u H' F_0 F_0' X_i' \right) + \frac{1}{N \sqrt{T}} \text{tr} (e F_0 H u')
\]

\( A = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{t=1}^{T} X_i u X_i' \), and \( X_i \) is an \( N \times K \) matrix with the \( i \)th row given by \( X_i' \). Let \( B = \left( F_0 F_0' \right) \otimes I_N \), \( C = \frac{1}{\sqrt{N}} \sum_{i=1}^{T} F_0' \otimes X_i' \), \( D = \frac{1}{N \sqrt{T}} \varepsilon e F_0 \), and \( \theta = \text{vec}(u H') / \sqrt{N} \). Following Bai (2009, p.1256), we have

\[
\Pi_{NT} (b, u) = b' Ab + \eta' B \eta + 2 b' C' \eta + \eta' \text{vec} (D)
\]

\[
= b' (A - C' B^{-1} C) b + (\eta' + b' C' B^{-1}) B (\eta + B^{-1} C b) + \eta' \text{vec} (D)
\]

\[
= [b' Ab - b' C' B^{-1} \text{vec} (D)] + \eta' B \eta + \eta' \text{vec} (D)
\]

where \( \tilde{A} = A - C' B^{-1} C \), \( \tilde{\eta} = \eta + B^{-1} C b \), and the first equality follows from the fact that \( \text{tr} (B_1 B_2) = \text{vec}(B_2)' \times \text{vec}(B_1) \), \( \text{tr} (B_1 B_2 B_3) = \text{vec}(B_1)' (B_2 \otimes I) \text{vec}(B_3)' \), and \( \text{tr} (B_1 B_2 B_3 B_4) = \text{vec}(B_1)' (B_2 \otimes B_3') \text{vec}(B_4) \) for any conformable matrices \( B_1, B_2, B_3, \) and an identity matrix \( I \) (see, e.g., Bernstein (2005, p.247, and p.253)). Assumptions A.1(viii) and (ii) ensure the positive definiteness of \( \tilde{A} \) and \( B \). We can verify that \( \|C\| = O_P (1) \) and \( \|D\| = O_P (1) \). By allowing \( ||b|| \) and \( ||\tilde{\eta}|| \) to be sufficiently large, the linear terms \( b' C' B^{-1} \text{vec} (D) \) and \( \tilde{\eta}' \text{vec} (D) \) are dominated by the quadratic terms \( b' Ab \) and \( \tilde{\eta}' B \tilde{\eta} \), respectively; so \( \Pi_{NT} (b, u) \) cannot be minimized in this case. Letting \( \hat{b} = \sqrt{T} (\hat{\beta} - \beta^0) \), \( \hat{u} = \sqrt{T} (\hat{\lambda} - \lambda^*) \), \( \hat{\eta} = \text{vec}(\hat{u} H') / \sqrt{N} \) and \( \tilde{\eta} = \hat{\eta} + B^{-1} C \hat{b} \), this implies that either \( ||\hat{b}|| \) or \( ||\tilde{\eta}|| \) (or both) has to be stochastically bounded. We consider two cases: (a) \( ||\hat{b}|| \) is stochastically bounded, and (b) \( ||\tilde{\eta}|| \) is stochastically bounded.

Suppose (a) holds. Then \( \hat{\beta} - \beta^0 = O_P (T^{-1/2}) \) and the first part of the theorem follows. To prove the second part of the theorem, observe that

\[
0 \geq T \left[ Q_\gamma (\hat{\beta}, \hat{\lambda}) - Q_\gamma (\beta^0, \lambda^*) \right] \geq \Pi_{NT} (\hat{b}, \hat{u}) + o_P (1)
\]

\[
= \hat{b}' \hat{A} \hat{b} + \frac{1}{\sqrt{N}} \text{tr} \left( \frac{F_0 F_0' H \hat{u}' \hat{u} H'}{N} \right) + 2 \hat{b}' \frac{1}{\sqrt{N}} \sum_{t=1}^{T} X_i' \hat{u} H' F_0 + \frac{1}{N \sqrt{T}} \text{tr} (e F_0 H \hat{u}') + o_P (1)
\]

where the last line follows from the \( \sqrt{T} \)-consistency of \( \hat{\beta} \), the positive definiteness of \( A \), the analysis of \( A_{22} (u) \) above, and the fact that \( \frac{1}{\sqrt{N}} \sum_{t=1}^{T} X_i' u H' F_0 = O_P (1) \) uniformly in \( u \). It follows that \( \text{tr} \left( \frac{F_0 F_0' H \hat{u}' \hat{u} H'}{N} \right) = O_P (1) \), which further implies that \( \text{tr} \left( \frac{1}{N} \hat{u} H' \hat{u} H' \right) = \frac{1}{N} \|\hat{u} H'\|^2 = O_P (1) \) as \( \text{tr} (B_1 B_2) \geq \mu_{\min} (B_1) \text{tr} (B_2) \) for any positive semidefinite matrix \( B_2 \) and symmetric matrix \( B_1 \) (see, e.g., Bernstein (2005, p.275)) and \( T^{-1} F_0 F_0' \) is asymptotically nonsingular by Assumption A.1(iii). Noting that \( \hat{u} = \sqrt{T} (\hat{\lambda} - \lambda^0 H') \) and \( H H' = I_{R_0} \), the last result implies that \( \frac{1}{N} \|\hat{\lambda} H' - \lambda^0\|^2 = O_P (T^{-1}) \).

Now, suppose (b) holds. Note that

\[
0 \geq T \left[ Q_\gamma (\hat{\beta}, \hat{\lambda}) - Q_\gamma (\beta^0, \lambda^*) \right] \geq [\hat{b}' \hat{A} \hat{b} - \hat{b}' C' B^{-1} \text{vec} (D)] + [\tilde{\eta}' B \tilde{\eta} + \tilde{\eta}' \text{vec} (D)] + o_P (1)
\]

\[
= [\hat{b}' \hat{A} \hat{b} - \hat{b}' C' B^{-1} \text{vec} (D)] + o_P (1),
\]

which further implies that \( ||\hat{b}|| \) is stochastically bounded by the positive definiteness of \( \hat{A} \) and stochastic boundedness of \( ||C|| \), \( ||D|| \), and \( ||B^{-1}||_{sp} \). Then using arguments like those used in the study of case (a), we can further demonstrate that \( \frac{1}{N} \|\hat{\lambda} H' - \lambda^0\|^2 = O_P (T^{-1}) \).
C Proof of Theorem 3.2

Let \( \hat{\lambda}_r \) denotes the rth column of \( \hat{\lambda} \) for \( r = 1, \ldots, R \). We want to demonstrate that

\[
P\left( \left| \beta_k \right| = 0 \text{ and } \left\| \hat{\lambda}_r \right\| = 0 \text{ for } k = K^0 + 1, \ldots, K, \text{ and } r = R^0 + 1, \ldots, R \right) \to 1 \text{ as } (N, T) \to \infty.
\]

Suppose that to the contrary, \( \hat{\beta}_k \neq 0 \) for some \( k \in {K^0 + 1, \ldots, K} \) or \( \left\| \hat{\lambda}_r \right\| \neq 0 \) for some \( r \in \{R^0 + 1, \ldots, R\} \) for sufficiently large \((N, T)\). Wlog assume that \( \hat{\beta}_K \neq 0 \) or \( \left\| \hat{\lambda}_K \right\| \neq 0 \). If \( \hat{\beta}_K \neq 0 \), by the first order condition (FOC) with respect to \( \hat{\beta}_K \) for our minimization problem, we have

\[
0 = -\frac{2}{NT^{1/2}} \text{tr} \left[ X_k^T \left( Y - \sum_{k=1}^{K} \beta_k X_k - \hat{\lambda} \hat{F} \right) \right] + \frac{T^{1/2} \gamma_{1NT}}{\left\| \beta_K \right\|} \hat{\beta}_K \tag{C.1}
\]

\[
= -\frac{2}{NT^{1/2}} \text{tr} (X_k^T \varepsilon) + \frac{2}{NT} \sum_{k=1}^{K} \text{tr} (X_k^T X_k) \sqrt{T} (\hat{\beta}_k - \beta_0) 
\]

\[
+ \frac{2}{NT^{1/2}} (X_k^T \hat{\lambda} \hat{F} + \lambda^0 F^0) + \frac{T^{1/2} \gamma_{1NT}}{\left\| \beta_K \right\|} \hat{\beta}_K
\]

\[
\equiv -2B_1 + 2B_2 + 2B_3 + 2B_4, \text{ say.}
\]

Note that \( B_1 = O_P(N^{-1/2}) \) by Assumption A.1(iii) and \( B_2 = O_P(1) \) by Assumption A.1(iv) and Theorem 3.1. In addition, \( B_3 \leq \frac{1}{NT^{1/2}} \text{tr} \left[ X_k^T \left( \hat{\lambda} H' - \lambda^0 F^0 \right) \right] + \frac{1}{NT^{1/2}} \text{tr} \left[ X_k^T \hat{\lambda} \left( \hat{F} - F^0 H' \right) \right] \equiv B_{31} + B_{32} \).

By Assumptions A.1(ii) and (iv) and Theorem 3.1, \( B_{31} \leq (NT)^{-1/2} \| X_k \| \| \lambda H' - \lambda^0 \| \times T^{-1/2} \| F^0 \| = O_P(1) \). Similarly, \( B_{32} = O_P(1) \) by Assumption A.1(iv), Lemma A.2(iii) and the remark after Theorem 3.1. It follows that \( B_3 = O_P(1) - B_1 + B_2 + B_3 = O_P(1) \). Noting that \( \hat{\beta}^c = O_P((NT)^{-1/2}) \), \( \| B_4 \| = \frac{T^{1/2} \gamma_{1NT}}{\left\| \beta_K \right\|} \) is explosive in probability provided that \((NT)^{1/2} T^{1/2} \gamma_{1NT} \to \infty \).

This implies that (C.1) cannot be true for sufficiently large \( N \) and \( T \). Consequently, w.p.a.1 \( \hat{\beta}_K \) must be in a position where \( \left\| \beta_K \right\| \) is not differentiable, i.e., \( P(\left\| \beta_K \right\| = 0) \to 1 \) as \((N, T) \to \infty \).

Let \( \lambda_i, r \) and \( \hat{\lambda}_i, r \) denote the rth elements of \( \lambda_i \) and \( \hat{\lambda}_i \), respectively. If \( \left\| \hat{\lambda}_R \right\| \neq 0 \), the FOC for \( \lambda_i, R \) implies that for all \( i = 1, \ldots, N \), we have

\[
0 = -\frac{2}{T^{1/2}} \sum_{i=1}^{T} (\hat{F}_i R(Y_i - X_i \hat{\beta}_k - \hat{\lambda}_i \hat{F}_i)) + \frac{\sqrt{NT} \gamma_{2NT}}{\left\| \beta_K \right\|} \hat{\lambda}_i, R \tag{C.2}
\]

\[
= -\frac{2}{T^{1/2}} \sum_{i=1}^{T} \hat{F}_i R \varepsilon_{it} + \frac{2}{T} \sum_{i=1}^{T} \hat{F}_i R X_i \sqrt{T} (\hat{\beta} - \beta^0) 
\]

\[
+ \frac{2}{T^{1/2}} \sum_{i=1}^{T} \hat{F}_i R (\hat{\lambda}_i \hat{F}_i - \lambda_i^0 H^t H' F^0) + \frac{\sqrt{NT} \gamma_{2NT}}{\left\| \beta_K \right\|} \hat{\lambda}_i, R
\]

\[
\equiv 2d_{i1} + 2d_{i2} + 2d_{i3} + d_{i4}, \text{ say.}
\]

Observe that \( N^{-1} \sum_{i=1}^{N} (d_{i1} + d_{i2} + d_{i3})^2 \leq 3N^{-1} \sum_{i=1}^{N} (d_{i1}^2 + d_{i2}^2 + d_{i3}^2) \). By the \( C_r \) inequality, Lemma A.2(i), Assumption A.2(i), and we have

\[
N^{-1} \sum_{i=1}^{N} d_{i1}^2 \leq 2N^{-1} T^{-1} \sum_{i=1}^{N} \left( \sum_{t=1}^{T} (\hat{F}_i - H' F^0) R \varepsilon_{it} \right)^2 + 2N^{-1} T^{-1} \sum_{i=1}^{N} \left( H' F^0 \right) R \varepsilon_{it}^2
\]

\[
\leq 2N^{-1} T^{-1} \sum_{i=1}^{N} \left\| \varepsilon_{i} (\hat{F}_i R - F^0 H) \right\|^2 + 2N^{-1} T^{-1} \sum_{i=1}^{N} \left\| \varepsilon_{i} F^0 \right\|^2 \| H R \| \equiv D_{i1} + D_{i2},
\]

\[
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\]
where \((A)_R\) and \((B)_R\) denote the \(R\)th element of the \(R \times 1\) vector \(A\) and the \(R\)th column of matrix \(B\), respectively. \(D_{12} = O_P \left( \delta_{N \tau}^{-1} \right)\) by the fact that \(N^{-1} T^{-1} \sum_{i=1}^{N} \| e_i^T F^0 \|^2 = O_P (1)\) under Assumption A.1(viii) and (iv) and \(\| H_{R, R} \| = O_P \left( \delta_{N \tau}^{-1} \right)\) by Lemma A.2(ii). Using the decomposition for \(\hat{F} - F^0 H\) in the proof of Lemma A.2(i), we can readily show that \(D_{11} = O_P (1)\). It follows that

\[
N^{-1} \sum_{i=1}^{N} d_{2i}^2 = O_P (1)\]

By Theorem 3.1 and Lemma A.2(i), one can readily show that \(N^{-1} \sum_{i=1}^{N} d_{2i}^2 \leq T \| \hat{\beta}^0 - \beta^0 \|^2 N^{-1} \sum_{i=1}^{N} \left( \frac{1}{T} \sum_{t=1}^{T} \| \hat{F}_{t, R} X_{it} \| \right)^2 = O_P (1)\) where \(\hat{F}_{t, R}\) denotes the \(\hat{F}\). Noting that \(\sum_{i=1}^{T} \hat{F}_{t, R} (\hat{\lambda}_i - H^+ \lambda_i^0) + \hat{F}_{t, R} (\hat{F} - F^0 H) H^+ \lambda_i^0\), we have

\[
N^{-1} \sum_{i=1}^{N} d_{2i}^2 \leq 2 N^{-1} \sum_{i=1}^{N} \left[ \hat{F}'_{R} \hat{F} \left( \hat{\lambda}_i - H^+ \lambda_i^0 \right) \right]^2 + 2 N^{-1} \sum_{i=1}^{N} \left[ \hat{F}'_{R} \left( \hat{F} - F^0 H \right) H^+ \lambda_i^0 \right]^2
\]

\[
= 2 D_{31} + 2 D_{32}, \text{ say.}
\]

Noting that \(\hat{F}'_{R} \hat{F} = (\hat{F}_R - F^0 H R)' (\hat{F} - F^0 H) + (\hat{F}_R - F^0 H R)' F^0 H + H_{R, R} F^0 (\hat{F} - F^0 H) + H_{R, R} F^0 F^0 H\), we have the C_{r} inequality,

\[
D_{31} \leq 4 N^{-1} T^{-1} \sum_{i=1}^{N} \left[ (\hat{F}_R - F^0 H R)' (\hat{F} - F^0 H) \left( \hat{\lambda}_i - H^+ \lambda_i^0 \right) \right]^2
\]

\[
+ 4 N^{-1} T^{-1} \sum_{i=1}^{N} \left[ (\hat{F}_R - F^0 H R)' F^0 H \left( \hat{\lambda}_i - H^+ \lambda_i^0 \right) \right]^2
\]

\[
+ 4 N^{-1} T^{-1} \sum_{i=1}^{N} \left[ H_{R, R} F^0 \left( \hat{F} - F^0 H \right) \left( \hat{\lambda}_i - H^+ \lambda_i^0 \right) \right]^2 + 4 N^{-1} T^{-1} \sum_{i=1}^{N} \left[ H_{R, R} F^0 F^0 H \left( \hat{\lambda}_i - H^+ \lambda_i^0 \right) \right]^2
\]

\[
= 4 D_{31,1} + 4 D_{31,2} + 4 D_{31,3} + 4 D_{31,4}, \text{ say.}
\]

By Lemma A.2(i) and the remark after Theorem 3.1, \(D_{31,1} \leq T \{ T^{-1} \| \hat{F}_R - F^0 H R \| \} \{ T^{-1} \| \hat{F} - F^0 H \| \} \{ N^{-1} \| \hat{\lambda} - \lambda^0 H^+ \| \} = T O_P \left( \delta_{N \tau}^{-2} \right) o_P \left( \delta_{N \tau}^{-2} \right) o_P (1) = o_P (1)\). Similarly, by Lemma A.2, Theorem 3.1 and the remark after it, we have

\[
D_{31,2} \leq T^{-1} \left\| (\hat{F}_R - F^0 H R)' F^0 H \right\|^2 N^{-1} \sum_{i=1}^{N} \left( H \hat{\lambda}_i - \lambda_i^0 \right)^2 = O_P \left( T \delta_{N \tau}^{-2} \right) o_P (T^{-1})
\]

\[
D_{31,3} \leq \left\| H_{R, R} \right\|^2 T^{-1} \left\| F^0 \left( \hat{F} - F^0 H \right) \right\|^2 N^{-1} \sum_{i=1}^{N} \left\| \hat{\lambda}_i - H^+ \lambda_i^0 \right\|^2 = O_P \left( \delta_{N \tau}^{-2} \right) o_P \left( T \delta_{N \tau}^{-2} \right) o_P (1)
\]

\[
D_{31,4} \leq \left\| H_{R, R} \right\|^2 T^{-1} \left\| F^0 F^0 \right\|^2 N^{-1} \sum_{i=1}^{N} \left\| H \hat{\lambda}_i - \lambda_i^0 \right\|^2 = O_P \left( \delta_{N \tau}^{-2} \right) o_P (T) o_P (T^{-1})
\]

It follows that \(D_{31} = o_P (1)\). In addition, by the triangle inequality and Lemma A.2(iv), \(D_{32} \leq T^{-1} \| \hat{F}'_{R} (\hat{F} - F^0 H) \| \{ N^{-1} \sum_{i=1}^{N} \| \lambda_i^0 \| \} \| H^+ \| \| \lambda_i^0 \| = O_P \left( T \delta_{N \tau}^{-2} \right) o_P (1) = o_P (1)\). It follows that \(N^{-1} \sum_{i=1}^{N} d_{3i}^2 = o_P (1)\). Consequently we have shown that

\[
N^{-1} \sum_{i=1}^{N} (d_{31} + d_{32} + d_{33})^2 = O_P (1)
\]

Now, we study \(N^{-1} \sum_{i=1}^{N} d_{4i}^2\). In view of the fact that \(\tau_R = O_P \left( N^{-1/2} \right)\) by Lemma A.2(viii), \(N^{-1} \sum_{i=1}^{N} d_{4i}^2 = \left( \frac{\sqrt{T} \tau_{2N}}{\sqrt{\tau_{2N}}} \right)^2\) is explosive in probability provided that \(N^{\alpha/2} T^{1/2} \gamma_{2N_T} \rightarrow \infty\). This, in conjunction with (C.3), implies that \(N^{-1} \sum_{i=1}^{N} d_{4i}^2 \gg 4 N^{-1} \sum_{i=1}^{N} (d_{31} + d_{32} + d_{33})^2\) so that (C.2) cannot be true for all \(i\) for sufficiently large \(N\) and \(T\). Then we conclude that w.p.a.1, \(\| \hat{\lambda}_{R} \|\) must be in a position where \(\| \lambda_{R} \|\) is not differentiable. Consequently \(P(\| \hat{\lambda}_{R} \| = 0) \rightarrow 1\) as \((N, T) \rightarrow \infty\).
D Oracle property

Let $D(\tilde{\beta}_1, \tilde{\lambda}_1) = \left( \frac{\gamma_{1NT}}{2|\beta_1|}, \ldots, \frac{\gamma_{1NT}}{2|\beta_{K_0}|} \right), \tilde{D}_i(\tilde{\lambda}_1; \tilde{\lambda}_1) = \left( \frac{\sqrt{N\gamma_{2NT}}}{2\tau_1}, \frac{\lambda_1}{\|\lambda_1\|}, \ldots, \frac{\sqrt{N\gamma_{2NT}}}{2\tau_{K_0}}, \frac{\lambda_{1, R_0}}{\|\lambda_{1, R_0}\|} \right)$, $\tilde{D}(\tilde{\beta}_1; \tilde{\beta}_1, \tilde{\lambda}_1)$ where, e.g., $\tilde{\lambda}_1 = (\lambda_1, \ldots, \lambda_{R_0})$. W.p.a.1, $\tilde{\beta}_k/|\tilde{\beta}_k| = \tilde{\beta}_k^e/|\tilde{\beta}_k^e|$ for $k = 1, \ldots, K_0$ by the consistency of $\beta_k$ and $\tilde{\beta}_k^e$ and the fact that $\beta_k^0 \neq 0$ for $k = 1, \ldots, K_0$. So $D(\tilde{\beta}_1; \tilde{\beta}_1) = D(\tilde{\beta}_1^e)$ w.p.a.1. It follows from Theorem 3.2 and the FOCs, w.p.a.1 we have

$$0 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} X_{it}(1) \left[ Y_{it} - X_{it}'(1) \hat{\beta}_1 - \hat{F}'_{it}(1) \hat{\lambda}_1 \right] - D(\tilde{\beta}_1^e),$$

$$0 = \frac{1}{T} \sum_{i=1}^{T} \hat{F}_{i1}(1) \left[ Y_{it} - X_{it}'(1) \hat{\beta}_1 - \hat{F}'_{it}(1) \hat{\lambda}_1 \right] - \hat{D}_i(\hat{\lambda}_1) \text{ for } i = 1, \ldots, N.$$

Solving for $\hat{\beta}_1$ and $\hat{\lambda}_1$ yields

$$\hat{\beta}_1 = D^{-1}_{F_1} \left[ \frac{1}{NT} \sum_{i=1}^{N} X_{i1}'(1) M_{F_1} Y_i - D(\tilde{\beta}_1^e) + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} X_{it}(1) \hat{F}'_{it} \tilde{\Sigma}^{-1}_{F_1} \hat{D}_i(\hat{\lambda}_1) \right],$$

$$\hat{\lambda}_i = \tilde{\Sigma}^{-1}_{F_1} \left[ \frac{1}{T} \sum_{i=1}^{T} \hat{F}_{i1}(1) \left( Y_{it} - X_{it}'(1) \hat{\beta}_1 \right) - \hat{D}_i(\hat{\lambda}_1) \right] \text{ for } i = 1, 2, \ldots, N,$$

where $\hat{D}_{F_1} = \frac{1}{NT} \sum_{i=1}^{N} X_{i1}'(1) M_{F_1} X_{i1}$ and $\tilde{\Sigma}_{F_1} = \frac{1}{N} \sum_{i=1}^{N} ||X_{i1}(1)||^2 ||P_{F_1} - P_{F^*}|| = O_P(\delta_{NT}^4)$. By Lemma A.2(vii), $||\hat{D}_{F_1} - D_{F_1}|| \leq \frac{1}{NT} \sum_{i=1}^{N} ||X_{it}(1)||^2 ||P_{F_1} - P_{F^*}|| = O_P(\delta_{NT}^4)$. In view of the fact that $M_{F^*} = M_{F^0}$, we have $\hat{D}_{F_1} = D_{F_1} + o_P(1)$. By Assumption A.4(i). It follows that $\hat{D}_{F_1} = D_{F_1} + o_P(1)$. By Assumption A.6 and the fact that $||\hat{\lambda}_i||^2 = O_P(N^{-1/2})$ for $r = 1, ..., R_0$, $D(\tilde{\beta}_1^e) = O_P(\gamma_{1NT}) = O_P((NT)^{-1/2})$ and $\hat{D}_i(\hat{\lambda}_1) = O_P(\gamma_{2NT}) = o_P((NT)^{-1/2})$. Consequently, we have

$$\hat{\beta}_1 = D^{-1}_{F_1} \frac{1}{NT} \sum_{i=1}^{N} X_{i1}'(1) M_{F_1} Y_i + o_P((NT)^{-1/2})$$

and

$$\hat{\lambda}_i = \tilde{\Sigma}^{-1}_{F_1} \frac{1}{T} \sum_{i=1}^{T} \hat{F}_{i1}(1) \left( Y_{it} - X_{it}'(1) \hat{\beta}_1 \right) + o_P((NT)^{-1/2}) \text{ for } i = 1, 2, \ldots, N.$$

Below we study the asymptotic distributions of $\hat{\beta}_1$ and $\hat{\lambda}_1$ in turn.

Asymptotic distribution of $\hat{\beta}_1$. Noting that $Y_i = \hat{F}_{i1}(1) \lambda^*_i + X_{i1}(1) \beta^0_i + \varepsilon_i + (F^*_i - \hat{F}_{i1}(1)) \lambda^*_i$, we have

$$\sqrt{NT} \left( \hat{\beta}_1 - \beta^0_1 \right) = D^{-1}_{F_1} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_{i1}'(1) M_{F^0} \varepsilon_i + D^{-1}_{F_1} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_{i1}'(1) M_{F_1}(F^*_i - \hat{F}_{i1}(1)) \lambda^*_i$$

$$+ D^{-1}_{F_1} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_{i1}'(1) \left( M_{F_1} - M_{F^*} \right) \varepsilon_i + o_P(1)$$

$$= S_{1NT} + S_{2NT} + S_{3NT} + o_P(1), \text{ say.}$$

By Propositions D.1-D.2 below and the fact that $D^{-1}_{F_1} = D^{-1}_{F_1} + o_P(1)$, we have $S_{2NT} = D^{-1}_{F_1} (B_{1NT} - B_{2NT} - V_{1NT} + V_{2NT}) + o_P(1)$, and $S_{3NT} = -D^{-1}_{F_1} B_{3NT} + o_P(1)$, where $B_{1NT} = O_P(\sqrt{T/N})$, $B_{2NT} = \ldots$.

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\( O_p(1 + \sqrt{NT}) \), \( E_3NT = O_p(\sqrt{T/N}) \), and \( V_{1NT} = O_p(1) \) for \( l = 1, 2 \). Consequently,
\[
\sqrt{NT} \left( \tilde{\beta}_{(1)} - \beta^0_{(1)} \right) = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) M_{F_0} e_i - V_{1NT} + V_{2NT} + o_p(1).
\]

Following Moon and Weidner (2010a, 2010b), we can show that \( \sqrt{NT}(\tilde{\beta} - \beta^0) = W^{-1}_{NT} V_{NT} + o_p(1) \), where \( W_{NT} = \frac{1}{N} \sum_{i=1}^{N} X'_i M_{F_0} X_i \), \( X_i = X_i - \frac{1}{N} \sum_{j=1}^{N} \lambda_i^0 N^{-1} \lambda_i^0 \lambda_j^0 X_j \), \( V_{NT} = \frac{1}{N} \sum_{i=1}^{N} [X_i - P_{F_0} E_D(X_i) - M_{F_0} X_{2NT}] \), and \( X_{2NT} = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^0 (N^{-1} \lambda_i^0 \lambda_j^0) X_j \). It follows that \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i M_{F_0} e_i - V_{1NT} + V_{2NT} = J_{NT} + o_p(1) \), where \( J_{NT} \) is defined in Remark 4. In the presence of lagged dependent variables, \( J_{NT} \) does not center around 0 asymptotically and it contributes to both the asymptotic bias and variance of \( \tilde{\beta}_{(1)} \). We make the following decomposition: \( J_{NT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z'_i e_i - B_{4NT} \), where \( Z_i = X_{i(1)} - P_{F_0} E_D(X_{i(1)}) - M_{F_0} X_{1NT} + [X_i - P_{F_0} E_D(X_i) - M_{F_0} X_{2NT}] W^{-1}_{NT} C_{NT} \) and \( B_{4NT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} [X_i(1) - E_D(X_{i(1)})] P_{F_0} e_i \). Then
\[
\sqrt{NT} \left( \tilde{\beta}_{(1)} - \beta^0_{(1)} \right) = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z'_i e_i + o_p(1) \]

It is easy to show that \( D^{-1}_{F_0} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z'_i e_i = D^{-1}_{F_0} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} Z'_i e_i + o_p(1) \) by Assumptions A.4(ii) and A.5(i).

**Asymptotic distribution of \( \tilde{\lambda}_{i(1)} \).** Noting that \( Y_i - X'_{i(1)} \tilde{\beta}_{(1)} = \hat{F}_i(1) \lambda^*_{i(1)} + e_i - X_{i(1)}(\tilde{\beta}_{(1)} - \beta^0_{(1)}) + (\hat{F}'_{(1)} - \hat{F}'_{(1)}) \lambda^*_{i(1)} \), we have
\[
\sqrt{T} \left( \tilde{\lambda}_{i(1)} - \lambda^*_{i(1)} \right) = \frac{\hat{\Sigma}^{-1}_{(1)}(1)}{\sqrt{T}} \sum_{t=1}^{T} \hat{F}_t(1) e_{it} - \tilde{\Sigma}^{-1}_{(1)}(1) \sum_{t=1}^{T} \hat{F}_t(1) X'_{i(1)} \left( \tilde{\beta}_{(1)} - \beta^0_{(1)} \right) + \frac{\hat{\Sigma}^{-1}_{(1)}(1)}{\sqrt{T}} \sum_{t=1}^{T} \hat{F}_t(1) \left( \hat{F}'_{(1)} - \hat{F}'_{(1)} \right)^{i} \lambda^*_{i(1)} + o_p \left( N^{-1/2} \right), \text{ say.}
\]

By Lemmas A.2(v) and A.1(iv), \( \hat{\Sigma}^{-1}_{(1)} = \frac{1}{T} \sum_{t=1}^{T} \hat{F}_t(1) X'_{i(1)} \hat{F}_t(1) + o_p \left( N^{-1/2} \right) \). By Lemma A.2(i) and Assumptions A.1(ii) and (iv), \( \frac{1}{T} \sum_{t=1}^{T} \hat{F}_t(1) X'_{i(1)} = H'(1) T \sum_{t=1}^{T} \hat{F}_t(1) X'_{i(1)} + o_p \left( N^{-1/2} \right) \). Let \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i M_{F_0} e_i \) be the kth element corresponds to \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i M_{F_0} e_i \) defined in Moon and Weidner (2010b, Theorem 3.1) which contributes both to the asymptotic bias and variance. Without their Assumption 5, we can make the following decomposition:
\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i M_{F_0} e_i = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i P_{F_0} e_i
\]

where the first term is \( V_{NT} \) and contributes to the asymptotic variance and the second term can be corrected as in Moon and Weidner (2010b); see also the proof of Corollary 3.4. In the absence of lagged dependent variables, one can replace \( V_{NT} \) by \( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \) as in Bai (2009).
$O_P(1)$. By the study of the asymptotic distribution of $\hat{\beta}_i(1)$, $\hat{\beta}_i(1) - \beta_0(1) = O_P(\delta^{-2}_{NT})$. It follows that $e_{2i} = O_P(T^{1/2}\delta^{-2}_{NT}) = o_P(1)$. By Lemma A.2(iv), $|e_{2i}| \leq \sqrt{T} ||\hat{\beta}_i(1) - \beta_0(1)|| ||\hat{F}_i(1)|| ||\hat{H}_i(1)|| \leq \sqrt{T} O_P(1) O_P(\delta^{-2}_{NT}) O_P(1) = o_P(1)$. In addition, by Lemma A.3(iv) and Assumption A.5(ii), $\frac{1}{\sqrt{T}} \sum_{i=1}^{T} \hat{F}_i \hat{e}_{i} = \frac{1}{\sqrt{T}} \sum_{i=1}^{T} F^0_i \hat{e}_{i} + o_P(1) \Rightarrow N(0, \Theta_i F_i)$. It follows that $\sqrt{T} (\hat{\lambda}_i(1) - \lambda_i(1)) \Rightarrow N(0, \sum_i \Theta_i F_i \sum_i F_i^{-1})$ by Slutsky theorem. □

**Proposition D.1** Suppose that the conditions of Theorem 3.3 hold. Then $s_{NT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_i^\prime (M F_i - M \hat{F}_i) \lambda_i(1) = \mathbb{B}_{1NT} - \mathbb{B}_{2NT} - \mathbb{V}_{1NT} + \mathbb{V}_{2NT} + o_P(1)$ where $\mathbb{B}_{1NT}, \mathbb{B}_{2NT}, \mathbb{V}_{1NT}$, and $\mathbb{V}_{2NT}$ are defined in Section 3.3.

**Proof.** Note that $s_{NT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_i^\prime (M F_i - M \hat{F}_i) \lambda_i(1) = \mathbb{B}_{1NT} + \mathbb{B}_{2NT} + o_P(1)$. By Lemma A.3(i), $s_{NT} = \mathbb{B}_{1NT} + o_P(1)$. By (F.2) in the supplementary appendix and the fact that $M F_0 = M F_i$, $s_{NT} = - \sum_{i=1}^{N} X_i^\prime (M F_0 a_i(1) \lambda_i(1)) = - \sum_{i=1}^{N} s_{2NT,i}$. Say. We dispense with the terms that are easy to analyze first. First, $s_{2NT,2} = s_{2NT,5} = 0$ as $M F_0 F_0 = 0$. Next, we want to show that $s_{2NT,l} = o_P(1)$ for $l = 4, 7, and 8$. By the triangle inequality and Assumptions A.1(i), (iii) and (iv),

$$\|s_{2NT,4}\| = (NT)^{-1/2} \left\| \sum_{i=1}^{N} X_i^\prime s_{i} \left( \beta_i(0) - \beta_i(0) \right) M F_0 X_i \sum_{i=1}^{K} \left( \beta_i(0) - \beta_i(0) \right) X_i \hat{F}(1) \right\| \leq (NT)^{-1/2} \left\| \beta - \beta \right\| \left\| \mathcal{X}_i \right\| \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \leq \left\| \beta - \beta \right\| (NT)^{-1/2} O_P \left( \left\| \mathcal{X}_i \right\| \right) \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \leq \left\| \beta - \beta \right\| (NT)^{-1/2} O_P \left( \left\| \mathcal{X}_i \right\| \right) \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\|$$

Let $c_{K_0} = (c_1, ..., c_{K_0})^\prime$ be an arbitrary $K_0 \times 1$ nonrandom vector with $\|c_{K_0}\| = 1$. Note that

$$|c_{K_0} s_{2NT,7}| = (NT)^{-1/2} \left\| \sum_{i=1}^{K} \left( \beta_i(0) - \beta_i(0) \right) \mathcal{X}_i \hat{F}(1) \right\| \leq R_0 (NT)^{-1/2} \left\| \beta - \beta \right\| \left\| \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\|$$

Let $c_{K_0} s_{2NT,8}$ be an arbitrary $K_0 \times 1$ nonrandom vector with $\|c_{K_0}\| = 1$. Note that

$$\|s_{2NT,8}\| = (NT)^{-1/2} \left\| \sum_{i=1}^{N} X_i^\prime \left( \beta_i(0) - \beta_i(0) \right) M F_0 X_i \hat{F}(1) \right\| \leq R_0 (NT)^{-1/2} \left\| \beta - \beta \right\| \left\| \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\|$$

where we use the fact that $|\text{tr}(A)| \leq \text{rank}(A) \|A\|_{sp}$, the submultiplicative property of the spectral norm, the fact that $\|M F_0\|_{sp} = 1$, and Assumptions A.1(i) and (iii)-(v). Similarly, by the fact that $\|A\|_{sp} \leq \|A\|_{sp}$ and Assumptions A.1(i) and (iii)-(v),

$$\|s_{2NT,8}\| = (NT)^{-1/2} \left\| \sum_{i=1}^{N} X_i^\prime \left( \beta_i(0) - \beta_i(0) \right) M F_0 X_i \hat{F}(1) \right\| \leq R_0 (NT)^{-1/2} \left\| \beta - \beta \right\| \left\| \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\|$$

Let $c_{K_0} s_{2NT,8}$ be an arbitrary $K_0 \times 1$ nonrandom vector with $\|c_{K_0}\| = 1$. Note that

$$\|s_{2NT,8}\| = (NT)^{-1/2} \left\| \sum_{i=1}^{N} X_i^\prime \left( \beta_i(0) - \beta_i(0) \right) M F_0 X_i \hat{F}(1) \right\| \leq R_0 (NT)^{-1/2} \left\| \beta - \beta \right\| \left\| \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\| \left\| \lambda_i(1) \right\| \left\| \sum_{i=1}^{K} \mathcal{X}_i \right\| \left\| \hat{F}(1) \right\|$$

which completes the proof. □
For $s_{2NT,1}$, $s_{2NT,3}$, and $s_{2NT,6}$, we have $s_{2NT,1} = (NT)^{-3/2} \sum_{i=1}^{N} X'_i(1) M_{F_0} \epsilon' \epsilon \hat{F}(1) \lambda^*_i(1) = \mathbb{B}_{2NT}$, $s_{2NT,3} = (NT)^{-1/2} \sum_{i=1}^{N} \left( \frac{1}{NT} \sum_{j=1}^{N} \lambda_j^0 F_{0j} \hat{F}(1) \lambda^*_j(1) X'_i(1) \right) M_{F_0} \epsilon_i \equiv \mathbb{V}_{1NT}$, and

$$- s_{2NT,6} = (NT)^{-3/2} \sum_{i=1}^{N} X'_i(1) \sum_{k=1}^{K} \left( \hat{\beta}_k^c - \beta_k^0 \right) M_{F_0} X_k^0 \lambda^0 F_{0k} \hat{F}(1) \lambda^*_k(1)$$

$$= \left( (NT)^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_j^0 F_{0j} \hat{F}(1) \lambda^*_j(1) X'_i(1) M_{F_0} X_j \right) \sqrt{NT}(\hat{\beta}^c - \beta^0)$$

$$= C_{NT} \sqrt{NT}(\hat{\beta}^c - \beta^0) \equiv \mathbb{V}_{2NT}.$$

Combining the above results yields the conclusion.

**Proposition D.2** Suppose that the conditions of Theorem 3.3 hold. Then $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) (M_{\hat{F}(1)} - M_{F_0}) \epsilon_i = - \mathbb{B}_{3NT} + o_P(1)$ where $\mathbb{B}_{3NT}$ is defined in Section 3.3.

**Proof.** Using (F.3) in the supplementary appendix, we have $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) (M_{\hat{F}(1)} - M_{F_0}) \epsilon_i = \sum_{i=1}^{N} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) b_i \epsilon_i = \sum_{i=1}^{N} B_{1NT}$, say. Let $c_{K_0} = (c_{1K_0}, \ldots, c_{K_0K_0})'$ be an arbitrary $K_0 \times 1$ nonrandom vector with $\|c_{K_0}\| = 1$. Then by the facts that $|\text{tr}(A)| \leq \text{rank}(A) \|A\|_F$ and $\|A\|_F \leq \|A\|$, the submultiplicative property of the spectral norm, Lemmas A.2(i) and (v), and Assumptions A.1(iv)-(v) and A.3(i)

$$|c_{K_0} B_{1NT}| = \frac{1}{\sqrt{NT}} \text{tr} \left( \left( \hat{F}(1) - F_0(1) \right) \left( \hat{F}(1) \hat{F}(1) \right)^{-1} \left( \hat{F}(1) - F_0(1) \right)^t \sum_{k=1}^{K_0} c_{K_0} \epsilon' X_k \right)$$

$$\leq \frac{K_0}{\sqrt{NT}} \left\| \hat{F}(1) \hat{F}(1) \right\|^{-1} \left\| \hat{F}(1) - F_0(1) \right\|^2 \|\epsilon\|_F \|c_{K_0}\| \sum_{k=1}^{K_0} \|X_k\|$$

$$= (NT)^{-1/2} o_P(T^{-1}) O_P(T^{3/2} \sigma_{NT}^{-2}) O_P \left( N^{1/2} + T^{1/2} \right) = o_P(1).$$

By Lemma A.3(ii), $B_{2NT} = o_P(1)$. By Lemma A.3(iii), $B_{3NT} = \mathbb{B}_{3NT} + o_P(1)$. Lastly, by Lemma A.2(v) and Assumptions A.1(ii), (iv) and (vii)

$$\|B_{4NT}\| \leq N^{-1/2} T^{-3/2} \left\| (T^{-1} \hat{F}(1) \hat{F}(1))^{-1} \right\| \left\| (T^{-1} F_0^t F_0(1))^{-1} \right\| \left\| F_0(1) \right\| \sum_{i=1}^{N} \|X'_i(1)\| \|F_0^t(1)\|$$

$$= N^{-1/2} T^{-3/2} O_P \left( \sigma_{NT}^{-2} \right) O_P \left( T^{1/2} \right) O_P(NT) = O_P \left( N^{1/2} \sigma_{NT}^{-2} \right) = o_P(1).$$

Consequently we have $\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_i(1) (M_{\hat{F}(1)} - M_{F_0}) \epsilon_i = - \mathbb{B}_{3NT} + o_P(1)$.

**E** Selection of the tuning parameters

By Theorem 3.2, we know that the shrinkage estimation based on $\gamma_{NT}$ can correctly select all relevant covariates and factors and shrink the coefficients of irrelevant covariates and factors to 0 w.p.a.1. This implies that $\gamma_{NT} \in \Omega_0$ and w.p.a.1

$$IC \left( \gamma_{NT} \right) = \hat{\sigma}^2 \left( \gamma_{NT} \right) + \rho_1 NT \left| S_{\beta} \left( \gamma_{NT} \right) \right| + \rho_{2NT} N \left| S_{\lambda} \left( \gamma_{NT} \right) \right| = \hat{\sigma}^2 S + o_P(1) \Rightarrow \sigma^2 S,$$
where the second equality holds by Theorem 3.2 and Assumption A.8, and the last convergence holds by Assumption A.7. We consider the cases of under- and over-fitted models separately.

**Case 1: Under-fitted model.** In this case, we have either \( S_\beta (\gamma) \not\supseteq S_{T,\beta} \) or \( S_\lambda (\gamma) \not\supseteq S_{T,\lambda} \). Noting that 
\[
\hat{\sigma}^2 (\gamma) \geq \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (Y_{it} - \hat{\beta}'_{S_\beta(\gamma)} X_{it,S_\beta(\gamma)} - \hat{\lambda}'_{S_\lambda(\gamma)} F_{i,t,S_\lambda(\gamma)})^2 = \tilde{\sigma}^2_S (\gamma),
\]
we have by Assumption A.7
\[
IC (\gamma) = \hat{\sigma}^2 (\gamma) + \rho_{1NT} |S_\beta (\gamma)| + \rho_{2NT} N |S_\lambda (\gamma)| > \tilde{\sigma}^2_S (\gamma)
\]
\[
\geq \min \left\{ \sigma^2_S \left\{ S : S_\beta \not\subseteq S_{T,\beta} \lor S_\lambda \not\subseteq S_{T,\lambda} \right\} \right\} \sigma^2_S > \sigma^2_{ST}.
\]
It follows that \( P (\inf_{\gamma \in \Omega_+} IC (\gamma) > IC (\gamma^0_{NT})) \rightarrow 1. \)

**Case 2: Over-fitted model.** Let \( S = \{ S_\beta \times S_\lambda : S_\beta \supseteq S_{T,\beta}, S_\lambda \supseteq S_{T,\lambda}, |S_\beta| + |S_\lambda| > |S_{T,\beta}| + |S_{T,\lambda}| \} \). Let \( \gamma \in \Omega_+ \) such that \( S (\gamma) = S_\beta (\gamma) \times S_\lambda (\gamma) \in S \). Let 
\[
L_{NT,S} (\beta_{S_\beta, S_\lambda}) = \frac{1}{NT} \sum_{i=1}^N \left\| Y_i - X_{i,S_\beta, S_\lambda} \beta_{S_\beta, S_\lambda} - \hat{F}_{S_\lambda} \lambda_{S_\lambda} \right\|^2,
\]
where \( S = S_\beta \times S_\lambda, \lambda_{S_\lambda} = \{ \lambda_{1,S_\lambda}, \ldots, \lambda_{N,S_\lambda} \} \), \( X_{i,S_\beta} \) denotes the \( T \times |S_\beta| \) submatrix of \( X_i \) with column indices given by \( S_\beta \), and \( \hat{F}_{S_\lambda} \) is analogously defined. By the definition of \( \tilde{\sigma}^2_S \) in (3.4), we have \( \tilde{\sigma}^2_S = L_{NT,S} (\beta_{S_\beta, S_\lambda}) \). In view of the facts that \( P (\hat{\sigma}^2 (\gamma^0_{NT}) = \tilde{\sigma}^2_{ST}) \rightarrow 1 \) as \( (N, T) \rightarrow \infty \) by Theorem 3.2 and that \( \tilde{\sigma}^2 (\gamma) \geq \tilde{\sigma}^2_S (\gamma) \), we have that w.p.a.1

\[
\delta^2_{NT} [IC (\gamma) - IC (\gamma^0_{NT})] = \delta^2_{NT} [\hat{\sigma}^2 (\gamma) - \sigma^2 (\gamma^0_{NT})] + \delta^2_{NT} [\sigma^2 (\gamma^0_{NT}) - \sigma^2_S (\gamma^0_{NT})] + \delta^2_{NT} [\sigma^2_S (\gamma^0_{NT}) - \sigma^2_S (\gamma^0_{NT})] + \delta^2_{NT} [\rho_{1NT} (|S_\beta (\gamma)| - K_0) + \rho_{2NT} N (|S_\lambda (\gamma)| - R_0)]
\]

By Assumption A.8, \( \delta^2_{NT} \rho_{1NT} \rightarrow \infty \) and \( \delta^2_{NT} \rho_{2NT} N \rightarrow \infty \). In addition, for any \( S = S_\beta \times S_\lambda \in S \) we have \( \max(|S_\beta| - K_0, |S_\lambda| - R_0) \geq 1 \) and \( \min(|S_\beta| - K_0, |S_\lambda| - R_0) \geq 0 \). Then by Lemma E.1 below

\[
P \left( \inf_{\gamma \in \Omega_+} IC (\gamma) > IC (\gamma^0_{NT}) \right) = P \left( \inf_{\gamma \in \Omega_+} \delta^2_{NT} [IC (\gamma) - IC (\gamma^0_{NT})] > 0 \right)
\]

\[
\geq P \left( \min_{S = S_\beta \times S_\lambda \in S} \delta^2_{NT} [\hat{\sigma}^2 - \sigma^2_S] + \delta^2_{NT} [\rho_{1NT} (|S_\beta| - K_0) + \rho_{2NT} N (|S_\lambda| - R_0)] > 0 \right)
\]

\[
\rightarrow 1 \text{ as } (N, T) \rightarrow \infty. \]

**Lemma E.1** Suppose that the conditions in Theorem 3.5 hold. Suppose that \( S = \{ S_\beta \times S_\lambda : S_\beta \supseteq S_{T,\beta}, S_\lambda \supseteq S_{T,\lambda}, |S_\beta| + |S_\lambda| > |S_{T,\beta}| + |S_{T,\lambda}| \} \). Then \( \delta^2_S - \hat{\delta}^2_S = O_P (\delta^2_{NT}). \)

**Proof.** Consider the minimization of the following objective function

\[
\tilde{L}_{NT,S_T} (\beta_{S_{T,\beta}, S_{T,\lambda}}) = \frac{1}{NT} \sum_{i=1}^N \left\| Y_i - X_{i,S_{T,\beta}, S_{T,\lambda}} \beta_{S_{T,\beta}, S_{T,\lambda}} - \hat{F}_{S_T} \lambda_{S_T} \right\|^2,
\]
where we pretend that the factors are observed. Let \( \beta_{S_{T,\beta}} \) and \( \lambda_{S_{T,\lambda}} \) denote the OLS solution to the above problem. Let \( \hat{\beta}_{S_{T,\beta}} \) denote the OLS solution to the above problem. Let \( \hat{\sigma}^2_{ST} = \tilde{L}_{NT,S_T} (\beta_{S_{T,\beta}}, \lambda_{S_{T,\lambda}}) \). Straightforward algebra shows that

\[
\hat{\sigma}^2_{ST} = \frac{1}{NT} \sum_{i=1}^N \left( Y_i - X_{i,S_{T,\beta}} \hat{\beta}_{S_{T,\beta}} \right)' M F_0 \left( Y_i - X_{i,S_{T,\beta}} \hat{\beta}_{S_{T,\beta}} \right)
\]

(3.3)
and
\[ \tilde{\beta}_{S_{T,\beta}} - \beta_{0S_{T,\beta}} = O_P \left( (NT)^{-1/2} \right). \] (E.4)

Noting that \(|\sigma_{S}^{2} - \sigma_{S_{T}}^{2}| \leq |\sigma_{S}^{2} - \sigma_{S_{T}}^{2}| + |\sigma_{S_{T}}^{2} - \sigma_{S_{T}}^{2}| \leq 2 \max_{S_{T}, S_{T} \supset S_{T}} |\sigma_{S}^{2} - \sigma_{S_{T}}^{2}| \), it suffices to prove that for each \( S \in \mathbb{S} \subseteq \mathbb{S} \cup S_{T} \) with \( S_{T} = S_{T,\beta} \times S_{T,\lambda} \), we have \( \tilde{\sigma}_{S}^{2} - \sigma_{S}^{2} = O_P(\delta^{-2}_{NT}) \).

Now, fix \( S \in \mathbb{S} \). We consider the minimization of the least square objective function in (E.1). Noting that \( T^{-1} \hat{F}_{S_{T}}^{\prime} \hat{F}_{S_{T}} \) is asymptotically singular when \( |S| \geq R_{0} \), the OLS estimate \( \hat{\lambda}_{S_{T}} \) is not necessarily unique whereas \( \hat{\beta}_{S_{T,\lambda}} \) is. Despite this, the minimum of \( L_{NT,S}(\beta_{S_{T,\beta}}, \lambda_{S_{T,\lambda}}) \) is uniquely determined. Standard algebra shows that \( \hat{\beta}_{S_{T,\beta}} = \left( \frac{1}{N} \sum_{i=1}^{N} X_{i,S_{T}}^{\prime} M_{F_{S_{T}}} X_{i,S_{T}} \right)^{-1} \frac{1}{N} \sum_{i=1}^{N} X_{i,S_{T}}^{\prime} M_{F_{S_{T}}} Y_{i} \), and \( \hat{\lambda}_{S_{T,\lambda}} = (\hat{F}_{S_{T}}^{\prime} \hat{F}_{S_{T}})^{-1} \hat{F}_{S_{T}}^{\prime} (Y_{i} - X_{i,S_{T}} \hat{\beta}_{S_{T,\beta}}) \), where \( A^{\prime} \) is any generalized inverse of \( A \) and \( M_{F_{S_{T}}} = I_{T} - \hat{F}_{S_{T}}^{\prime} (\hat{F}_{S_{T}}^{\prime} \hat{F}_{S_{T}})^{-1} \hat{F}_{S_{T}} \). It follows that
\[ \tilde{\sigma}_{S}^{2} = \frac{1}{NT} L_{NT,S}(\beta_{S_{T,\beta}}, \lambda_{S_{T,\lambda}}) = \frac{1}{NT} \sum_{i=1}^{N} \left( Y_{i} - X_{i,S_{T}} \beta_{S_{T,\beta}} \right)^{\prime} M_{F_{S_{T}}} \left( Y_{i} - X_{i,S_{T}} \beta_{S_{T,\beta}} \right). \] (E.5)

In addition, using Lemma A.3 we can readily show that
\[ \hat{\beta}_{S_{T,\beta}} - \beta_{0S_{T,\beta}} = O_P \left( \delta^{-2}_{NT} \right). \] (E.6)

Recall \( F^{\ast} = F^{0} H \) and \( \lambda_{S_{T}}^{\ast} = H^{\ast} \lambda_{S_{T}} \). Define \( F_{S}^{\ast} \) as an \( N \times |S_{T}| \) submatrix of \( F^{\ast} \) whose column indices are given in \( S_{T} \). Similarly define \( \lambda_{S_{T}}^{\ast} \). Noting that \( Y_{i} - X_{i,S_{T}} \hat{\beta}_{S_{T,\beta}} = (X_{i,S_{T}} \hat{\beta}_{S_{T,\beta}} + \varepsilon_{i}) - X_{i,S_{T}} \beta_{S_{T,\beta}} = \hat{F}_{S_{T}} \lambda_{S_{T}}^{\ast} + \varepsilon_{i} \), where \( \varepsilon_{i} = \varepsilon_{i} + X_{i,S_{T}} (\beta_{0S_{T,\beta}} - \beta_{S_{T,\beta}}) + (\hat{F}_{S_{T}} - \hat{F}_{S_{T}}) \lambda_{S_{T}}^{\ast} \), we can decompose \( \tilde{\sigma}_{S}^{2} = \frac{1}{NT} \sum_{i=1}^{N} \varepsilon_{i}^{\prime} M_{F_{S_{T}}} \varepsilon_{i} \) as follows
\begin{align*}
\tilde{\sigma}_{S}^{2} &= \frac{1}{NT} \sum_{i=1}^{N} \varepsilon_{i}^{\prime} M_{F_{S_{T}}} \varepsilon_{i} + \frac{1}{NT} \sum_{i=1}^{N} (\beta_{0S_{T,\beta}} - \hat{\beta}_{S_{T,\beta}})^{\prime} X_{i,S_{T}}^{\prime} M_{F_{S_{T}}} X_{i,S_{T}} (\beta_{0S_{T,\beta}} - \hat{\beta}_{S_{T,\beta}}) \\
&\quad + \frac{1}{NT} \sum_{i=1}^{N} \lambda_{S_{T}}^{\ast} (F_{S_{T}}^{\ast} - \hat{F}_{S_{T}})^{\prime} M_{F_{S_{T}}} (F_{S_{T}}^{\ast} - \hat{F}_{S_{T}}) \lambda_{S_{T}}^{\ast} + \frac{2}{NT} \sum_{i=1}^{N} \varepsilon_{i}^{\prime} M_{F_{S_{T}}} X_{i,S_{T}} (\beta_{0S_{T,\beta}} - \hat{\beta}_{S_{T,\beta}}) \\
&\quad + \frac{2}{NT} \sum_{i=1}^{N} \varepsilon_{i}^{\prime} M_{F_{S_{T}}} (F_{S_{T}}^{\ast} - \hat{F}_{S_{T}}) \lambda_{S_{T}}^{\ast} \lambda_{S_{T}}^{\ast} + \frac{2}{NT} \sum_{i=1}^{N} (\beta_{0S_{T,\beta}} - \hat{\beta}_{S_{T,\beta}}) X_{i,S_{T}}^{\prime} M_{F_{S_{T}}} (F_{S_{T}}^{\ast} - \hat{F}_{S_{T}}) \lambda_{S_{T}}^{\ast} \\
&= \Delta_{1NT} + \Delta_{2NT} + \Delta_{3NT} + 2 \Delta_{4NT} + 2 \Delta_{5NT} + 2 \Delta_{6NT}, \text{ say.}
\end{align*}

By (E.6), Lemma A.2(i) and the fact that \( \mu_{\max}(M_{F_{S_{T}}}) = 1 \), \( \Delta_{2NT} \leq ||\beta_{0S_{T,\beta}} - \beta_{S_{T,\beta}}||^{2} \frac{1}{NT} \sum_{i=1}^{N} ||X_{i,S_{T}}||^{2} = O_P(\delta_{NT}^{-2}) \), and \( \Delta_{3NT} \leq \frac{1}{T} ||F_{S_{T}} - \hat{F}_{S_{T}}||^{2} \frac{1}{NT} \sum_{i=1}^{N} ||\lambda_{S_{T}}^{\ast}||^{2} = O_P(\delta_{NT}^{-2}) \). By Cauchy-Schwarz inequality, \( |\Delta_{6NT}| \leq \{ \Delta_{2NT} \Delta_{3NT} \}^{1/2} = O_P(\delta_{NT}^{-1}) \). In addition, \( |\Delta_{4NT}| \leq ||\beta_{0S_{T,\beta}} - \beta_{S_{T,\beta}}|| \frac{1}{NT} \sum_{i=1}^{N} ||\varepsilon_{i}|| ||X_{i,S_{T}}|| = O_P(\delta_{NT}^{-1}) \) by (E.6), and
\begin{align*}
|\Delta_{5NT}| &= \frac{1}{NT} \left| \text{tr} \left( M_{F_{S_{T}}} (F_{S_{T}}^{\ast} - \hat{F}_{S_{T}}) \sum_{i=1}^{N} \lambda_{S_{T}}^{\ast} \varepsilon_{i}^{\prime} \right) \right| \\
&\leq \frac{|S_{T}|}{\sqrt{N}} \frac{1}{\sqrt{T}} ||F_{S_{T}} - \hat{F}_{S_{T}}|| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \lambda_{S_{T}}^{\ast} \varepsilon_{i}^{\prime} \varepsilon_{i} = O_P \left( N^{-1/2} \delta_{NT}^{-1} \right)
\end{align*}
by Lemma A.2(i), the fact that \( |\text{tr}(A)| \leq r ||A|| \) for any \( r \times r \) matrix \( A \), the submultiplicative property of the Frobenius norm, and the facts that \( \mu_{\max}(M_{F_{S_{T}}}) = 1 \) and \( \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \lambda_{S_{T}}^{\ast} \varepsilon_{i}^{\prime} \varepsilon_{i} \right\| = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \lambda_{S_{T}}^{\ast} \varepsilon_{i}^{\prime} \varepsilon_{i} \).
Noting that we have shown that we are left to show that

\[ \hat{\sigma}_S^2 - \sigma_S^2 = \frac{1}{N^2} \sum_{i=1}^{N} \varepsilon_i^T P_{\hat{S}_S} \varepsilon_i + O_P(\delta_{NT}^2). \]

Analogously, using \( Y_i - X_i \tilde{\beta}_{S,\alpha} = F^0 \eta_i^0 + \varepsilon_i + X_i \beta_{S,\alpha} (\beta_{S,\beta}^0 - \tilde{\beta}_{S,\beta}), \) (E.3), and (E.4) we can readily show that

\[ \hat{\sigma}_S^2 = \frac{1}{N^2} \sum_{i=1}^{N} \varepsilon_i^T P_{\hat{S}_S} \varepsilon_i + O_P((NT)^{-1}). \]

It follows that

\[ \hat{\sigma}_S^2 - \sigma_S^2 = \frac{1}{N^2} \sum_{i=1}^{N} (\varepsilon_i^T P_{\hat{S}_S} \varepsilon_i - \varepsilon_i^T P_{\hat{S}_S} \varepsilon_i) + O_P(\delta_{NT}^2). \]

Noting that \( \frac{1}{N^2} \sum_{i=1}^{N} \varepsilon_i^T P_{\hat{S}_S} \varepsilon_i = O_P(T^{-1}) \), we are left to show that \( R_{NT} \equiv \frac{1}{N^2} \sum_{i=1}^{N} \varepsilon_i^T P_{\hat{S}_S} \varepsilon_i = O_P(\delta_{NT}^2) \). Note that

\[ R_{NT} = \frac{1}{N^2} \text{tr}(P_{\hat{S}_S} \varepsilon^T \varepsilon) \leq \frac{1}{N^2} \| \varepsilon \|_{sp}^2 \text{tr}(P_{\hat{S}_S}) = \left( \frac{\| \varepsilon \|_{sp}}{N^2} \right)^2 \| \varepsilon \|_{sp} = O_P(\delta_{NT}^2). \]

Consequently, we have shown that \( \hat{\sigma}_S^2 - \sigma_S^2 = O_P(\delta_{NT}^2) \) for any \( S \in \hat{S} \). □

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(a) Eigenvalues: DGP1, SN ratio=1, N=20, T=20

(b) Eigenvalues: DGP1, SN ratio=1, N=40, T=40

(c) Number of factors selected: DGP1, SN ratio=1, N=20, T=20

(d) Number of factors selected: DGP1, SN ratio=1, N=40, T=40

(e) Information criterion: DGP1, SN ratio=1, N=20, T=20

(f) Information criterion: DGP1, SN ratio=1, N=40, T=40

Figure 1: Illustration of the main ideas

(a) Eigenvalues: Estimation without regressors

(b) Eigenvalues: Linear estimation

(c) Eigenvalues: Nonlinear estimation I

(d) Eigenvalues: Nonlinear estimation II

Figure 2: Eigenvalues used in estimation
### Table 1: MSEs of estimates of \((\beta_1, \beta_2)\) for different numbers of factors included in the model

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<tr>
<th>DGP</th>
<th>N</th>
<th>T</th>
<th>(\beta_1) MSEs</th>
<th>(\beta_2) MSEs</th>
<th>(\beta_1) MSEs</th>
<th>(\beta_2) MSEs</th>
<th>(\beta_1) MSEs</th>
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<td>20</td>
<td>18.75 4.43 1.22 1.89 2.98 4.54</td>
<td>19.37 6.47 3.77 4.43 6.49 9.96</td>
<td>17.92 4.17 0.25 0.28 0.35 0.43</td>
<td>18.25 4.68 0.57 0.62 0.76 0.92</td>
<td>17.64 3.36 0.19 0.23 0.32 0.40</td>
<td>18.00 4.21 0.42 0.48 0.66 0.85</td>
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<td>16.90 3.44 0.30 0.39 0.52 0.76</td>
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<td>60</td>
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<td>17.16 3.61 0.11 0.12 0.14 0.13</td>
<td>17.16 3.89 0.19 0.21 0.24 0.26</td>
<td>17.16 3.61 0.11 0.12 0.14 0.13</td>
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<td>24.78 5.14 0.28 0.36 0.41 0.50</td>
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<td>16.51 4.81 0.60 0.73 0.86 1.10</td>
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Note: Numbers in the main entries are 100 times MSEs of the estimates of \(\beta_1\) or \(\beta_2\).
Table 2: Selection of the number of factors

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<th>SN ratio=1</th>
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Note: Numbers in the main entries are the proportions of the replications in which the selected number of factors is less than, equal to, or greater than the true number of factors (i.e., 2) out of total 250 replications. Bai and Ng refers to Bai and Ng (2002) and Ona refers to Onatski (2009).
Table 2: Selection of the number of factors (cont’d)

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Table 3: Selection of regressors

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Note: Numbers in the main entries are the proportions of the replications in which the estimates of β’s are shrunk to zeros out of total 250 replications.
Table 4: MSEs of the estimates of $\beta_1$ and $\beta_2$

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<td>$\beta_2$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>PC1</td>
<td>PC2</td>
<td>IC1</td>
<td>IC2</td>
<td>Ona</td>
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<td>2 20 20</td>
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<td>4.48</td>
<td>3.41</td>
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<td>40 40 20</td>
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<tr>
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<td>0.50</td>
<td>0.41</td>
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<td>10.35</td>
<td>0.84</td>
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<tr>
<td>60 60 60</td>
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<td>0.11</td>
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<td>0.11</td>
<td>0.11</td>
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Note: Numbers in the main entries are 100×MSEs of the estimates of $\beta_1$ or $\beta_2$. Bai and Ng refers to Bai and Ng (2002) and Ona refers to Onatski (2009).
Table 4: MSEs of the estimates of $\beta_1$ and $\beta_2$ (cont’d)

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<th>DGP</th>
<th>N</th>
<th>T</th>
<th>MSE (SN ratio=1)</th>
<th>MSE (SN ratio=2)</th>
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<td>$\beta_2$</td>
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<td>$\beta_1$</td>
<td>$\beta_2$</td>
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<td>60</td>
<td>60</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
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<td>Description</td>
<td>Mean</td>
<td>Median</td>
<td>SD</td>
</tr>
<tr>
<td>--------------</td>
<td>------------------------------------------</td>
<td>---------</td>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>Growth</td>
<td>Annual growth rate of real GDP per capita</td>
<td>1.57</td>
<td>1.84</td>
<td>6.12</td>
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</table>

**Independent variables:**

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<th>Description</th>
<th>Mean</th>
<th>Median</th>
<th>SD</th>
<th>Min</th>
<th>Max</th>
<th>Data sources</th>
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<tr>
<td>Young</td>
<td>Age dependency ratio, young</td>
<td>66.02</td>
<td>72.92</td>
<td>23.15</td>
<td>19.34</td>
<td>106.43</td>
<td>WDI</td>
</tr>
<tr>
<td>Fert</td>
<td>Fertility rate (births per woman)</td>
<td>4.27</td>
<td>4.25</td>
<td>1.98</td>
<td>0.90</td>
<td>8.29</td>
<td>WDI</td>
</tr>
<tr>
<td>Life</td>
<td>Life expectancy at birth (years)</td>
<td>62.21</td>
<td>63.98</td>
<td>11.78</td>
<td>26.82</td>
<td>82.03</td>
<td>WDI</td>
</tr>
<tr>
<td>Popu</td>
<td>Population growth</td>
<td>1.92</td>
<td>2.05</td>
<td>1.38</td>
<td>-17.28</td>
<td>17.91</td>
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<tr>
<td>Invpri</td>
<td>Price level of investment</td>
<td>88.36</td>
<td>62.33</td>
<td>174.75</td>
<td>8.88</td>
<td>2612.60</td>
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</tr>
<tr>
<td>Con</td>
<td>Consumption share</td>
<td>71.77</td>
<td>71.08</td>
<td>17.89</td>
<td>6.4</td>
<td>193.96</td>
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<tr>
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<td>Government consumption share</td>
<td>10.54</td>
<td>8.40</td>
<td>7.63</td>
<td>0.73</td>
<td>58.64</td>
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<tr>
<td>Inv</td>
<td>Investment share</td>
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<td>21.24</td>
<td>10.43</td>
<td>11.50</td>
<td>80.12</td>
<td>Penn Table</td>
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<tr>
<td>Open</td>
<td>Openness</td>
<td>59.56</td>
<td>51.97</td>
<td>37.21</td>
<td>3.78</td>
<td>377.79</td>
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</table>

**Table 6: Linear estimation**

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<tr>
<th>Young</th>
<th>Fert</th>
<th>Life</th>
<th>Popu</th>
<th>Invpri</th>
<th>Con</th>
<th>Gov</th>
<th>Inv</th>
<th>Open</th>
<th>Lag1</th>
<th>Lag2</th>
<th>Lag3</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimate</td>
<td>0.018</td>
<td>-0.001</td>
<td>0.019</td>
<td>-0.446</td>
<td>-0.062</td>
<td>0.003</td>
<td>0.128</td>
<td>0.002</td>
<td>0.143</td>
<td>0.041</td>
<td>0.031</td>
</tr>
<tr>
<td>t-stat</td>
<td>0.987</td>
<td>-0.003</td>
<td>1.754</td>
<td>-1.103</td>
<td>1.215</td>
<td>-2.907*</td>
<td>0.732</td>
<td>1.521</td>
<td>1.160</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Number of factors=0**

| estimate | 0.026  | -0.680 | -0.011 | -0.161 | 0.002  | -0.083 | -0.159 | 0.146  | -0.011 | 0.060  | 0.017  |
| t-stat   | 1.241  | -2.382*| -0.390 | -0.981 | 1.717  | -2.015* | 6.195* | -1.842 | 2.094* | 0.753  |

**Number of factors=3**

| estimate | 0.006  | 0.090  | 0.005  | -0.441 | 0.000  | -0.009 | -0.157 | 0.264  | -0.004 | 0.067  | -0.064 |
| t-stat   | 0.127  | 0.159  | 0.148  | -0.881 | 0.058  | -0.319 | -2.089*| 5.002* | -0.360 | 1.277  | -1.428 |

**Number of factors=5**

| estimate | -0.001 | -0.040 | -0.033 | -0.234 | 0.003  | -0.048 | 0.032  | 0.006  | 0.092  | -0.061 | -0.046 |
| t-stat   | -0.032 | -0.094 | -1.379 | -1.209 | 1.461  | 0.153  | 0.875  | 9.215* | -0.894 | 2.652* | -2.052*|

**Number of factors=8**

| estimate | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  |
| t-stat   | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  |

**Note:** BC-est. denotes the bias-corrected estimate. * denotes significance at the 5% level. Lag1, Lag2, and Lag3 refer to the first, second, and third lag of economic growth, respectively.

**Table 7: Nonlinear estimation I**

<table>
<thead>
<tr>
<th>Fert</th>
<th>Popu</th>
<th>Con</th>
<th>Inv</th>
<th>Lag1</th>
<th>Lag2</th>
<th>Lag3</th>
<th>Lag1²</th>
<th>Gov × Inv</th>
</tr>
</thead>
<tbody>
<tr>
<td>estimate</td>
<td>-0.145</td>
<td>-0.350</td>
<td>-0.021</td>
<td>0.128</td>
<td>-0.004</td>
<td>0.031</td>
<td>0.009</td>
<td>-0.002</td>
</tr>
<tr>
<td>t-stat</td>
<td>-0.741</td>
<td>-0.729</td>
<td>-2.223*</td>
<td>2.067*</td>
<td>1.476</td>
<td>-0.045</td>
<td>1.306</td>
<td>3.103*</td>
</tr>
</tbody>
</table>

**Number of factors=0**

| estimate | -0.101 | 0.052  | -0.094 | 0.180  | 0.145  | -0.047 | 0.014   | 0.006   |
| t-stat  | -0.352 | 0.122  | -5.525*| 2.431* | 2.070* | -0.763 | 0.663   | 3.090*   |

**Number of factors=3**

| estimate | -0.223 | 0.037  | -0.027 | 0.350  | 0.082  | -0.136 | 0.015   | 0.006   |
| t-stat  | -0.572 | 0.069  | -1.371 | 3.513* | 0.937  | -1.687 | 0.515   | 2.629*   |

**Number of factors=5**

| estimate | 0.187  | -0.151 | 0.029  | 0.362  | 0.057  | -0.251 | 0.036   | 0.005   |
| t-stat  | 0.332  | -0.211 | -0.996 | 2.745* | 0.520  | -2.638*| 0.983   | 1.292   |

**Number of factors=8**

| estimate | 0.579  | -0.318 | -0.038 | 0.303  | 0.015  | -0.087 | -0.009  | 0.003   |
| t-stat  | 0.441  | -0.359 | -0.034 | 0.292  | 0.051  | -0.044 | -0.014  | 0.003   |

**AgLasso: Number of factors=3**

| estimate | 0.579  | -0.318 | -0.038 | 0.303  | 0.015  | -0.087 | -0.009  | 0.003   |
| t-stat  | 0.579  | -0.318 | -0.038 | 0.303  | 0.015  | -0.087 | -0.009  | 0.003   |

**Post-agLasso: Number of factors=3**

| estimate | -0.151 | -0.286 | -0.104 | 0.219  | 0.052  | -0.012 | -0.003  | 0.003   |
| t-stat  | -0.709 | -0.859 | -5.434*| 7.423* | 1.703  | -0.499 | -0.115  | 1.224   |

| estimate | -0.151 | -0.286 | -0.104 | 0.219  | 0.052  | -0.012 | -0.003  | 0.003   |
| t-stat  | -0.709 | -0.859 | -5.434*| 7.423* | 1.703  | -0.499 | -0.115  | 1.224   |

**Note:** BC-est. denotes the bias-corrected estimate. * denotes significance at the 5% level. Lag1, Lag2, and Lag3 refer to the first, second, and third lag of economic growth, respectively.
<table>
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<th></th>
<th>Life</th>
<th>Con</th>
<th>Gov</th>
<th>Inv</th>
<th>Lag1</th>
<th>Fert</th>
<th>Fert</th>
<th>Fert</th>
<th>Fert</th>
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<tr>
<td>estimate</td>
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<td></td>
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</tr>
<tr>
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<td>-0.082</td>
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<td>6.169*</td>
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<td>-1.341</td>
<td>-4.250*</td>
<td>0.936</td>
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</tr>
</tbody>
</table>

Note: BC-est. denotes the bias-corrected estimate. * denotes significance at the 5% level. Lag1 and Lag2 refer to the first and second lag of economic growth, respectively.
Supplementary Material for
“Shrinkage Estimation of Dynamic Panel Data Models with Interactive Fixed Effects”
Xun Lu\textsuperscript{a}, Liangjun Su\textsuperscript{b}
\textsuperscript{a} Department of Economics, HKUST
\textsuperscript{b} School of Economics, Singapore Management University

THIS SUPPLEMENTARY MATERIAL PROVIDES PROOFS FOR THE TECHNICAL LEMMAS AND COROLLARY 3.4 IN THE ABOVE PAPER. WE ALSO PRESENT SOME PRIMITIVE CONDITIONS TO VERIFY SOME HIGH LEVEL CONDITIONS IN ASSUMPTIONS A1, A2, A4 AND A5 IN THE TEXT.

F Proof of the Technical Lemmas in Appendix A

Proof of Lemma A.1. (i) From the principal component analysis, we have the identity \((NT)^{-1} \hat{Y}' \hat{Y} \hat{F} = \hat{F} V_{NT}\). Pre-multiplying both sides by \(T^{-1} \hat{F}'\) and using the normalization \(T^{-1} \hat{F}' \hat{F} = I_R\) yields \(T^{-1} \hat{F}' (NT)^{-1} \hat{Y}' \hat{Y} \hat{F} = V_{NT}\). (i) follows provided \(V = \text{plim} V_{NT}\), which we show below.

(ii) Let \(e = \epsilon + \sum_{k=1}^{K} (\beta_0^k - \hat{\beta}_k^c) X_k\). Then \(\|e\|_{sp} \leq \|\epsilon\|_{sp} + \sum_{k=1}^{K} |\beta_0^k - \hat{\beta}_k^c| \|X_k\|_{sp} = O_P(\sqrt{N} + \sqrt{T}) + O_P(1) = O_P(\sqrt{N} + \sqrt{T})\) by Assumptions A.1(i), (iv) and (v). Noting that \(\hat{Y} = Y - \sum_{k=1}^{K} \hat{\beta}_k^c X_k = \lambda^0 F^0 + e\), (i) implies that

\[
(T^{-1} \hat{F}' F^0) (N^{-1} \lambda^0 \lambda^0) (T^{-1} F^0 \hat{F}) + d_{NT} = V_{NT} \xrightarrow{P} V
\]  

where \(d_{NT} = N^{-2} T^{-1} \hat{F}' e' e \hat{F} + (T^{-1} \hat{F}' F^0) (N^{-1} T^{-1} \lambda^0 e' e) + (N^{-1} T^{-1} \hat{F}' \lambda^0) (T^{-1} F^0 \hat{F})\). Noting that \(N^{-2} T^{-1} ||\hat{F}' e' e \hat{F}|| \leq \mathbb{R} N^{-2} \{T^{-1} ||\hat{F}'||^2\} ||e||_{sp}^2 = N^{-2} O_P(N + T) = O_P(N^{-1} + N^{-2} T)\), \(N^{-1} T^{-1} ||\lambda^0 e|| \leq N^{-1/2} \{T^{-1/2} ||\hat{F}'|| \{N^{-1/2} T^{-1/2} ||\lambda^0|| \} = O_P(N^{-1/2})\), and \(T^{-1} ||F^0 \hat{F}|| = O_P(1)\), we have \(d_{NT} = O_P(N^{-2} T + N^{-1/2}) = o_P(1)\). It follows that \((\hat{F}' F^0 / T) (\lambda^0 \lambda^0 / N) (F^0 \hat{F} / T) \xrightarrow{P} V\).

We are left with showing that \(V\) is the probability limit of \(V_{NT}\). We discuss two cases: \(R = R_0\) and \(R > R_0\). The first case is studied in Bai (2003, Lemma A.3) who characterizes \(V\) as a diagonal matrix consisting of the \(R_0\) eigenvalues of \(\Sigma_{\hat{\lambda}} \Sigma_{F^0}\). In the second case, observe that \((T^{-1} \hat{F}' F^0) (N^{-1} \lambda^0 \lambda^0) (T^{-1} F^0 \hat{F})\) has rank \(R_0\) at most in both finite and large samples. Let \(\Delta_{lNT} = T^{-1} F^0 \hat{F} (l)\) for \(l = 1, 2\), and \(\hat{\Sigma}_{\hat{\lambda}} = N^{-1} \lambda^0 \lambda^0\). Then

\[
(T^{-1} \hat{F}' F^0) (N^{-1} \lambda^0 \lambda^0) (T^{-1} F^0 \hat{F}) = \left[ \begin{array}{c} \hat{\Delta}_{1NT} \hat{\Sigma}_{\hat{\lambda}} \Delta_{1NT} \\ \hat{\Delta}_{2NT} \hat{\Sigma}_{\hat{\lambda}} \Delta_{2NT} \end{array} \right].
\]

By Lemma A.3(ii) of Bai (2003), \(\Delta_{1NT} \hat{\Sigma}_{\hat{\lambda}} \Delta_{1NT} \xrightarrow{P} V_{11}\), which has full rank \(R_0\) under Assumptions A.1(ii) and (iii). This ensures that \((\hat{F}' F^0 / T) (\lambda^0 \lambda^0 / N) (F^0 \hat{F} / T)\) has rank \(R_0\) in large samples and \(\Delta_{2NT} \hat{\Sigma}_{\hat{\lambda}} \Delta_{2NT} \xrightarrow{P} 0\). Then \(\Delta_{1NT} \hat{\Sigma}_{\hat{\lambda}} \Delta_{2NT} \xrightarrow{P} 0\) by Cauchy-Schwarz inequality. It follows that \(V = \left[ \begin{array}{cc} V_{11} & 0 \\ 0 & 0 \end{array} \right]\).

(iii) From the above proof and the fact that \(\hat{\Sigma}_{\hat{\lambda}}\) is asymptotically nonsingular by Assumption A.1(iii), we have \(\Delta_{2NT} \xrightarrow{P} 0\) and \(\Delta_{1NT} \xrightarrow{P} \Delta_1\) where \(\Delta_1\) is an \(R_0 \times R_0\) full rank matrix.

(iv) This follows from (iii), Assumption A.1(iii), and Slutsky lemma. \(\blacksquare\)

Proof of Lemma A.2. (i) Using \(Y - \sum_{k=1}^{K} \hat{\beta}_k^c X_k = \lambda^0 F^0 + \epsilon + \sum_{k=1}^{K} (\beta_0^k - \hat{\beta}_k^c) X_k\) and \(F^0 H =
\((NT)^{-1} F^0 \lambda^0 F^0 \hat{F}\) yields

\[
\hat{F} - F^0 H = (NT)^{-1} \left( Y - \sum_{k=1}^{K} \beta_k^Y X_k \right)' \left( Y - \sum_{k=1}^{K} \beta_k^Y X_k \right) \hat{F} - F^0 H
\]

\[
= (NT)^{-1} \varepsilon' \varepsilon \hat{F} + (NT)^{-1} F^0 \lambda^0 \varepsilon \hat{F} + (NT)^{-1} \varepsilon' \lambda^0 F^0 \hat{F}
\]

\[
+ (NT)^{-1} \sum_{k=1}^{K} \left( \beta_k^0 - \beta_k^\gamma \right) X_k \sum_{l=1}^{K} \left( \beta_l^0 - \beta_l^\gamma \right) X_l \hat{F}
\]

\[
+ (NT)^{-1} F^0 \lambda^0 \sum_{k=1}^{K} \left( \beta_k^0 - \beta_k^\gamma \right) X_k \hat{F} + (NT)^{-1} \sum_{k=1}^{K} \left( \beta_k^0 - \beta_k^\gamma \right) X_k \lambda^0 F^0 \hat{F}
\]

\[
+ (NT)^{-1} \varepsilon' \sum_{k=1}^{K} \left( \beta_k^0 - \beta_k^\gamma \right) X_k \hat{F} + (NT)^{-1} \sum_{k=1}^{K} \left( \beta_k^0 - \beta_k^\gamma \right) X_k \varepsilon \hat{F}
\]

\[
\equiv a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8, \quad \text{say.} \quad (F.2)
\]

It follows that \(T^{-1} \| \hat{F} - F^0 H \|^2 \leq 8T^{-1} \sum_{l=1}^{8} \| a_l \|^2 \). Similarly, we can write \(\hat{F}^{(1)} - F^0 H^{(1)} = \sum_{l=1}^{8} a^{(1)}_l \), where \(a_l = (a_l^{(1)})^{(1)}(a_l^{(2)})^{(1)} \) and \(a^{(1)}_l \) denotes a \(T \times R_0 \) submatrix of \(a_l \) for \(l = 1, 2, ..., 8\).

By Bai and Ng (2002, pp.213-214), \(T^{-1} \|a_l\|^2 = O_P(\delta^2_{NT}) \) for \(l = 1, 2, 3 \) under Assumptions A.1(ii)-(iii) and A.2(i)-(iii) as our assumptions also ensure their Lemma 1 to hold. By the facts that \(\| \beta^0 - \beta^\gamma \| = O_P((NT)^{-1/2}), \| \hat{F} \|^2 = O_P(1), \| F^0 \| = O_P(1), \| \lambda^0 \|^2 = O_P(1), \| \hat{F}' F^0 \| = O_P(1) \), and that \(\sum_{k=1}^{K} \| X_k \|_{\infty} = O_P((NT)^{1/2} (N^{1/2} + T^{1/2})) \) under Assumption A.1, we have

\[
T^{-1} \| a_4 \|^2 \leq \| \beta^0 - \beta^\gamma \|^4 \left\{ T^{-1} \| \hat{F} \|^2 \right\} \left\{ (NT)^{-1/2} \sum_{k=1}^{K} \| X_k \| \right\}^4 = O_P \left( (NT)^{-2} \right),
\]

\[
T^{-1} \| a_5 \|^2 \leq \| \beta^0 - \beta^\gamma \|^2 \left\{ T^{-1} \| \hat{F} \|^2 \right\} \left\{ (NT)^{-1/2} \sum_{k=1}^{K} \| X_k \| \right\}^2 = O_P \left( (NT)^{-1} \right),
\]

\[
T^{-1} \| a_6 \|^2 \leq \| \beta^0 - \beta^\gamma \|^2 \left\{ T^{-1} \| \hat{F}' F^0 \|^2 \right\} \left\{ N^{-1} \| \lambda^0 \|^2 \right\} \left\{ (NT)^{-1/2} \sum_{k=1}^{K} \| X_k \| \right\}^2 = O_P \left( (NT)^{-1} \right),
\]

\[
T^{-1} \| a_7 \|^2 \leq N^{-2} T^{-2} \eta_{NT}^2 \| \beta^0 - \beta^\gamma \|^2 \left\{ T^{-1} \| \hat{F} \|^2 \right\} \left\{ \eta_{NT}^{-1} \sum_{k=1}^{K} \| X_k \| \right\}^2 = O_P \left( (NT)^{-2} (N + T) \right),
\]

\[
T^{-1} \| a_8 \|^2 \leq N^{-2} T^{-2} \eta_{NT}^2 \| \beta^0 - \beta^\gamma \|^2 \left\{ T^{-1} \| \hat{F} \|^2 \right\} \left\{ \eta_{NT}^{-1} \sum_{k=1}^{K} \| X_k \| \right\}^2 = O_P \left( (NT)^{-2} (N + T) \right),
\]

where \(\eta_{NT} = (NT)^{1/2} (N^{1/2} + T^{1/2})\).

Hence \(T^{-1} \| \hat{F} - F^0 H \|^2 = O_P(\delta^2_{NT})\).

(ii) By the definition \(\hat{F} = (NT)^{-1} Y'\tilde{Y} \tilde{F} \) and the identity \((NT)^{-1} Y'\tilde{Y} \tilde{F} = \tilde{F} V_{NT} \), \(T^{-1} \| \hat{F} - F^0 H \|^2 = T^{-1} \| \tilde{F} V_{NT} - F^0 H \|^2 = T^{-1} \| \tilde{F} V_{NT,11} - F^0 H(1) \|^2 + T^{-1} \| \tilde{F} V_{NT,22} - F^0 H(2) \|^2 \). Then (i) implies that \(T^{-1} \| \tilde{F} V_{NT,11} - F^0 H(1) \|^2 = O_P(\delta^2_{NT}^2) \) and \(T^{-1} \| \tilde{F} V_{NT,22} - F^0 H(2) \|^2 = O_P(\delta^2_{NT}^2) \). Noting that \(V_{NT,11} \) is asymptotically nonsingular by Lemma A.1(i), we have \(T^{-1} \| \tilde{F} V_{NT,11} - F^0 H(1) \|^2 = O_P(\delta^2_{NT}^2) \). In addition,

\[
T^{-1} \| F^0 H(2) \|^2 \leq 2T^{-1} \| \tilde{F} V_{NT,22} - F^0 H(2) \|^2 + 2T^{-1} \| \tilde{F} V_{NT,22} \|^2 = O_P(\delta^2_{NT}^2) + O_P(N^{-1} T^2 + N^{-1}) = O_P(\delta^2_{NT}^2) \quad \text{under Assumption A.3(i)}
\]
because \( T^{-1} \left\| \tilde{F}_{(2)} V_{NT,22} \right\|^2 \leq \left( \mu_{\text{max}}(V_{NT,22}) \right)^2 T^{-1} \left\| \tilde{F}_{(2)} \right\|^2 = (R - R_0) \left[ \mu_{\text{max}}(V_{NT,22}) \right]^2 \) and
\[
\mu_{\text{max}}(V_{NT,22}) = \mu_{R_0+1}(V_{NT}) \leq \mu_{R_0+1} \left( (T^{-1} \tilde{F}' F^0) (N^{-1} \lambda^0 \lambda^0) (T^{-1} F^0 \tilde{F}) + \|d_{NT}\| \right) = 0 + O_P \left( N^{-2} T + N^{-1/2} \right) = O_P \left( N^{-2} T + N^{-1/2} \right)
\]
by (F.1), the calculation below it, Weyl's inequality and the fact that \( \|A\|_{sp} \leq \|A\| \). Observing that
\( T^{-1} \|F^0 H_{(2)}\|^2 = T^{-1} \text{tr}(H_{(2)} H_{(2)}' F^0 F^0) \geq \mu_{\text{min}} \left( T^{-1} F^0 F^0 \right) \|H_{(2)}\|^2 \), it follows that \( \|H_{(2)}\| \leq O_P \left( \delta_{NT}^{-1} \right) \)/\( \left\| \mu_{\text{min}} \left( T^{-1} F^0 F^0 \right) \right\|^{1/2} = O_P \left( \delta_{NT}^{-1} \right) \).

**(iii)** Writing \( T^{-1}(\tilde{F} - F^0 H)'F^0 = \sum_{i=1}^{8} T^{-1} \eta_i F^0 H = \sum_{i=1}^{8} A_i \), it suffices to show that \( \|A_i\| = O_P(\delta_{NT}^{-2}) \) for \( l = 1, 3, \ldots, 8 \). \( \|A_2\| = O_P(\delta_{NT}^{-1}) \), and \( \|A_{2(1)}\| = O_P(\delta_{NT}^{-1}) \) where \( A_{2(1)} \) is defined analogously as \( A_2 \) with \( a_2 \) being replaced by \( a_{2(1)} \). By the definitions of \( a_i \)'s in (F.2) and Assumption A.1, we can readily show that \( \|A_1\| = O_P(\delta_{NT}^{-2}) \), \( \|A_2\| = O_P(\delta_{NT}^{-1}) \), \( \|A_i\| = O_P(\delta_{NT}^{-1}) \) for \( l = 3, 5, 6 \), \( \|A_4\| = O_P((NT)^{-1}) \), and \( \|A_i\| = O_P((NT)^{-1}) \) for \( l = 7, 8 \). It follows that \( T^{-1}(\tilde{F} - F^0 H)'F^0 = O_P(\delta_{NT}^{-1}) \), implying the second part of (iii). In addition, by (ii) we have
\[
\|A_{2(1)}\| = N^{-1} T^{-2} \left\| \tilde{F}_{(1)} \left( \epsilon^0 F^0 \right) \right\|
\leq N^{-1} T^{-2} \left\{ \left\| \tilde{F}_{(1)} - F^0 H_{(1)} \right\| V_{NT,11}^{-1} \right\| \| \epsilon^0 \| \left\| F^0 \right\| + \left\| V_{NT,11}^{-1} H_{(1)} \right\| \left\| F^0 \right\| \right\}
\leq N^{-1} T^{-2} \left\{ O_P(T^{-1} \delta_{NT}^{-1} ) O_P(N^{-1/2} T^{-1/2} ) + O_P(1) O_P(N^{-1/2} T^{-1/2} ) O_P(T) \right\} = O_P(\delta_{NT}^{-2}).
\]

It follows that \( T^{-1}(\tilde{F}_{(1)} - F^0 H_{(1)})'F^0 = O_P(\delta_{NT}^{-2}) \).

**(iv)** Noting that \( T^{-1}(\tilde{F} - F^0 H)'F^0 \) is \( T^{-1}(\tilde{F} - F^0 H)'(\tilde{F} - F^0 H) + T^{-1}(\tilde{F} - F^0 H)'F^0 H \), the first two parts of the result follow from (i) and (iii). To prove the third part, noting that \( T^{-1}(\tilde{F} - F^0 H)'F^0 = (T^{-1}(\tilde{F}_{(1)} - F^0 H_{(1)})'F^0, T^{-1}(\tilde{F}_{(2)} - F^0 H_{(2)})'F^0) \), it suffices to show \( T^{-1}(\tilde{F}_{(2)} - F^0 H_{(2)})'F^0 = O_P(\delta_{NT}^{-2}) \). By the triangle inequality and the submultiplicative property of the Frobenius norm
\[
T^{-1} \left\| (\tilde{F}_{(2)} - F^0 H_{(2)})'F^0 \right\| \leq T^{-1} \left\| (\tilde{F}_{(2)} - F^0 H_{(2)})'F^0 + F^{(2)} H_{(2)} \right\| H_{(2)} \right\|
+ T^{-1} \left\| (\tilde{F}_{(2)} - F^0 H_{(2)})'F^0 \right\| H_{(2)} \right\|
= O_P(\delta_{NT}^{-2}) + O_P(\delta_{NT}^{-1} \delta_{NT}^{-1} ) = O_P(\delta_{NT}^{-2}).
\]

This proves (iv).

**(v)** In view of the fact that \( T^{-1}(\tilde{F}' F - H' F^0 F^0 H) = T^{-1}(\tilde{F} - F^0 H)'(\tilde{F} - F^0 H) + T^{-1}(\tilde{F} - F^0 H)'F^0 H \), the result follows from (i) and (iii).

**(vi)** Observe that
\[
P_{F_{(1)}} - P_{F_{(1)}} = \tilde{F}_{(1)} \left( F_{(1)} F_{(1)}' \right)^{-1} F_{(1)} - F_{(1)} \left( F_{(1)}' F_{(1)} \right)^{-1} F_{(1)}' = \left( \tilde{F}_{(1)} - F_{(1)} \right) \left( F_{(1)}' F_{(1)} \right)^{-1} \left( \tilde{F}_{(1)} - F_{(1)} \right)' + \left( \tilde{F}_{(1)} - F_{(1)} \right)' \left( \tilde{F}_{(1)} F_{(1)}' \right)^{-1} \left( \tilde{F}_{(1)} - F_{(1)} \right) = F_{(1)} + F_{(1)}' \left( \tilde{F}_{(1)} F_{(1)}' \right)^{-1} \left( \tilde{F}_{(1)} F_{(1)}' \right)^{-1} \left( \tilde{F}_{(1)} - F_{(1)} \right)
\equiv b_1 + b_2 + b_3 + b_4, \text{ say.} \quad \text{(F.3)}
\]

By (v), \( T^{-1} ||\tilde{F}_{(1)} - F_{(1)}||^2 = O_P(\delta_{NT}^{-2}) \) and \( T^{-1} ||\tilde{F}_{(1)}' F_{(1)} - F_{(1)}' F_{(1)}|| = O_P(\delta_{NT}^{-2}) \). With these, one can readily show that \( \|b_s\| = O_P(\delta_{NT}^{-2}) \) for \( s = 1, 4 \). For \( b_2 \), we have \( \|b_2\| \leq T^{-1/2} ||\tilde{F}_{(1)} - F_{(1)}||\).
\[||(T^{-1}\hat{F}^t(1)\hat{F}(1))^{-1}||T^{-1/2}||F(1)|| = O_P(\delta^{-1}_{NT}).\]

Noticing that \(b_3 = \beta_2\), we have completed the proof of (vi).

(vii) Note that \(T^{-1}H'F^oF^0H = (\hat{F}^tF^o/T) (\hat{X}^t\hat{X}/N) (F^oF^0/T) (\hat{X}^t\hat{X}/N) (F^o\hat{F}/T)\) has rank \(R_0\) in large samples by Lemma A.1(iii). As a result, \(\tau_{\hat{r}}\) converges in probability to some positive number \(\tau^*_l\) for \(l = 1, \ldots, R_0\) and \(\tau_{\hat{r}} = 0\) for \(l = R_0 + 1, \ldots, R\). In addition, by (iv) and the continuity of eigenvalues, \(\tau_{\hat{r}} - \tau^*_l = O_P(\delta^{-2}_{NT})\) for \(l = 1, \ldots, R_0\) and \(= O_P(N^{-1/2})\) for \(l = R_0 + 1, \ldots, R\). It follows that \(\tau_{\hat{r}} = \tau^*_l + O_P(1)\) for \(l = 1, \ldots, R_0\) and \(\tau_{\hat{r}} = O_P(N^{-1/2})\) for \(l = R_0 + 1, \ldots, R\). □

**Proof of Lemma A.3.** (i) By (F.3) we have

\[
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'(i) \left( P_{(1)} - P_{(1)}^* \right) \left( \hat{F}(1) - F(1) \right) \lambda^*(i) = \sum_{i=1}^{4} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X(i) b(i) \left( \hat{F}(1) - F(1) \right) \lambda^*(i) = \sum_{i=1}^{4} B_{iNT}.
\]

For \(l = 1\) and 4, we use the fact that \(\|b(l)\| = O_P(\delta^{-2}_{NT})\), Lemma A.2(i), and Assumptions A.1(iii)-(iv) and A.3(i) to obtain

\[
\|B_{iNT}\| \leq (NT)^{-1/2} \left\| \hat{F}(1) - F(1) \right\| \sum_{i=1}^{N} \|X(i)\| \|\lambda^*(i)\| = (NT)^{-1/2} O_P(\delta^{-2}_{NT}) \quad \frac{(T^{1/2}\delta^{-1}_{NT})}{O_P(NT^{1/2})} = O_P(NT^{1/2})
\]

By Lemmas A.2(i), (iii) and (iv) and Assumptions A.1(iii)-(iv),

\[
\left\| B_{2NT} \right\| = \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'(i) \left( \hat{F}(1) - F(1) \right) \left( \hat{F}(1) - F(1) \right)^{-1} \right\|
\]

\[
\leq (NT)^{-1/2} \left\| \hat{F}(1) - F(1) \right\| \left\| F(1) \left( \hat{F}(1) - F(1) \right)^{-1} \right\| \leq (NT)^{-1/2} O_P(T^{1/2}\delta^{-1}_{NT}) \quad \frac{(T\delta^{-2}_{NT})}{O_P(T^{-1})} \quad O_P(NT^{1/2}) = O_P(N^{1/2}T^{1/2}\delta^{-3}_{NT}) = o_P(1).
\]

For \(B_{3NT}\), using Lemmas A.2(i) and (v), we can readily show that

\[
B_{3NT} = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'(i) F(1) \left( \hat{F}(1) - F(1) \right)^{-1} \left( \hat{F}(1) - F(1) \right) \lambda^*(i) + O_P\left( N^{1/2}T^{1/2}\delta^{-4}_{NT} \right)
\]

\[
= \tilde{B}_{3NT} + O_P\left( N^{1/2}T^{1/2}\delta^{-4}_{NT} \right), \quad \text{say.}
\]

By (F.2), we can write \(\tilde{B}_{3NT}\) as \(\tilde{B}_{3NT} = \sum_{k=1}^{8} \sum_{l=1}^{K_0} \sum_{i=1}^{N} X'(i) F(1) \left( \hat{F}(1) - F(1) \right) \left( \hat{F}(1) - F(1) \right) \lambda^*(i) \equiv \sum_{k=1}^{8} \sum_{l=1}^{K_0} \tilde{B}_{3NT}(k,l), \quad \text{say.}
\]

Note that \(\tilde{B}_{3NT}(3,3) = \frac{1}{N^{5/2}T^{5/2}} \sum_{i=1}^{N} F(1) \left( \hat{F}(1) - F(1) \right) \left( \hat{F}(1) - F(1) \right) \lambda^*(i) \equiv \sum_{k=1}^{8} \sum_{l=1}^{K_0} \tilde{B}_{3NT}(k,l), \quad \text{say.}
\]

In addition, one can readily show that \(\tilde{B}_{3NT}(k,l) = o_P(1)\) for \(k,l = 1, \ldots, 8\) with \(k \neq l\) or \(l = \neq 1\). For example,

\[
\left\| \tilde{B}_{3NT}(1,1) \right\| = \frac{1}{N^{5/2}T^{5/2}} \sum_{i=1}^{N} X'(i) F(1) \left( \hat{F}(1) - F(1) \right) \left( \hat{F}(1) - F(1) \right) \lambda^*(i) \equiv \frac{K_0}{N^{5/2}T^{5/2}} \sum_{i=1}^{N} \left\| \hat{F}(1) \right\| \left\| \hat{F}(1) \right\| \lambda^*(i) \equiv \frac{N^{5/2}T^{-5/2}O_P(NT)O_P(N+T)}{O_P(NT^{1/2})} = O_P(1).
\]
where we use the fact that 
\[ \| \hat{F}'(1) \epsilon' \| \leq \| V_{NT,11}^{-1} H'(1) F^0 \epsilon' \| + \| (\hat{F}'(1) - F^0 H(1) V_{NT,11}^{-1}) \epsilon' \| \leq \| V_{NT,11}^{-1} \| \times \| H(1) \| \| F^0 \epsilon' \| + R_0 \| \hat{F}'(1) - F^0 H(1) V_{NT,11}^{-1} \| \| \epsilon' \|_{sp} = O_P(N^{1/2} T^{1/2} + T^{1/2} \delta_{NT}^{-1} (N^{1/2} + T^{1/2})) = O_P(N^{1/2} T^{1/2}) \] by Lemmas A.1(iii) and A.2(ii) and Assumptions A.1(v) and (vii). Similarly

\[
\| \tilde{B}_3 NT (2, 2) \| = \frac{1}{N^{\tau/2} T^{\tau/2}} \sum_{i=1}^{N} \| X_i(1) (F'_i(1) F_i(1))^{-1} (\hat{F}'(1) \epsilon' \lambda^0 F^0 \lambda^0 \epsilon \hat{F}(1) \lambda_i(1)) \| 
\leq \frac{1}{N^{\tau/2} T^{\tau/2}} \| F'_i(1) \epsilon' \lambda^0 \| \| F^0 \epsilon' \| \| F'_i(1) \epsilon' \lambda^0 \| \| \left( \sum_{i=1}^{N} \| X_i(1) \| \lambda^0_i(1) \| \right) \|
= N^{-\tau/2} T^{-\tau/2} O_P(N^{1/2} T^{1/2} + T^{1/2} \delta_{NT}^{-1} N^{1/2} T^{1/2}) = O_P(N^{1/2} T \delta_{NT}^{-1})
\]
by Lemmas A.1(iii) and A.2(ii) and Assumption A.1(vii). Consequently, \[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X_i(1) (P_{F'_i(1)} - P_{F_i(1)}) (\hat{F}'(1) - F'_i(1) \lambda^0_i(1)) = B_{3 NT} + o_p(1). \]

(ii) Letting \( \hat{F}'_i(1) = T^{-1} \hat{F}'(1) \), we can write the kth element of \( \frac{1}{NT} \sum_{i=1}^{N} X_i(1) (\hat{F}'_i(1) - F'_i(1) \lambda_i(1)) \) \( F'_i(1) \epsilon_i(1) \) as \[ \frac{1}{NT} \text{tr} \left[ \hat{F}'(1) \epsilon(1) X_k (\hat{F}'(1) - F'_i(1))^{-1} \right]. \] Noting that \[ \text{tr} \left[ \hat{F}'(1) \epsilon(1) X_k (\hat{F}'(1) - F'_i(1))^{-1} \right] \leq R_0 \| \hat{F}'(1) \epsilon(1) X_k \| \| \hat{F}'(1) - F'_i(1) \| \| H(1) \| = O_P(1), \]
and \( \hat{F}'(1) \) is asymptotically nonsingular by Lemma A.2(v) and Assumption A.1(ii), it suffices to prove (ii) by showing that 
\[ \frac{1}{NT} \| F^0 \epsilon' X_k (\hat{F}'(1) - F'_i(1)) \| = O_P(N^{-1/2} T^{1/2} \delta_{NT}^{-1}). \]

By (F.2), \[ \frac{1}{NT} \| F^0 \epsilon' X_k (\hat{F}'(1) - F'_i(1)) \| = \sum_{i=1}^{8} \frac{1}{NT} \| F^0 \epsilon' X_k a_i(1) \| \equiv \sum_{i=1}^{8} A_{1, NT}, \]

\[ A_{1, NT} = \frac{1}{(NT)^2} \| F^0 \epsilon' X_k \| \| \epsilon' \hat{F}'(1) \| \leq \frac{R_0}{(NT)^2} \| F^0 \epsilon' \| \| X_k \| \| \epsilon' \|_{sp} \| \hat{F}'(1) \| = (NT)^{-2} O_P(N^{1/2} T^{1/2}) O_P(N^{1/2} T^{1/2}) O_P(N + T) O_P(T^{1/2}) = O_P(N^{1/2} T^{1/2} + T^{-1/2}). \]

Let \( H_B \equiv H(1) V_{NT,11}^{-1} \), which corresponds to the H matrix in Bai (2009). By the triangle inequality, 
\[ A_{2, NT} = (NT)^{-2} \| F^0 \epsilon' X_k F^0 \lambda^0 \epsilon \hat{F}'(1) \| \leq (NT)^{-2} \| F^0 \epsilon' X_k F^0 \lambda^0 \epsilon H_B \| + (NT)^{-2} \| F^0 \epsilon' X_k F^0 \lambda^0 \epsilon (\hat{F}'(1) - F^0 H_B) \| \equiv A_{2, NT, 1} + A_{2, NT, 2}, \]

By Assumptions A.1(ii), (iv), and (vii) and the fact \( \| H_B \| = O_P(1) \)
\[ A_{2, NT, 1} \leq (NT)^{-2} \| F^0 \epsilon' X_k F^0 \lambda^0 \epsilon H_B \| = (NT)^{-2} O_P(N^{1/2} T^{1/2}) O_P(N^{1/2} T^{1/2}) O_P(T^{1/2}) O_P(T^{1/2}) O_P(1) = O_P(N^{-1/2}). \]

Similarly, by Lemma A.2(ii) and Assumptions A.1(iv), (iv) and (vii) 
\[ A_{2, NT, 2} \leq (NT)^{-2} \| F^0 \epsilon' X_k F^0 \lambda^0 \epsilon (\hat{F}'(1) - F^0 H_B) \| = (NT)^{-2} O_P(N^{1/2} T^{1/2}) O_P(T^{1/2} \delta_{NT}^{-1}) = O_P(N^{-1/2} T^{1/2} \delta_{NT}^{-1}). \]
It follows that $A_{2NT} = O_P(N^{-1/2} + N^{-1/2}T^{1/2}\delta_{NT}^{-1}) = O_P(N^{-1/2}T^{1/2}\delta_{NT}^{-1})$. By Assumption A.4(iii),

$$A_{3NT} = (NT)^{-2} \| F^0\varepsilon'X_k\varepsilon'X_k^0F^0\bar{F}_1 \| \leq (NT)^{-2} \| F^0\varepsilon'X_k\varepsilon'X_k^0 \| \| F^0\bar{F}_1 \|$$

$$= (NT)^{-2} O_P \left( NT \left( N^{1/2} + T^{1/2} \right) \right) = O_P \left( N^{-1/2} + N^{-1}T^{1/2} \right).$$

Noting that $\| F^0\varepsilon'X_k \| \leq \| F^0\varepsilon' \| \| X_k \| = O_P(NT)$ by Assumptions A.1(iv) and (vii), we can readily apply Assumption A.1 to obtain $A_{2NT} = O_P(N^{-1}T^{-1/2})$, $A_{sNT} = O_P(N^{-1})$ for $s = 5, 6$, and $A_{sNT} = O_P(N^{-1/2}T^{-1/2} + N^{-1})$ for $s = 7, 8$. Consequently, $\frac{1}{\sqrt{T}} \| F^0\varepsilon'X_k(\bar{F}_1 - F^*_1) \| = O_P(N^{-1/2}T^{1/2}\delta_{NT}^{-1})$ and (iv) follows.

(iii) Note that

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)}F^*_1 \left( \bar{F}'_1 F^*_1 \right)^{-1} \left( \bar{F}_1 - F^*_1 \right)' \varepsilon_i$$

$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)}F^*_1 \left( F'^*_1 F^*_1 \right)^{-1} \left( \bar{F}_1 - F^*_1 \right)' \varepsilon_i$$

$$+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)}F^*_1 \left( \bar{F}'_1 F^*_1 \right)^{-1} \left( F'^*_1 F^*_1 \right)^{-1} \left( \bar{F}_1 - F^*_1 \right)' \varepsilon_i$$

$$= J_1 + J_2,$$

say.

By Lemmas A.2(i) and (iii), Assumptions A.1(iv)-(v), for any arbitrary $K_0 \times 1$ vector $c_{K_0}$ with $\|c_{K_0}\| = 1$,

$$|c'_{K_0} J_2| = \frac{1}{\sqrt{NT}} \left| \text{tr} \left( F^*_1 \left[ \left( \bar{F}'_1 F^*_1 \right)^{-1} - \left( F'^*_1 F^*_1 \right)^{-1} \right] \left( F_1 - F^*_1 \right) \left( \sum_{k=1}^{K_0} c_{K_0} \varepsilon'X_k \right) \right) \right|$$

$$\leq R_0 (NT)^{-1/2} \left( \| F^*_1 \| \left\| \left( \bar{F}'_1 F^*_1 \right)^{-1} - \left( F'^*_1 F^*_1 \right)^{-1} \right\| \left\| F_1 - F^*_1 \right\| \| \varepsilon \| \| X_k \|_{sp} \sum_{k=1}^{K_0} \right)$$

$$= (NT)^{-1/2} O_P(T^{1/2})O_P(T^{-1}\delta_{NT}^{-2})O_P(T^{1/2}\delta_{NT}^{-1})O_P(N^{1/2} + T^{1/2})O_P(N^{1/2}T^{1/2}) = O_P(\delta_{NT}^{-2}).$$

Hence $J_2 = O_P(\delta_{NT}^{-2})$. For $J_1$, by (F.2) we have

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} X'_{i(1)}F^*_1 \left( F'^*_1 F^*_1 \right)^{-1} \left( \bar{F}_1 - F^*_1 \right)' \varepsilon_i$$

$$= \sum_{i=1}^{8} \frac{1}{\sqrt{N^{1/2}T^{1/2}}} \sum_{i=1}^{N} X'_{i(1)}F^*_1 \left( F'^*_1 F^*_1 \right)^{-1} \left( \bar{F}_1 - F^*_1 \right)' \varepsilon_i$$

$$\equiv \sum_{i=1}^{8} A_{NT} (l),$$

say.

One can readily show that $A_{NT} (l) = O_P(\delta_{NT}^{-1})$ for $l = 4, 5, ..., 8$. For $A_{NT} (1)$, we apply Assumptions A.1(iv)-(v) to obtain

$$|c'_{K_0} A_{NT} (1)| = \frac{1}{N^{3/2}T^{3/2}} \left| \text{tr} \left( F^*_1 \left[ \left( F'^*_1 F^*_1 \right)^{-1} \right] \left( \bar{F}_1 \right)' \sum_{k=1}^{K_0} c_{K_0} \varepsilon'X_k \right) \right|$$

$$\leq \frac{R_0}{N^{3/2}T^{3/2}} \left( \| F^*_1 \| \left\| \left( F'^*_1 F^*_1 \right)^{-1} \right\| \left\| \bar{F}_1 \right\| \| \varepsilon \| \| X_k \|_{sp} \sum_{k=1}^{K_0} \right)$$

$$= (NT)^{-3/2} O_P(1) O_P(N^{3/2} + T^{3/2})O_P(N^{1/2}T^{1/2}) = O_P(\delta_{NT}^{-1}).$$
where we also use the fact that $|\text{tr} (A)| \leq \text{rank}(A) \|A\|_{sp}$, and the submultiplicative property of the spectral norm, and the triangle inequality. Hence $A_{NT} (1) = O_P (\delta_{NT}^{-1})$. Next, we decompose $A_{NT} (2)$ as follows:

$$A_{NT} (2) = \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N} X'_{i(1)} F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} H_B' F^{00} \varepsilon' \lambda^0 F^{00} \varepsilon_i$$

$$+ \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N} X'_{i(1)} F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} \left( \tilde{F}_{(1)} - F^{0} H_B \right)' \varepsilon' \lambda^0 F^{00} \varepsilon_i$$

$$= A_{NT} (2, 1) + A_{NT} (2, 2),$$

By Assumptions A.1(ii), (iv), (v) and (vii),

$$|c'_{K_0} A_{NT} (2, 1)| = \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N} c'_{K_0} X'_{i(1)} F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} H_B' F^{00} \varepsilon' \lambda^0 F^{00} \varepsilon_i$$

$$= \frac{1}{N^{3/2}T^{3/2}} \left| \text{tr} \left( F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} H_B' F^{00} \varepsilon' \lambda^0 F^{00} \varepsilon \sum_{k=1}^{K_0} c_k K_0 \varepsilon' X_k \right) \right|$$

$$\leq \frac{R_0}{N^{3/2}T^{3/2}} \left\| F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} H_B' \right\| \| F^{00} \varepsilon' \lambda^0 F^{00} \| \| \varepsilon \|_{sp} \sum_{k=1}^{K_0} \| X_k \|$$

$$= (NT)^{-3/2} O_P \left( T^{-1/2} \right) O_P \left( N^{1/2} T^{1/2} \right) O_P \left( T^{1/2} \right) O_P \left( N^{1/2} + T^{1/2} \right) O_P \left( N^{1/2} T^{1/2} \right)$$

$$= O_P \left( \delta_{NT}^{-1} \right).$$

Similarly,

$$|c'_{K_0} A_{NT} (2, 2)| = \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N} c'_{K_0} X'_{i(1)} F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} \left( \tilde{F}_{(1)} - F^{0} H_B \right)' \varepsilon' \lambda^0 F^{00} \varepsilon_i$$

$$= \frac{1}{N^{3/2}T^{3/2}} \left| \text{tr} \left( F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} \left( \tilde{F}_{(1)} - F^{0} H_B \right)' \varepsilon' \lambda^0 F^{00} \varepsilon \sum_{k=1}^{K_0} c_k K_0 \varepsilon' X_k \right) \right|$$

$$\leq \frac{R_0}{N^{3/2}T^{3/2}} \left\| F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} \right\| \| \tilde{F}_{(1)} - F^{0} H_B \| \| \varepsilon' \lambda^0 F^{00} \| \| \varepsilon \|_{sp} \sum_{k=1}^{K_0} \| X_k \|$$

$$= (NT)^{-3/2} O_P \left( T^{-1/2} \right) O_P \left( T^{1/2} \delta_{NT}^{-1} \right) O_P \left( N^{1/2} T^{1/2} \right) O_P \left( T^{1/2} \right) O_P \left( N^{1/2} + T^{1/2} \right)$$

$$\times O_P \left( N^{1/2} T^{1/2} \right)$$

$$= O_P \left( \delta_{NT}^{-1} \left( 1 + (T/N)^{1/2} \right) \right).$$

It follows that $A_{NT} (2) = O_P \left( \delta_{NT}^{-1} \left( 1 + (T/N)^{1/2} \right) \right)$ . For $A_{NT} (3)$, we have

$$A_{NT} (3) = \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N} X'_{i(1)} F_{(1)} \left( F'_{(1)} F_{(1)} \right)^{-1} \tilde{F}_{(1)} F^{00} \lambda^0 \varepsilon_i$$

$$= \sqrt{\frac{T}{N}} \left\{ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( T^{-1} X'_{i(1)} F_{(1)} \right) \left( T^{-1} F'_{(1)} F_{(1)} \right) \left( T^{-1} \tilde{F}_{(1)} F^{00} \lambda^0 \left( T^{-1} \varepsilon_i \varepsilon_i \right) \right) \right\} = \sqrt{\frac{T}{N}} b_{3NT},$$

where $b_{3NT} = O_P (1)$ . This completes the proof of (iii).

(iv) By (F.2), $\frac{1}{N} (\tilde{F}_{(1)} - F^{0} H) \varepsilon_i = \sum_{j=1}^{N} \frac{1}{N} \tilde{F}_{(1)} \varepsilon_i = \sum_{l=1}^{N} A_{NT} (l)$ , say. The proof is analogous to that of Lemma A.2(iii) and we only show that $A_{NT} (l) = O_P \left( N^{-1} \right)$ for $l = 1, 2, 3$ as the analysis of the other
terms is simpler. By the triangle inequality, the relationship between the spectral and Frobenius norms
and their submultiplicative property, and Assumptions A.1(ii), (v) and (vii), A.2(i), and A.4(iv)
\[
\|A_{NT}(1)\| = \frac{N^{-1/2}}{2} \left\| \tilde{F}\epsilon' \epsilon_i \right\| \leq \|R_0 N^{-1/2} \| \|\epsilon\|_p \|\epsilon_i\|
\]
\[
= N^{-1/2}O_p \left( \frac{T^{1/2}}{N} \right) O_p \left( T^{1/2} \right) = O_p \left( \eta_{NT}^2 \right),
\]
\[
\|A_{NT}(2)\| = N^{-1/2} \left\| \tilde{F}_\epsilon' \epsilon_i \lambda^0 F_\omega \epsilon_i \right\| \leq N^{-1/2} \left\| \tilde{F}_\epsilon' \epsilon_i \right\| \|\epsilon_i\| \lambda^0 \|F_\omega\| \|F_\omega\| = N^{-1/2} O_p \left( T^{1/2} \right) = O_p \left( N^{-1/2} T^{-1/2} \right),
\]
and
\[
\|A_{NT}(3)\| = N^{-1/2} \left\| \tilde{F}_\epsilon' \epsilon_i \lambda^0 \epsilon_i \right\| \leq N^{-1/2} \left\| \tilde{F}_\epsilon' \epsilon_i \lambda^0 \|\epsilon_i\| \|\epsilon_i\| = N^{-1/2} O_p \left( T^{1/2} \right) = O_p \left( N^{-1} + N^{-1/2} T^{-1/2} \right).
\]
This completes the proof of (iv). □

G Some Primitive Assumptions and Technical Lemmas

In this appendix we present two assumptions that replace some high level conditions in Assumptions
A1, A2, A4 and A5 in the text. They are also used to prove the technical lemmas in this appendix and
Corollary 3.4 in the next appendix.

Recall that \( D \equiv \sigma \left( F_\omega, \lambda^0 \right) \) and \( E_D (A) = E(A|D) \). Let \( \|A\|_{q,D} \equiv \left[ E_D (\|A\|_q^q) \right]^{1/q} \).

**Assumption B.1** (i) \( \max_{1 \leq t \leq T} E \|F_\omega^t \|^{8+4(1+q)} \| \leq C \) for some \( \sigma > 0 \) and \( C < \infty \) and \( T^{-1} F_\omega^t F_\omega^s P \to \Sigma_F > 0 \) as \( T \to \infty \).

(ii) \( \max_{1 \leq i \leq N} E \|\lambda_i^0 \|^{8+4q} \| \leq C \) and \( N^{-1} \lambda_i^0 \lambda_i^0 \to P \Sigma_\lambda > 0 \) as \( N \to \infty \).

(iii) \( \max_{1 \leq i \leq N, 1 \leq t \leq T} E \|\Sigma_{it} \|^{8+4q} \| \leq C \) for \( \sigma = X_{it} \) and \( \epsilon_{it} \).

(iv) \( \max_{1 \leq i \leq N} T^{-1} \sum_{t=1}^T E \|\Sigma_{it} \|^{8+4q} \| \leq C \) for \( \sigma = X_{it} \) and \( \epsilon_{it} \).

(v) \( \max_{1 \leq i \leq N} T^{-1} \sum_{t=1}^T E \|\Sigma_{it} \|^{8+4q} \| \leq C \) for \( \sigma = X_{it} \) and \( \epsilon_{it} \).

Note that Assumptions B.1(i)-(iii) strengthen the moment conditions in Assumptions A.1(ii)-(iv) and A.2(i) and require finite eighth plus moments for \( F_\omega^t, \lambda_i^0, X_{it}, \) and \( \epsilon_{it} \) to derive the asymptotic distribution of our adaptive group Lasso estimator and to estimate the asymptotic bias and variance terms. Admittedly, our moment conditions are generally different and may sometimes be stronger than those assumed in the literature (e.g., Bai, 2009). For example, Bai (2009) only requires finite fourth moments for \( F_\omega^t, \lambda_i^0, \) and \( X_{it} \) and finite eighth moments for \( \epsilon_{it} \); but he assumes independence between \( \epsilon_{it} \) and \( (X_{js}, F_s^t, \lambda_j^0) \) for all \( i, j, t, s, \) and thus rules out dynamics in the model. Moon and Weidner (2010) assume eighth moments for \( \epsilon_{it} \); but they also assume that both the factors and factor loadings are uniformly bounded. In addition, they assume that the error terms \( \epsilon_{it} \) are independent across both \( i \) and \( t \), which may rule out conditional heteroskedasticity in dynamic panels. Assumptions B.1(iv)-(v) are needed to show some uniform results below.

To state the next assumption, we first provide the definition of conditional strong mixing processes.

**Definition G.1** Let \( (\Omega, \mathcal{A}, P) \) be a probability space and \( \mathcal{B} \) be a sub-\( \sigma \)-algebra of \( \mathcal{A} \). Let \( P_B (\cdot) \equiv P (\cdot|B) \). Let \( \{\xi_t, t \geq 1\} \) be a sequence of random variables defined on \( (\Omega, \mathcal{A}, P) \). The sequence \( \{\xi_t, t \geq 1\} \) is said
to be conditionally strong mixing given \( B \) (or B-strong-mixing) if there exists a nonnegative B-measurable random variable \( \alpha^B(t) \) converging to 0 a.s. as \( t \to \infty \) such that

\[
|P_B (A \cap B) - P_B (A) P_B (B)| \leq \alpha^B(t) \text{ a.s.} \tag{G.1}
\]

for all \( A \in \sigma (\xi_1, \ldots, \xi_k), B \in \sigma (\xi_{k+t}, \xi_{k+t+1}, \ldots) \) and \( k \geq 1, t \geq 1 \).

The above definition is due to Prakasa Rao (2009); see also Roussas (2008). When one takes \( \alpha^B(t) \) as the supremum of the left hand side object in (G.1) over the set \( \{ A \in \sigma (\xi_1, \ldots, \xi_k), \sigma (\xi_{k+t}, \xi_{k+t+1}, \ldots) \}, \) we refer to it as the B-strong-mixing coefficient.

**Assumption B.2** (i) For each \( i = 1, \ldots, N \), \( \{(X_{it}, \varepsilon_{it}) : t = 1, 2, \ldots\} \) is conditionally strong mixing given \( D \) with mixing coefficients \( \{\alpha_{NT,i}^D(\cdot)\} \). \( \alpha_D(\cdot) \equiv \alpha_{NT,i}^D(\cdot) \equiv \max_{1 \leq i \leq N} \alpha_{NT,i}^D(\cdot) \) satisfies \( \alpha_D(s) = O_{a.s.} (s^{-\rho}) \) where \( \rho = (2 + \sigma)/(1 + \sigma) + \epsilon \) for some arbitrarily small \( \epsilon > 0 \) and \( \sigma \) is as defined in Assumption B.1(ii). In addition, there exist integers \( \tau_0, \tau_* \in (1, T) \) such that \( NT \alpha_D(\tau_0) = o_{a.s.}(1) \), \( T(T + N^{1/2})\alpha_D(\tau_0)/(1 + \sigma) = o_{a.s.}(1) \), and \( N^{1/2}T^{-1}r_*^2 = o(1) \).

(ii) For each \( i = 1, \ldots, N \), \( \{\varepsilon_{it}, X_{it}\} \) are mutually independent of each other conditional on \( D \).

(iii) For each \( i = 1, \ldots, N \), \( E(\varepsilon_{it}|\mathcal{F}_{NT,t-1}) = 0 \) a.s., where \( \mathcal{F}_{NT,t} = \sigma(D, \{X_{i,t-1}, X_{it}, \varepsilon_{it}, X_{i,t-1}, \varepsilon_{i,t-1}, \ldots, \varepsilon_{it} \}) \).

(iv) As \( (N,T) \to \infty \), \( MN^{1/2}T^{-1/2}(\alpha_{NT}^{-1} + \alpha_D) \to 0 \) and \( NT \alpha_D(M + 1)^{(3 + 2\sigma)/(2 + \sigma)} = o_{a.s.}(1) \).

B.2(ii) requires that each individual time series \( \{(X_{it}, \varepsilon_{it}) : t = 1, 2, \ldots\} \) be \( D \)-strong-mixing. To appreciate the importance of conditioning, we take the simple panel AR(1) model considered by Su and Chen (2013) as an example:

\[
Y_{it} = \rho_0 Y_{i,t-1} + \lambda_i^0 F_i^0 + \varepsilon_{it}, \quad i = 1, \ldots, N, \quad t = 1, \ldots, T. \tag{G.2}
\]

Even if \( \{(\varepsilon_{it}, F_i^0) : t \geq 1\} \) is a strong mixing process, \( \{Y_{it}, t \geq 1\} \) is generally not unless \( \lambda_i^0 \) is nonstochastic. For this reason, Hahn and Kuersteiner (2011) assume that the individual fixed effects are nonrandom and uniformly bounded in their study of nonlinear dynamic panel data models. In the case of random fixed effects, they suggest adopting the concept of conditional strong mixing where the mixing coefficient is defined by conditioning on the fixed effects. Our spirit is similar to theirs as we define the conditional strong mixing processes by conditioning on both factors and factor loadings. The dependence of the mixing rate on \( \sigma \) defined in B.1 reflects the trade-off between the degree of dependence and the moment bounds of the process \( \{(X_{it}, \varepsilon_{it}), t \geq 1\} \). As Su and Chen (2013) remark, Assumption B.2(ii) does not rule out cross sectional dependence among \( \{X_{it}, \varepsilon_{it}\} \). When \( X_{it} = Y_{i,t-1} \) and \( \varepsilon_{it} \) exhibits conditional heteroskedasticity (e.g., \( \varepsilon_{it} = \sigma_0(\varepsilon_{i,t-1}) \varepsilon_{it} \) where \( \varepsilon_{it} \sim \text{IID}(0,1) \) and \( \sigma_0(\cdot) \) is an unknown smooth function) as in (G.2), \( \{X_{it}, \varepsilon_{it}\} \) are not independent across \( i \) because of the presence of common factors irrespective of whether one allows \( \lambda_i^0 \) to be independent across \( i \) or not. Nevertheless, conditional on \( D \), it is possible that \( \{X_{it}, \varepsilon_{it}\} \) is independent across \( i \) such that A.2(ii) is still satisfied. Note that here the cross sectional dependence is similar to the type of cross sectional dependence generated by common shocks studied by Andrews (2005), but the latter author assumes IID observations conditional on the \( \sigma \)-field generated by the common shocks in a cross-section framework. B.2(iii) requires that the error term \( \varepsilon_{it} \) be a martingale difference sequence (m.d.s.) with respect to the filter \( \mathcal{F}_{NT,t} \) which allows for lagged dependent variables in \( X_{it} \), and conditional heteroskedasticity, skewness, or kurtosis of an unknown form in \( \varepsilon_{it} \). In contrast, both Bai (2009) and Pesaran (2006) assume that \( \varepsilon_{it} \) is independent of \( X_{js}, \lambda_j \), and \( F_s \) for all \( i, t, j \) and \( s \); Moon and Weidner (2010) allow dynamics but assume that \( \varepsilon_{it} \)'s are independent across both \( i \) and \( t \). The allowance of lagged dependent variables broadens the potential applicability of our shrinkage estimation method. B.2(iv) requires that \( M \) should not grow too fast.

To proceed, we remark that with Assumptions B.1-B.2, the high level conditions in Assumptions A.1(vi)-(viii), A.2(ii)-(iii), A.4(iii)-(iv) and A.5(i)-(ii) can be easily verified. Assumption A.1(vi) follows
from Assumptions B.1(iii) and B.2(ii)-(iii) and Chebyshev inequality. Assumptions A.1(viii) holds because $E \left[ \left\| F_\theta' \varepsilon \lambda^0 \right\|^2 \right] = \sum_{1 \leq i,j \leq N} \sum_{1 \leq t,s \leq T} E \left[ E_D ( \varepsilon_{it} \varepsilon_{js}) F_\theta' F_\theta' \lambda^0_j \lambda^0_j \right] = \sum_{i=1}^N \sum_{t=1}^T E \left[ E_D ( \varepsilon_{it}^2) \right] \left\| F_\theta' \right\|^2 \left\| \lambda^0 \right\|^2 = O \left( NT \right)$ under Assumptions B.1(i)-(iii) and B.2(ii)-(iii). Assumption A.2(ii) is trivially satisfied under Assumption B.1(iii) and B.2(ii)-(iii) and A.2(iii) can be verified under Assumptions B.1(iii) and B.2(ii) by the law of iterated expectations. Assumptions A.4(iii)-(iv) follow because we can show that

$$E_D \left[ \left\| F_\theta' X_k \varepsilon^0 \lambda^0 \right\|^2 \right] = \sum_{1 \leq i,j \leq N} \sum_{1 \leq t,s \leq T} E \left[ E_D ( \varepsilon_{it} \varepsilon_{js}) F_\theta' F_\theta' \lambda^0_j \lambda^0_j \right] = O_P \left( N^2 T^2 (N + T) \right),$$

$$E_D \left[ \left\| \varepsilon_i \right\|^2 \right] = \sum_{j=1}^N \sum_{k=1}^N \sum_{t=1}^T \sum_{s=1}^T E_D \left[ \varepsilon_{it} \varepsilon_{ks} \right] \lambda^0_k \lambda^0_j = O_P \left( (N + T) \right)$$

under Assumptions B.2(i)-(iii) by the use of Davydov inequality for conditional strong mixing processes; see, e.g., the proof of Lemma G.3(vii) below. Finally, Assumptions A.5(i)-(ii) follow under B.1-B.2 by straightforward verification of the moment conditions for the martingale central limit theorem (e.g., Pollard (1984, p.171)).

To prove Corollary 3.4, we need several lemmas.

**Lemma G.2** Suppose that the conditions in Corollary 3.4 hold. Then

(i) $\max_{1 \leq i \leq N} \left[ T^{-1} \sum_{t=1}^T \left[ \varepsilon_{it}^2 - E_D ( \varepsilon_{it}^2) \right] \right] = O_P (\alpha_{NT})$;

(ii) $\max_{1 \leq i \leq N} T^{-1} \left\| \varepsilon_i \right\|^2 = O_P (1)$;

(iii) $\max_{1 \leq i \leq N} T^{-1} F_\theta' \varepsilon_i = O_P (\alpha_{NT})$;

(iv) $\max_{1 \leq i \leq N} T^{-1} X_i (1) = O_P (\alpha_{NT})$;

(v) $\max_{1 \leq i \leq N} T^{-1} X_i (1) = O_P (1)$;

(vi) $\max_{1 \leq i \leq N} \lambda^+_i = O_P \left( N^{1/(8+4\sigma)} \right)$;

(vii) $\max_{1 \leq i \leq N} T^{-1} \left\| X_i (1) - E_D (X_i (1)) \right\| F_\theta^0 = O_P (\alpha_{NT})$;

where $\alpha_{NT} = \max \{ (NT)^{1/2} \log (NT) / T, (\log (NT) / T)^{1/2} \}$.

**Proof.** (i) The proof is analogous to that of Lemma A.7(iii) in Su and Chen (2013) by using Bernstein inequality for conditional strong mixing processes (see, e.g., Lemma A.4 in Su and Chen (2013)).

(ii) $\max_{1 \leq i \leq N} T^{-1} \left\| \varepsilon_i \right\|^2 \leq \max_{1 \leq i \leq N} T^{-1} \sum_{t=1}^T \left[ \varepsilon_{it}^2 - E_D ( \varepsilon_{it}^2) \right] + \max_{1 \leq i \leq N} T^{-1} \sum_{t=1}^T E_D ( \varepsilon_{it}^2) = O_P (\alpha_{NT}) + O_P (1) = O_P (\alpha_{NT})$ by (i) and Assumption B.1(iv).

(iii) The proof is analogous to that of (i).

(iv) The proof is analogous to that of (i).

(v) Following the proof of (i), we can show that $\max_{1 \leq i \leq N} T^{-1} \sum_{t=1}^T X'_i (1) X'_i (1) - E_D (X'_i (1) X'_i (1)) = O_P (\alpha_{NT})$. Then the result follows from this, Assumption B.1(iv) and the triangle inequality.15

(vi) By Boole and Markov inequalities, for any $\epsilon > 0$ we have

$$P \left( \max_{1 \leq i \leq N} \left\| \lambda_i^0 \right\| \geq N^{1/(8+4\sigma)} \epsilon \right) \leq N \max_{1 \leq i \leq N} P \left( \left\| \lambda^0_i \right\| \geq N^{1/(8+4\sigma)} \epsilon \right) \leq \epsilon^{-8+4\sigma} \max_{1 \leq i \leq N} E \left\{ \left\| \lambda^0_i \right\|^{8+4\sigma} \right\} \leq 0.$$  

---

15Alternatively, under Assumption B.1(iii), the fourth moment of $T^{-1} \left\| X_i (1) \right\|^2 = T^{-1} \sum_{t=1}^T X'_i (1) X'_i (1)$ is finite. Then following the proof of (vi) below, we have $\max_{1 \leq i \leq N} T^{-1} \left\| X_i (1) \right\|^2 = o_P \left( N^{1/4} \right)$, a rough bound that also suffices for our purpose, but stringent conditions are required on the relative rates at which $N$ and $T$ pass to infinity.
where the last line follows from Assumption B.1(ii) and the dominated convergence theorem. It follows that \( \max_{1 \leq i \leq N} \| \lambda_i^0 \| = o_p(N^{1/(8+4\alpha)}) \). The conclusion follows as one can write \( \lambda_i^*(1) = H_{(1)}^{-1} \lambda_i^0 \) and \( H(1) \) is asymptotically nonsingular.

(vii) The proof is analogous to that of (i).

Let \( \Psi_{NT} = \text{diag}(\psi_{1T}, ..., \psi_{NT}) \) and \( \psi_{IT} = T^{-1} \sum_{t=1}^{T} E_D [\varepsilon_{it}^2] \). Recall \( \hat{\Psi}_{NT} = \text{diag}(\hat{\psi}_{1T}, ..., \hat{\psi}_{NT}) \) and \( \hat{\psi}_{IT} = T^{-1} \sum_{t=1}^{T} \hat{\varepsilon}_{it}^2 \), where \( \hat{\varepsilon}_i = (\hat{\varepsilon}_{i1}, ..., \hat{\varepsilon}_{iT})' \).

**Lemma G.3** Suppose that the conditions in Corollary 3.4 hold. Then

(i) \( \lambda^0 \Psi_{NT} \lambda^0 = O_p(N) \);

(ii) \( \lambda^0 (T^{-1} \varepsilon \varepsilon' - \Psi_{NT}) \lambda^0 = O_p \left( N^{1/2} + NT^{-1/2} \right) \);

(iii) \( \| \Psi_{NT} - \Psi_{NT} \|_s = \max_{1 \leq i \leq N} \| \psi_{IT} - \hat{\psi}_{IT} \| = o_p \left( \delta^{-1}_{NT} N^{1/(8+4\alpha)} \right) \);

(iv) \( NT^{-1} \left\| \hat{F}(1) \hat{\Psi}_{NT} \hat{F}(1)' - F^0 \lambda^0 \Psi_{NT} \lambda^0 F^0 \right\| = o_p \left( \delta^{-1}_{NT} N^{1/(8+4\alpha)} \right) \);

(v) \( NT^{-1} (\hat{\Psi}_{NT} X_k \hat{F}(1) - H^+ \lambda^0 \Psi_{NT} X_k F^*) = O_p (\delta^{-1}_{NT}) \);

(vi) \( N^{-3} \| \lambda^0 (T^{-1} \varepsilon \varepsilon' - \Psi_{NT}) X_k \|^2 = O_p \left( N^{-1} + T^{-1} \right) \) for \( k = 1, ..., K_0 \).

**Proof.** (i) By Assumption B.1(ii) and (iv) \( \lambda^0 \Psi_{NT} \lambda^0 \leq \max_{1 \leq i \leq N} \psi_{IT} \| \lambda^0 \|^2 = O_p (1) O_p (N) = O_p (N) \).

(ii) Under Assumption B.1(iv), \( T^{-1} E_D [\lambda^0 \varepsilon \varepsilon' \lambda^0] = \sum_{i=1}^{N} \lambda_i^0 \lambda_i^0 T^{-1} \sum_{t=1}^{T} E_D [\varepsilon_{it}^2] = \lambda^0 \Psi_{NT} \lambda^0 \). Let \( \Xi_{NT} = c'_R \left( T^{-1} \lambda^0 \varepsilon \varepsilon' \lambda^0 - \lambda^0 \Psi_{NT} \lambda^0 \right) c_R \). Then \( E_D (\Xi_{NT}) = 0 \). Noting that by Assumptions B.2(ii)-(iii) we have

\[
T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} E_D [\varepsilon_{it}^2 \varepsilon_{js}^2] = \begin{cases} \psi_{IT} \psi_{KT} & \text{if } i = j \neq k = l \\ T^{-2} \sum_{t=1}^{T} E_D [\varepsilon_{it}^2 \varepsilon_{jt}^2] & \text{if } i = k \neq j \text{ or } i = l \neq j = k \\ T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} E_D [\varepsilon_{it}^2 \varepsilon_{js}^2] & \text{if } i = j = k = l \\ 0 & \text{otherwise} \end{cases}
\]

(G.3)

we have

\[
\text{Var}_D (\Xi_{NT}) = E_D \left[ \left( c'_R \left( T^{-1} \lambda^0 \varepsilon \varepsilon' \lambda^0 c_R \right) \right)^2 \right] - \left[ c'_R \lambda^0 \Psi_{NT} \lambda^0 c_R \right]^2 = T^{-2} E_D \left[ \varepsilon \varepsilon' \lambda^0 \varepsilon \varepsilon' \lambda^0 \right] - \left[ c'_R \lambda^0 \Psi_{NT} \lambda^0 c_R \right]^2
\]

\[
= T^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} c'_R \lambda_i^0 \lambda_j^0 \varepsilon_i c_R c'_R \lambda_k^0 \lambda_l^0 \varepsilon_s c_R E_D [\varepsilon_{it} \varepsilon_{js} \varepsilon_{ks} \varepsilon_{ls}] - \left[ \sum_{i=1}^{N} c'_R \lambda_i^0 \lambda_i^0 \varepsilon_i c_R \psi_{IT} \right]^2
\]

\[
= T^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( c'_R \lambda_i^0 \lambda_j^0 \varepsilon_i c_R \right)^2 E_D [\varepsilon_{it}^2 \varepsilon_{is}^2]
\]

\[
+ T^{-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} c'_R \lambda_i^0 \lambda_j^0 \varepsilon_i c_R \left[ c'_R \lambda_i^0 \lambda_j^0 \varepsilon_i c_R + c'_R \lambda_i^0 \lambda_j^0 \varepsilon_i c_R \right] E_D [\varepsilon_{it}^2 \varepsilon_{js}^2] = O_p (N + N^2 T^{-1})
\]

It follows that \( T^{-1} \lambda^0 \varepsilon \varepsilon' \lambda^0 - \lambda^0 \Psi_{NT} \lambda^0 = O_p \left( N^{1/2} + NT^{-1/2} \right) \) by conditional Chebyshev inequality.

(iii) Noting that \( \hat{\xi}_i = M_{\hat{F}(1)} (Y_i - X_i(1) \hat{\beta}(1)) \) and \( Y_i - X_i(1) \hat{\beta}(1) = (X_i(1) \beta(1) + F^* \lambda_i^* (1) + \xi_i) - X_i(1) \hat{\beta}(1) = \hat{F}(1) \lambda_i^* (1) + \xi_i \) where \( \xi_i = \varepsilon_i + X_i(1) (\beta(1) - \hat{\beta}(1)) + (F^* - \hat{F}(1)) \lambda_i^* (1) \), we have

\[
\hat{\xi}_i - \xi_i = M_{\hat{F}(1)} \varepsilon_i - \xi_i = -P_{\hat{F}(1)} \varepsilon_i + M_{\hat{F}(1)} \left[ X_i(1) (\beta(1) - \hat{\beta}(1)) + (F^* - \hat{F}(1)) \lambda_i^* (1) \right]
\]

\[
\xi_i - \xi_i = M_{\hat{F}(1)} \xi_i - \xi_i = -P_{\hat{F}(1)} \xi_i + M_{\hat{F}(1)} \left[ X_i(1) (\beta(1) - \hat{\beta}(1)) + (F^* - \hat{F}(1)) \lambda_i^* (1) \right]
\]

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It follows that
\[
\hat{\psi}_{IT} - \psi_{IT} = T^{-1} \left( \hat{e}_i - e_i \right) + T^{-1} \sum_{t=1}^T \left[ \varepsilon^2_H - E_D(\varepsilon^2_H) \right]
\]
\[
= T^{-1} \left( \hat{e}_i M^*_{F(i)} e_i - e_i \right) + T^{-1} \sum_{t=1}^T \left[ \varepsilon^2_H - E_D(\varepsilon^2_H) \right]
\]
\[
= -T^{-1} \hat{e}_i P^*_{F(i)} e_i + T^{-1} \left( \beta(0) - \hat{\beta}(1) \right)' X_{i(1)} M^*_{F(i)} X_{i(1)} (\beta(0) - \hat{\beta}(1))
+ T^{-1} \lambda^*_i (F^*_{(1)} - \hat{F}_{(1)}) M^*_{F(i)} (F^*_{(1)} - \hat{F}_{(1)}) \lambda^*_i + 2T^{-1} \hat{e}_i M^*_{F(i)} X_{i(1)} (\beta(0) - \hat{\beta}(1))
+ 2T^{-1} \hat{e}_i M^*_{F(i)} (F^*_{(1)} - \hat{F}_{(1)}) \lambda^*_i + 2(\beta(0) - \hat{\beta}(1))' T^{-1} X_{i(1)} M^*_{F(i)} (F^*_{(1)} - \hat{F}_{(1)}) \lambda^*_i
+ T^{-1} \sum_{t=1}^T \left[ \varepsilon^2_D - E_D(\varepsilon^2_D) \right]
\]
\[
= -D_{\psi_{IT,1}} + D_{\psi_{IT,2}} + D_{\psi_{IT,3}} + 2D_{\psi_{IT,4}} + 2D_{\psi_{IT,5}} + 2D_{\psi_{IT,6}} + D_{\psi_{IT,7}}, \text{ say.}
\]
We bound each term in the last expression in order. First, by Lemmas A.2(vi) and G.2(iii),
\[
\max_{1 \leq i \leq N} \left| D_{\psi_{IT,1}} \right| = \max_{1 \leq i \leq N} \left| T^{-1} \hat{e}_i P^*_{F(i)} - P_{F(i)} \right| \leq \left( P_{F(i)} - P_{F(i)} \right) \max_{1 \leq i \leq N} T^{-1} \left| \varepsilon_i \right| \max_{1 \leq i \leq N} T^{-1} \left| X_i \right| = O_P \left( \delta_{NT}^2 \right) O_P \left( 1 \right) + O_P \left( 1 \right) O_P \left( \alpha_{NT}^2 \right) = O_P \left( \delta_{NT}^2 \right).
\]
By Theorem 3.3 and Lemma G.2(v),
\[
\max_{1 \leq i \leq N} \left| D_{\psi_{IT,2}} \right| \leq \left( \beta(0) - \hat{\beta}(1) \right) \max_{1 \leq i \leq N} T^{-1} \left| X_i \right| = O_P \left( \delta_{NT}^2 \right) O_P \left( 1 \right) = O_P \left( \delta_{NT}^2 \right).
\]
By Lemmas A.2(i) and G.2(vi),
\[
\max_{1 \leq i \leq N} \left| D_{\psi_{IT,3}} \right| \leq T^{-1} \left( F_{(1)}^* - \hat{F}_{(1)} \right) \max_{1 \leq i \leq N} \left( \lambda_i^* \right) \max_{1 \leq i \leq N} T^{-1} \left| X_i \right| = O_P \left( \delta_{NT}^2 \right) O_P \left( N^{1/(4+2\sigma)} \right) = O_P \left( \delta_{NT}^{-2} N^{1/(4+2\sigma)} \right).
\]
Using \( M_{F(i)} = I_T - P_{F(0)} - (P_{F(i)} - P_{F(0)}) \) and \( P_{F(0)} = P_{F(i)} \), we have that by Lemmas A.2(vi) and G.2(ii)-(v) and Assumption B.1(i),
\[
\max_{1 \leq i \leq N} \left| D_{\psi_{IT,4}} \right| \leq \max_{1 \leq i \leq N} \left( \beta(0) - \hat{\beta}(1) \right) \max_{1 \leq i \leq N} T^{-1} \left( e_i X_i \right) \left( \beta(0) - \hat{\beta}(1) \right) \max_{1 \leq i \leq N} T^{-1} \left( e_i P_{F(0)} X_i \right) \left( \beta(0) - \hat{\beta}(1) \right) \max_{1 \leq i \leq N} T^{-1} \left( e_i P_{F(0)} X_i \right) \left( \beta(0) - \hat{\beta}(1) \right)
+ \max_{1 \leq i \leq N} \left( \max_{1 \leq i \leq N} T^{-1} \left( e_i X_i \right) \right) \max_{1 \leq i \leq N} T^{-1} \left( e_i P_{F(0)} X_i \right) \left( \beta(0) - \hat{\beta}(1) \right)
\]
\[
\leq \left( \beta(0) - \hat{\beta}(1) \right) \left( \max_{1 \leq i \leq N} T^{-1} \left( e_i X_i \right) \right) \left( \max_{1 \leq i \leq N} T^{-1} \left( e_i P_{F(0)} X_i \right) \right)
+ \left( \max_{1 \leq i \leq N} T^{-1} \left( e_i X_i \right) \right) \max_{1 \leq i \leq N} T^{-1} \left( e_i P_{F(0)} X_i \right) \left( \beta(0) - \hat{\beta}(1) \right)
\]
\[
\leq O_P \left( \delta_{NT}^2 \right) \left( O_P \left( \alpha_{NT} \right) O_P \left( \alpha_{NT} \right) O_P \left( \alpha_{NT} \right) O_P \left( \alpha_{NT} \right) O_P \left( \alpha_{NT} \right) \right) \leq O_P \left( \delta_{NT}^{-2} \right) O_P \left( \alpha_{NT} + \delta_{NT}^{-1} \right).
\]
Using \( M_{F(i)} = I_T - P_{F(i)} = I_T - P_{F(0)} + (P_{F(0)} - P_{F(i)}) \), the fact that \( P_{F(0)} = P_{F(i)} \), and \( P_{F(0)} = F(0)^{-1} F(0) \),
we have that by Lemmas A.2(i) and (vi) and G.2(ii)-(v) and Assumption B.1(i),

\[
\max_{1 \leq i \leq N} \|Dy_{i,T,5}\| \leq \max_{1 \leq i \leq N} \left| T^{-1} \varepsilon_i (F_i^* - \hat{F}_i) \lambda_i^* \right| + \max_{1 \leq i \leq N} \left| T^{-1} \varepsilon_i P_{\psi_0}(F_i^* - \hat{F}_i) \lambda_i^* \right| \\
+ \max_{1 \leq i \leq N} T^{-1} \left| \varepsilon_i (P_{\psi_i}(F_i^* - \hat{F}_i) \lambda_i^* \right| \\
\leq \max_{1 \leq i \leq N} T^{-1/2} \|\varepsilon_i\| T^{-1/2} \left\| F_i^* - \hat{F}_i \right\| \left\| \lambda_i^* \right\| \\
+ [\mu_{\min} (T^{-1} F_0^* F_0^*)]^{-1} \max_{1 \leq i \leq N} T^{-1} \left| \varepsilon_i F_0^* \left\| F_i^* - \hat{F}_i \right\| \left\| F_0^* \right\| \max_{1 \leq i \leq N} \left\| \lambda_i^* \right\| \\
= O_P(1) O_P(\delta_{NT}^{-1}) O_P\left( N^{1/(8+4\sigma)} \right) + O_P(1) O_P(\alpha_{NT}) O_P(\delta_{NT}^{-2}) O_P\left( N^{1/(8+4\sigma)} \right) \\
+ O_P(1) O_P(\delta_{NT}^{-1}) O_P(\delta_{NT}^{-1}) O_P\left( N^{1/(8+4\sigma)} \right) \\
= O_P(\delta_{NT}^{-1} N^{1/(8+4\sigma)}).
\]

By Cauchy-Schwarz inequality, \(\max_{1 \leq i \leq N} |Dy_{i,T,6}| \leq \left\{ \max_{1 \leq i \leq N} |Dy_{i,T,2}|, \max_{1 \leq i \leq N} |Dy_{i,T,3}| \right\}^{1/2} = o_P(\delta_{NT}^{-2} N^{1/(8+4\sigma)})\). Lastly, \(\max_{1 \leq i \leq N} |Dy_{i,T,7}| = O_P(\alpha_{NT})\) by Lemma G.2(i). Consequently, (iii) follows.

(iv) Observe that

\[
(NT)^{-1} \left( \hat{F}_i (\hat{\lambda}_i^*) \hat{\psi}_{NT} \lambda_i^* \right) - F_0^* \lambda^0 \Psi_{NT} \lambda^0 F_0^* \\
= (NT)^{-1} \left( \hat{F}_i (\hat{\lambda}_i^*) - F_0^* \lambda^0 \right) \Psi_{NT} \lambda^0 F_0^* + (NT)^{-1} F_0^* \lambda^0 \Psi_{NT} \left( \hat{F}_i (\hat{\lambda}_i^*) - F_0^* \lambda^0 \right) \\
+ (NT)^{-1} \left( \hat{F}_i (\hat{\lambda}_i^*) - F_0^* \lambda^0 \right) \Psi_{NT} \hat{F}_i (\hat{\lambda}_i^*) - F_0^* \lambda^0 \Psi_{NT} \hat{F}_i (\hat{\lambda}_i^*) \\
= I + II + III + IV, \text{ say.}
\]

By the facts that

\[
\left\| \hat{F}_i (\hat{\lambda}_i^*) - F_0^* \lambda^0 \right\| \leq \left\| \hat{F}_i - F_0^* \right\| \left\| \hat{\lambda}_i^* \right\| + \left\| \hat{F}_i \right\| \left\| \lambda_i^* \right\| \\
\leq \left\| \hat{F}_i - F_0^* \right\| \left\| \lambda_i^* \right\| + \left\| \hat{F}_i \right\| \left\| \lambda_i^* \right\| \\
= O_P\left( N^{1/2 T^{1/2} / \delta_{NT}^{-1}} \right) + O_P\left( N^{1/2 T^{1/2} / \delta_{NT}^{-1}} \right)
\]

by Lemma A.2(i) and Theorem 3.1, \(\|A\| \leq \text{rank}(A) \|A\|_{sp}\), and that \(\|A\|_{sp} \leq \|A\|\), we have

\[
\|I\| \leq R_0 (NT)^{-1} \max_{1 \leq i \leq N} \|\hat{\psi}_{iT}\| \left\| \hat{F}_i \right\| \left\| \lambda_i^* \right\| \left\| F_0^* \right\| \\
= (NT)^{-1} O_P(1) O_P\left( N^{1/2 T^{1/2} / \delta_{NT}^{-1}} \right) + O_P\left( N^{1/2 T^{1/2} / \delta_{NT}^{-1}} \right) = O_P(\delta_{NT}^{-1}).
\]

Similarly, \(\|II\| = \|I\| = O_P(\delta_{NT}^{-1})\) and \(\|III\| = O_P(\delta_{NT}^{-2})\). In addition, \(\|IV\| \leq R_0 (NT)^{-1} \|\hat{F}_i \| \|\hat{\lambda}_i^*\|\|^2\max_{1 \leq i \leq N} |\hat{\psi}_{iT} - \psi_{iT}| = o_P(\delta_{NT}^{-1} N^{1/(8+4\sigma)}), \text{ by (iii)}.\) Hence (vi) follows.

(v) Observe that \( (NT)^{-1} \left( \hat{\lambda}_i^* \hat{\psi}_{NT} \lambda_i^* \right) - H^0 \lambda^0 \psi_{NT} \lambda_i^* \phi_{F_i} = (NT)^{-1} \left( \hat{\lambda}_i^* - \lambda_0^0 H^0 \psi_{NT} \lambda_i^* \phi_{F_i} + \right) (NT)^{-1} \hat{\lambda}_i^* \hat{\psi}_{NT} \lambda_i^* \phi_{F_i} = (NT)^{-1} \left( \hat{\lambda}_i^* - \lambda_0^0 H^0 \psi_{NT} \lambda_i^* \phi_{F_i} + \right) (NT)^{-1} \hat{\lambda}_i^* \psi_{NT} \phi_{F_i} = J_1 + J_2 + J_3, \text{ say.}\) By Theorem 3.1 and Lemmas A.2(i) and G.3(iii), we can show that \(J_s = O_P(\delta_{NT}^{-1})\) for \(s = 1,2,3\). For example,

\[
\|J_1\| \leq (NT)^{-1} \max_{1 \leq i \leq N} \|\hat{\psi}_{iT}\| \|\hat{\lambda}_i^* - \lambda_0^0 H^0 \| \|\phi_{F_i}\| \|\phi_{F_i}\| \\
= (NT)^{-1} O_P(1) O_P(\delta_{NT}^{-1} N^{1/2 T^{1/2}} / \delta_{NT}^{-1} T^{1/2}) = O_P(\delta_{NT}^{-1}).
\]
It follows that $(NT)^{-1} (\lambda' \hat{\Psi}_{NT} X_k \hat{F} (1) - H + \lambda^0 \Psi_{NT} X_k F^* (1)) = O_P (\delta^{-1}_{NT})$.

(vi) Let $ED \equiv N^{-3} E_D \| \lambda^0 (T^{-1} \varepsilon - \Psi_{NT}) X_k \|^2$. It suffices to show that $ED = O_P (N^{-1} + T^{-1})$. Note that

$$ED = N^{-3} E_D \{ \lambda^0 (T^{-1} \varepsilon - \Psi_{NT}) X_k (T^{-1} \varepsilon - \Psi_{NT}) \lambda^0 \} = N^{-3} T^{-2} \sum_{1 \leq i, j, l, m \leq N} \sum_{1 \leq t, n, r \leq T} \zeta_{ijlm, nsr} \lambda^0_i \lambda^0_l.$$

where $\zeta_{ijlm, nsr} = E_D \{ [\varepsilon_i \varepsilon_j - E_D (\varepsilon_i \varepsilon_j)] [\varepsilon_l \varepsilon_r - E_D (\varepsilon_l \varepsilon_r)] X_{ks,k} X_{ls,k} \}$. We consider four cases for the individual indices $\{i, j, l, m\}$: (a) $\# \{i, j, l, m\} = 4$, (b) $\# \{i, j, l, m\} = 3$, (c) $\# \{i, j, l, m\} = 2$, and (d) $\# \{i, j, l, m\} = 1$. We use $ED_a, ED_b, ED_c, \text{ and } ED_d$ to denote $ED$ when the individual indices in the summation are restricted to cases (a), (b), (c), and (d), respectively. In case (a), we can readily verify that $\zeta_{ijlm, nsr} = E_D [\varepsilon_i \varepsilon_j] E_D [\varepsilon_l \varepsilon_r] E_D [\varepsilon_{js,k} X_{ks,k}] E_D [\varepsilon_{ls,k}] E_D [\varepsilon_{mr}] = 0$ by Assumption B.2(ii)-(iii) when $\# \{i, j, l, m\} = 4$. In case (b), we consider three subcases: (b1) $i = j$, (b2) $i = l$, and (b3) $i = m$ as the other cases can be analyzed analogously, and write the corresponding summations as $ED_{b1}, ED_{b2}, \text{ and } ED_{b3}$, respectively. In subcase (b1), $\zeta_{ijlm, nsr} = E_D \{ [\varepsilon_i^2 - E_D (\varepsilon_i^2)] X_{js,k} \} E_D (\varepsilon_l \varepsilon_r) E_D (\varepsilon_{is,l} X_{ks,k}) E_D (\varepsilon_{mr}) = 0$ by Assumption B.2(ii)-(iii) and thus $ED_{b1} = 0$. In subcase (b2) $\zeta_{ijlm, nsr} = E_D (\varepsilon_i \varepsilon_r) E_D (\varepsilon_j \varepsilon_{js,k}) E_D (\varepsilon_{ls,k})$ by Assumption B.2(ii). In view of the facts that $E_D (\varepsilon_i \varepsilon_{ir}) = 0$ if $t \neq r$, $E_D (\varepsilon_j X_{js,k}) = 0$ if $t \geq s$, and that $E_D (\varepsilon_{is,l} X_{ks,k}) = 0$ if $r \geq s$ by Assumption B.2(ii), we have

$$ED_{b3} = N^{-3} T^{-2} \sum_{1 \leq i \neq j \leq l \neq m \leq N} \sum_{1 \leq t, n, r \leq T} E_D (\varepsilon_i \varepsilon_j) E_D (\varepsilon_j X_{js,k}) E_D (\varepsilon_{is,l} X_{ks,k}) \| \lambda^0_i \|^2.$$

By the fact that $ED (\varepsilon_j) = 0$ and Davydov inequality for conditional strong mixing processes (e.g., Su and Chen (2013, Lemma A.3)), we have $|ED (\varepsilon_j X_{js,k})| \leq 8 \| \varepsilon_j \|_{s_4, \mathcal{D}} \| X_{js,k} \|_{s_4, \mathcal{D}} \alpha_D (t-s)^{(3+2\sigma)/(4+2\sigma)}$ for any $t < s$ and any $j$. It follows that

$$|ED_{b3}| \leq 64 N^{-1} T^{-2} \sum_{i=1}^N \| \lambda^0_i \|^2 \sum_{1 \leq t < s \leq T} E_D (\varepsilon_i^2) \left\{ \sum_{j=1}^N \| \varepsilon_j \|_{s_4, \mathcal{D}} \| X_{js,k} \|_{s_4, \mathcal{D}} \right\}^2 \alpha_D (s-t)^{(3+2\sigma)/(2+\sigma)}$$

$$\leq 64 c_{1, NT} \left\{ N^{-1} T^{-2} \sum_{i=1}^N \sum_{t=1}^T E_D (\varepsilon_i^2) \| \lambda^0_i \|^2 \sum_{j=1}^N \| \varepsilon_j \|_{s_4, \mathcal{D}} \| X_{js,k} \|_{s_4, \mathcal{D}} \right\} \sum_{t=1}^T \alpha_D (t)^{(3+2\sigma)/(2+\sigma)}$$

$$= O_P (1) O_P (T^{-1}) O_P (1) = O_P (T^{-1}),$$

where we use the facts that $c_{1, NT} \equiv \max_{1 \leq t, s \leq T} N^{-1} \sum_{j=1}^N \| \varepsilon_j \|_{s_4, \mathcal{D}} \| X_{js,k} \|_{s_4, \mathcal{D}} = O_P (1)$ by Assumption B.1(v) and Cauchy-Schwarz inequality and that $\sum_{t=1}^\infty \alpha_D (t)^{(3+2\sigma)/(2+2\sigma)} < \infty$ by Assumption B.2(i). It follows that $ED_{b} = O_P (T^{-1})$.

Now we consider case (c). We consider three subcases: (c1) $i = j \neq l = m$, (c2) $i = l \neq j = m$, and (c3) $i = m \neq j = l$, and use $ED_{c1}, ED_{c2}$ and $ED_{c3}$ to denote $ED$ when the individual indices in its summation are restricted to these three subcases respectively. By Davydov inequality, we have

$$|ED_{c1}| = N^{-3} T^{-2} \sum_{1 \leq i \neq l \leq N} \sum_{1 \leq t, s, r \leq T} E_D \{ [\varepsilon_i^2 - E_D (\varepsilon_i^2)] [\varepsilon_l^2 - E_D (\varepsilon_l^2)] X_{is,k} X_{ls,k} \} \lambda^0_i \lambda^0_l \leq N^{-3} T^{-2} \sum_{1 \leq i \neq l \leq N} \sum_{1 \leq t, s, r \leq T} |E_D \{ [\varepsilon_i^2 - E_D (\varepsilon_i^2)] X_{is,k} \} | \| E_D \{ [\varepsilon_l^2 - E_D (\varepsilon_l^2)] X_{is,k} \} \| \| \lambda^0_i \| \| \lambda^0_l \| + O_P (N^{-1}).$$
\[
\begin{align*}
&\leq 64N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq s \neq r \leq T} \| \varepsilon_i^2_{ir} \|_{4+2\sigma, D} \| X_{is,k} \|_{4+2\sigma, D} \| \varepsilon_j^2_{jr} \|_{4+2\sigma, D} \| X_{js,k} \|_{4+2\sigma, D} \| \lambda_i^0 \| \| \lambda_j^0 \| \\
&\times \alpha_D ((s-t)^{(1+\sigma)/(2+\sigma)}) \alpha_D ((s-r)^{(1+\sigma)/(2+\sigma)}) + O_P (N^{-1}) \\
&\leq 8c_{2NT} \max_{i,s} \sum_{1 \leq i \neq j \leq N} \| \varepsilon_i^2_{ir} \|_{4+2\sigma, D} \| X_{is,k} \|_{4+2\sigma, D} \left( \sum_{r=1}^T \alpha_D (r)^{(1+\sigma)/(2+\sigma)} \right)^2 + O_P (N^{-1}) \\
&= O_P (1) o_P \left((NT)^{1/(4+2\sigma)}\right) O_P (N^{-1}-1) + O_P (N^{-1}) = o_P \left((NT)^{-1+1/(4+2\sigma)}\right) + O_P (N^{-1}),
\end{align*}
\]
where we use the facts that \(c_{2NT} \equiv \max_{1 \leq r, s \leq T} N^{-1} \sum_{i=1}^N \| \varepsilon_i^2_{ir} \|_{4+2\sigma, D} \| X_{is,k} \|_{4+2\sigma, D} \| \lambda_i^0 \| = O_P (1)\) by Assumption B.1(v) and Cauchy-Schwarz inequality, that \(\max_{i,t} \| \varepsilon_i^2_{it} \|_{4+2\sigma, D} = o_P ((NT)^{1/(4+2\sigma)})\) as \(E[\| \varepsilon_i^2_{it} \|_{4+2\sigma, D}]^{1+2\sigma} \leq E[|\varepsilon_i|^{8+4\sigma}] < \infty\) by Assumption B.1(iii), and that \(\sum_{r=1}^T \alpha_D (r)^{(1+\sigma)/(2+\sigma)} = O_P (1)\) by Assumption B.2(i). Next, noting that \(E[\varepsilon_i \varepsilon_{ir}] = 0\) if \(t \neq r\) and \(E[\varepsilon_i \varepsilon_{ir} X_{is,k}] = 0\) of \(t \neq r\) and \(\max (t, r) \geq s\) by Assumption B.2(iii), we apply Davydov inequality to obtain

\[
|ED_{c2}| = N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} ED (\varepsilon_i \varepsilon_{ir} X_{is,k}) ED (\varepsilon_j \varepsilon_{jr} X_{js,k}) \lambda_j^0 \lambda_i^0.
\]

\[
\leq N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} \| ED (\varepsilon_i \varepsilon_{ir} X_{is,k}) \| \| ED (\varepsilon_j \varepsilon_{jr} X_{js,k}) \| \lambda_j^0 \lambda_i^0
\]

\[
\leq 2N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} \| ED (\varepsilon_i \varepsilon_{ir} X_{is,k}) \| \| ED (\varepsilon_j \varepsilon_{jr} X_{js,k}) \| \lambda_j^0 \lambda_i^0 + O_P (N^{-1})
\]

\[
\leq 128N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} \| \varepsilon_i \varepsilon_{ir} \|_{4+2\sigma, D} \| X_{is,k} \|_{4+2\sigma, D} \| \varepsilon_j \varepsilon_{jr} \|_{4+2\sigma, D} \| X_{js,k} \|_{4+2\sigma, D} \lambda_j^0 \lambda_i^0 \lambda_j^0 \lambda_i^0
\]

\[
\times \alpha_D ((s-t)^{(5+4\sigma)/(8+4\sigma)}) \alpha_D ((s-r)^{(5+4\sigma)/(8+4\sigma)}) + O_P (N^{-1})
\]

\[
\leq 128c_{3NT} \max_{i} \max_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} \| \varepsilon_i \varepsilon_{ir} \|_{8+4\sigma, D} \| X_{is,k} \|_{8+4\sigma, D} \left( N^{-2}T^{-2} \sum_{r=1}^T \sum_{i=1}^N \| \varepsilon_i \varepsilon_{ir} \|_{8+4\sigma, D} \| \lambda_i^0 \| \right)
\]

\[
\times \left\{ \sum_{r=1}^T \alpha_D (r)^{(5+4\sigma)/(8+4\sigma)} \right\}^2 + O_P (N^{-1})
\]

\[
= O_P (1) o_P \left((NT)^{1/(8+4\sigma)}\right) o_P \left((NT)^{1/(8+4\sigma)}\right) O_P (N^{-1}-1) + O_P (N^{-1}) = o_P \left((NT)^{-1+1/(4+2\sigma)}\right) + O_P (N^{-1}),
\]

where \(c_{3NT} = \max_{t, s, r} N^{-1} \sum_{j=1}^N \| \varepsilon_j \varepsilon_{jr} \|_{8+4\sigma, D} \| x_{js,k} \|_{8+4\sigma, D} \| \lambda_j^0 \| = O_P (1)\) by Assumption B.1(v) and Cauchy-Schwarz inequality. In addition, noting that \(ED (\varepsilon_i \varepsilon_{ir} \varepsilon_{jr} X_{js,k}) = ED (\varepsilon_i \varepsilon_{ir} \varepsilon_{jr} X_{js,k}) ED (\varepsilon_j \varepsilon_{jr} X_{js,k})\) and the last expression is zero if \(t \neq r\) or \(\max (t, r) \geq s\) under Assumptions B.2(iii),

we have \(|ED_{c3}| = N^{-3}T^{-2} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, r, s \leq T} ED (\varepsilon_i^2_{ir} \varepsilon_j^2_{jr} X_{js,k}^2) \| \lambda_i^0 \| \| \lambda_j^0 \|^2 = O_P (N^{-1})\). In sum, \(ED_c = O_P (N^{-1}+T^{-1}) + o_p((NT)^{-1+1/(4+2\sigma)}) = O_P (N^{-1}+T^{-1}).\)

For case (d), we can also consider the application of Davydov inequality when the time indices \((t, s, r)\) are all distinct. Straightforward calculation shows that \(ED_d = o_P \left(N^{-2}(NT)^{-1/(4+2\sigma)}\right) = o_P (N^{-1})\). (Note that one can obtain a rough bound for \(ED_d\) by \(N^{-3}T^{-2}O_P (NT^3) = O_P (N^{-2}T)\) without the need to apply Davydov inequality.) Consequently, we have shown that \(ED = O_P (N^{-1}+T^{-1})\).  

Let \(\Phi_{NT} = \text{diag}(\varphi_{1N}, ..., \varphi_{TN})\) and \(\varphi_{1N} = N^{-1} \sum_{i=1}^N ED (\varepsilon_i^2_{ir})\). Recall \(\Phi_{NT} = \text{diag}(\varphi_{1N}, ..., \varphi_{TN})\) and \(\varphi_{1N} = N^{-1} \sum_{i=1}^N \varepsilon_i^2_{ir}\).
Lemma G.4 Suppose the conditions in Corollary 3.4 hold. Then

(i) \( F^0 (N^{-1} \varepsilon' \epsilon - \Phi_{NT}) F^0 = O_P \left( T^{1/2} + TN^{-1/2} \right) \);

(ii) \( \| \Phi_{NT} - \Phi_{NT} \|_{sp} = \max_{1 \leq i \leq T} | \hat{\gamma}_{i,N} - \varphi_{i,N} | = O_P ( \delta_{NT}^{-1} T^{1/(8+4\sigma)} ) \);

(iii) \( T^{-3} \| X_k (N^{-1} \varepsilon' \epsilon - \Phi_{NT}) F^0 \|^2 = O_P (N^{-1}) + O_P \left( T^{-(7+4\sigma)/(8+4\sigma)} \right) \).

Proof. (i) The proof is analogous to that of Lemma G.3(ii) and thus omitted.

(ii) The proof is analogous to that of Lemma G.3(iii) and thus omitted.

(iii) Note that

\[
T^{-3} E_D \left\| X_k (N^{-1} \varepsilon' \epsilon - \Phi_{NT}) F^0 \right\|^2
= T^{-3} \text{tr} \left[ X_k (N^{-1} \varepsilon' \epsilon - \Phi_{NT}) F^0 F^0' (N^{-1} \varepsilon' \epsilon - \Phi_{NT}) X_k \right]
=
T^{-3} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, s, r, q \leq T} E_D \left\{ [\varepsilon_{jt} \varepsilon_{js} - E_D (\varepsilon_{jt} \varepsilon_{js})] [\varepsilon_{jr} \varepsilon_{jq} - E_D (\varepsilon_{jr} \varepsilon_{jq})] \right\} E_D \left[ X_{it,k} X_{ir,k} \right] F^0_s F^0_r
+ T^{-3} \sum_{i=1}^{N} \sum_{1 \leq t, s, r, q \leq T} E_D \left\{ [\varepsilon_{it} \varepsilon_{is} - E_D (\varepsilon_{it} \varepsilon_{is})] [\varepsilon_{ir} \varepsilon_{iq} - E_D (\varepsilon_{ir} \varepsilon_{iq})] \right\} X_{it,k} X_{ir,k} F^0_s F^0_r
\equiv I + II ,
\]

where the second equality follows because \( \varepsilon_{ij,t,rstq} = E_D \left\{ [\varepsilon_{jt} \varepsilon_{js} - E_D (\varepsilon_{jt} \varepsilon_{js})] [\varepsilon_{jr} \varepsilon_{jq} - E_D (\varepsilon_{jr} \varepsilon_{jq})] X_{it,k} X_{ir,k} \right\} = 0 \) under Assumption B.2(ii)-(iii) if \( \# \{ i,j,k \} = 3 \) or \( j = i \neq l \) or \( l = i \neq j \). To study \( I \), we consider three cases for the time indices \( \{ t, s, r, q \} \) inside the summation: (a) \( \# \{ t, s, r, q \} = 4 \), (b) \( \# \{ t, s, r, q \} = 3 \), and (c) all other cases. We use \( I_a \), \( I_b \), and \( I_c \) to denote \( I \) when the time indices in the summation are restricted to cases (a), (b), and (c), respectively. Apparently, \( I_a = 0 \) and \( I_c = O_P (T^{-1}) \) under Assumptions B.2(ii)-(iii) and B.1(i) and (iii). In case (b) noting that under Assumption B.2(iii) we always have

\[
E_D \left\{ [\varepsilon_{jt} \varepsilon_{js} - E_D (\varepsilon_{jt} \varepsilon_{js})] [\varepsilon_{jr} \varepsilon_{jq} - E_D (\varepsilon_{jr} \varepsilon_{jq})] \right\}
= E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}) - E_D (\varepsilon_{jt} \varepsilon_{js} ) E_D (\varepsilon_{jr} \varepsilon_{jq}) = E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}),
\]

wlog we can assume \( r = q \) and then consider two subcases (b1) \( \max(t,s) > r \) and (b2) \( r > s > t \).

Accordingly, we define \( I_{b1} \) as \( I_b \) but with the time indices restricted to subcase (bs) for \( s = 1, 2 \). Noting that \( E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}) = E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}) = 0 \) in subcase (b1), \( I_{b1} = 0 \). For subcase (b2) we apply the Davydov inequality for conditional strong mixing processes to obtain

\[
| E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}) | \leq 8 \| \varepsilon_{jt} \varepsilon_{js} \|_{4+2\sigma,D} \| \varepsilon_{jr} \|_{4+2\sigma,D} \| \varepsilon_{jq} \|_{4+2\sigma,D} \alpha_D (r-s)^{(1+\sigma)/(2+\sigma)}.\]

Consequently, we have

\[
|I_{b2}| = T^{-3} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, s, r, q \leq T} E_D (\varepsilon_{jt} \varepsilon_{js} \varepsilon_{jr} \varepsilon_{jq}) E_D \left[ X_{it,k} X_{ir,k} \right] F^0_s F^0_r
\leq 8 T^{-3} \sum_{1 \leq i \neq j \leq N} \sum_{1 \leq t, s, r, q \leq T} \| \varepsilon_{jt} \varepsilon_{js} \|_{4+2\sigma,D} \| \varepsilon_{jr} \|_{4+2\sigma,D} \| \varepsilon_{jq} \|_{4+2\sigma,D} \alpha_D (r-s)^{(1+\sigma)/(2+\sigma)} \| E_D \left[ X_{it,k} X_{ir,k} \right] F^0_s F^0_r \|
\leq 8 T^{-3} \sum_{1 \leq i \neq j \leq N} \max_{1 \leq s \leq T} \| F^0_s \|^2 \left\{ T^{-2} N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \| X_{it,k} \|_{2,D} \| X_{ir,k} \|_{2,D} \| F^0_r \|ight\} \sum_{r=1}^{\infty} \alpha_D (r)^{(1+\sigma)/(2+\sigma)}
= T^{-1} O_P \left( T^{1/(8+4\sigma)} \right) O_P (1) O_P (1) = O_P \left( T^{-(7+4\sigma)/(8+4\sigma)} \right),
\]
where we use Lemma G.5(i) below, the fact that 
\[ c_{4NT} = \max_{1 \leq t, s, r \leq T} N^{-1} \sum_{j=1}^{N} \| \varepsilon_{js} \|_{4+\alpha, D} \| \varepsilon_{js} \|_{4+\alpha, D} \times \| \varepsilon_{jr} \|_{4+2\alpha, D} = O_P(1) \]
by Assumption B.1(v) and Cauchy-Schwarz inequality, and the fact that
\[
T^{-2}N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} \| X_{it,k} \|_{2,D} \sum_{r=1}^{T} \| X_{ir,k} \|_{2,D} \| F^0_r \| \\
\leq \frac{1}{2}T^{-1}N^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} E_D \left( X^2_{it,k} \right) + \frac{1}{2}T^{-1}N^{-1} \sum_{i=1}^{N} \sum_{r=1}^{T} E_D \left( X^2_{ir,k} \right) \| F^0_r \| ^2 = O_P(1) + O_P(1) = O_P(1)
\]
by Cauchy-Schwarz and Markov inequalities. It follows that 
\[ I = O_P(T^{- (7 + 4\alpha) / (8 + 4\alpha)}) \]
For II, we consider two cases: (a) \# \{t, s, r, q\} = 4 and (b) \# \{t, s, r, q\} \leq 3, and write II = II_a + II_b where II_b is defined as II but with the time indices restricted to case (l) for \(l = a, b\). Apparently, \(II_b = T^{-3}N^{-2}O_P(NT^2) = O_P(N^{-1})\). For case (a), \(E_D \left[ \| \varepsilon_{it} \|_{4+\alpha} \right] = O_P(1)\) and \(E_D \left( \| X_{it,k} \|_{2,D} \right) = O_P(1)\) for any \(k\), and \(E_D \left( \| X_{it,k} \|_{4+\alpha} \right) = O_P(1)\) for any \(k\). Thus, \(I = O_P(1)\),

**Lemma G.5** Suppose the conditions in Corollary 3.4 hold. Let \( \varepsilon_{it} \equiv (\varepsilon_{11}, \ldots, \varepsilon_{NT}) \) and \( \hat{\varepsilon}_{it} \equiv (\hat{\varepsilon}_{11}, \ldots, \hat{\varepsilon}_{NT}) \). Let \( \bar{X}_{s,k} \) denote the \(k\)th column of the \(N \times T\) matrix \( \bar{X}_k \). Then

- (i) max\(_{1 \leq t \leq T} \| F^0_t \| = O_P(T^{1/(8+4\alpha)})
- (ii) max\(_{1 \leq t \leq T} N^{-1} \| \varepsilon_t \|^2 = O_P(1)
- (iii) max\(_{1 \leq t \leq T} N^{-1} \| X_t \|^2 = O_P(1)
- (iv) max\(_{1 \leq t \leq T} \| X_{it,k} \|^2 = O_P(1)
- (v) max\(_{1 \leq t \leq T} N^{-1} \sum_{s=1}^{N} \| \bar{X}_{s,k} - E_D (\varepsilon_t X_{s,k}) \| = O_P(1)
- (vi) max\(_{1 \leq t \leq T} \| F_t (1) - F^*_t (1) \| = O_P(\eta_{1NT})
- (vii) max\(_{1 \leq t \leq T} \| \hat{F}_t (1) \| = O_P(T^{1/2} \delta^{-1}_{NT}) + O_P(T^{1/(8+4\alpha)})
- (viii) max\(_{1 \leq t \leq T} \| \bar{X}_{s,k} - X_{s,k} \| = O_P(1)
- (ix) max\(_{1 \leq t \leq T} \| \hat{\varepsilon}_t - \varepsilon_t \|^2 = O_P(1)

where \( \eta_{1NT} = O_P(\delta^{-1}_{NT}) + O_P(\alpha_{TN} + N^{-1/2}T^{1/(8+4\alpha)}) \) and \( \alpha_{TN} \) is analogously defined as \( \alpha_{NT} \) by interchanging \( N \) and \( T \).

**Proof.**

- (i) The proof is analogous to that of Lemma G.2(vi).
- (ii) Note that max\(_{1 \leq t \leq T} N^{-1} \| \varepsilon_t \|^2 \leq \max\(_{1 \leq t \leq T} \| X_t \|^2 \) by the proof of Lemma G.2(ii), we can show the first term is \( O_P(\alpha_{TN}) \). The second term is \( O_P(1) \) by Assumption B.1(iv). Thus (ii) follows.
- (iii) The proof is analogous to that of Lemma G.2(ii).
- (iv) The proof is analogous to that of (ii).
- (v) The proof is analogous to that of Lemma G.2(ii).
- (vi) Write \( \hat{F}_t (1) - F^*_t (1) = \sum_{l=1}^{T} a_l(t) \) where \( a_l(t) \) denotes the \(l\)th element of \( a_l(1) \) and recall \( a_l = (a_l(1), a_l(2), \ldots) \). By (i)-(iv), we can readily show that max\(_{1 \leq t \leq T} \| a_l(t) \| = O_P(\delta^{-1}_{NT}) \), max\(_{1 \leq t \leq T} \| a_2(t) \| = O_P(N^{-1/2}T^{1/(8+4\alpha)}) \), max\(_{1 \leq t \leq T} \| a_3(t) \| = O_P(\alpha_{TN}) \), max\(_{1 \leq t \leq T} \| a_4(t) \| = O_P(\alpha_{NT}^{-1}) \), max\(_{1 \leq t \leq T} \| a_5(t) \| = O_P(N^{-1/2}T^{1/(8+4\alpha)}) \), max\(_{1 \leq t \leq T} \| a_6(t) \| = O_P(N^{-1/2}T^{-1/2}) \), max\(_{1 \leq t \leq T} \| a_7(t) \| = O_P(N^{-1/2}T^{1/(8+4\alpha)}) \), and max\(_{1 \leq t \leq T} \| a_8(t) \| = O_P((\alpha_{NT}^{-1} - (N^{1/2} + T^{1/2})) \), where, e.g.,

\[
\max_{1 \leq t \leq T} \| a_1(t) \| = \max_{1 \leq t \leq T} (\alpha_{NT})^{-1} \| \hat{F}_t (1) \| \| \varepsilon_t \| \| \bar{X}_{s,k} - X_{s,k} \| \leq R_0 (\alpha_{NT})^{-1} \| \hat{F}_t (1) \| \| \varepsilon_t \| = (\alpha_{NT})^{-1} \| \bar{X}_{s,k} - X_{s,k} \| = O_P(\alpha_{NT}^{-1})
\]

It follows that \( \max_{1 \leq t \leq T} \| \hat{F}_t (1) - F^*_t (1) \| = O_P(\eta_{1NT}) \).
(vii) By Lemmas A.2(i) and G.5(i), \( \max_i \| \hat{F}_i(1) \| \leq \max_i \| \hat{F}_i(1) - F_i^0 H(1) \| + \max_i \| F_i^0 H(1) \| = O_P(T^{1/2} \delta_{NT}^{-1}) + o_P(T^{1/(8+4\sigma)}) \).

(viii) Noting that \( \hat{\varepsilon}_i = M_{\hat{F}_i(1)}(Y_i - X_i(1)\hat{\beta}_i) \) and \( Y_i - X_i(1)\hat{\beta}_i = (X_i(1)\hat{\beta}_i^0 + F_i^* \lambda^*_i(1) + \varepsilon_i) - X_i(1)\hat{\beta}_i = \hat{F}_i(1)\lambda^*_i(1) + \varepsilon_i \) where \( \varepsilon_i = \varepsilon_i + X_i(1)\hat{\beta}_i^0 - \hat{\beta}_i(1) + (F_i^* - \hat{F}_i(1))\lambda^*_i(1) \), we have

\[
\hat{\varepsilon}_i = \varepsilon_i = M_{\hat{F}_i(1)} \varepsilon_i - \varepsilon_i = -P_{\hat{F}_i(1)} \varepsilon_i + M_{\hat{F}_i(1)} \left[ \varepsilon_i + X_i(1)(\hat{\beta}_i^0 - \hat{\beta}_i(1)) + (F_i^* - \hat{F}_i(1))\lambda^*_i(1) \right].
\]

It follows that

\[
\hat{\varepsilon}_{it} - \varepsilon_{it} = \hat{F}_i'(1) \left( \hat{F}_i(1) \hat{F}_i(1) \right)^{-1} \hat{F}_i(1) \varepsilon_i + X_{it}(1)(\hat{\beta}_i^0 - \hat{\beta}_i(1)) + (F_i^* - \hat{F}_i(1))\lambda^*_i(1)
\]

\[
= \varepsilon_{it} + \hat{\varepsilon}_{it} + \varepsilon_{it} - \hat{\varepsilon}_{it},
\]

Noting that \( \vartheta_{1it} = \hat{F}_i'(1) \hat{F}_i(1) \left( \hat{F}_i(1) - F_i^0 H(1) \right) \varepsilon_i + \hat{F}_i'(1) \hat{F}_i(1) \left( \hat{F}_i(1) - F_i^0 H(1) \right) \varepsilon_i \), we have by Lemmas G.2(ii)-(iii) and G.5(vi)

\[
\max_{i,t} |\vartheta_{1it}| \leq T^{-1} \max_i \| \hat{F}_i(1) \| \left\| \left( T^{-1} \hat{F}_i'(1) \hat{F}_i(1) \right)^{-1} \right\| \left\| \hat{F}_i(1) - F_i^0 H(1) \right\| \max_i \| F_i^0 \varepsilon_i \|
\]

\[
= T^{-1} \left[ O_P \left( T^{1/2} \delta_{NT}^{-1} \right) + o_P \left( T^{1/(8+4\sigma)} \right) \right] O_P(1) \max_i \| \hat{F}_i(1) \| \leq \left[ O_P \left( T^{1/2} \delta_{NT}^{-1} \right) + o_P \left( T^{1/(8+4\sigma)} \right) \right] O_P(1) \max_i \| H(1) \| \leq O_P \left( T^{1/2} \delta_{NT}^{-2} + T^{1/2} \delta_{NT}^{-1} \alpha_{NT} \right) + o_P \left( T^{1/(8+4\sigma)} \right) O_P \left( \delta_{NT}^{-1} + \alpha_{NT} \right) = o_P(1).
\]

Noting that \( \max_{i,t} \| X_{it}(1) \| = O_P \left( (NT)^{1/(8+4\sigma)} \right) \) by Assumption B.1(iii) and Markov inequality, by Theorem 3.1 we have

\[
\max_{i,t} |\vartheta_{2it}| \leq \left\| \beta_i^0(1) - \hat{\beta}_i(1) \right\| \max_{i,t} \| X_{it}(1) \| = O_P \left( \delta_{NT}^{-2} (NT)^{1/(8+4\sigma)} \right) = o_P(1).
\]

By Lemmas A.2(i) and (iv), Lemmas G.2(v)-(vi), and Theorem 3.1, we have

\[
\max_{i,t} |\vartheta_{3it}| \leq \max_i \| F_i^* \| - \hat{F}_i(1) \| \max_i \| \lambda^*_i(1) \|
\]

\[
\leq \left[ O_P \left( \delta_{NT}^{-1} \right) + o_P \left( \alpha_{NT} + N^{-1/2} T^{1/(8+4\sigma)} \right) \right] O_P \left( N^{1/(8+4\sigma)} \right) = o_P(1),
\]

\[
\max_{i,t} |\vartheta_{4it}| \leq \left\| \beta_i^0(1) - \hat{\beta}_i(1) \right\| \max_i \| \hat{F}_i(1) \| \left\| \left( \hat{F}_i'(1) \hat{F}_i(1) \right)^{-1} \hat{F}_i(1) \right\| \max_i \| X_{it}(1) \|
\]

\[
= O_P \left( \delta_{NT}^{-2} \right) \left[ O_P \left( T^{1/2} \delta_{NT}^{-1} \right) + o_P \left( T^{1/(8+4\sigma)} \right) \right] O_P \left( T^{-1/2} \right) O_P \left( T^{1/2} \right) = o_P(1),
\]

and

\[
\max_{i,t} |\vartheta_{5it}| \leq \max_i \| \hat{F}_i(1) \| \left\| \left( \hat{F}_i'(1) \hat{F}_i(1) \right)^{-1} \right\| \left\| \hat{F}_i'(1) (F_i^0 - \hat{F}_i(1)) \right\| \max_i \| \lambda^*_i(1) \|
\]

\[
= \left[ O_P \left( T^{1/2} \delta_{NT}^{-1} \right) + o_P \left( T^{1/(8+4\sigma)} \right) \right] O_P \left( T^{-1} \right) O_P \left( T \delta_{NT}^{-2} \right) o_P \left( N^{1/(8+4\sigma)} \right) = o_P(1).
\]

(ix) \( \max_{1 \leq t \leq T} N^{-1} \| \hat{\varepsilon}_t - \varepsilon_t \| \leq \max_{1 \leq t \leq T} N^{-1} \sum_{i=1}^N (\hat{\varepsilon}_{it} - \varepsilon_{it})^2 \leq \max_{1 \leq i \leq N} \max_{1 \leq t \leq T} (\hat{\varepsilon}_{it} - \varepsilon_{it})^2 \)

\( = o_P(1) \) by (viii).
H Proof of Corollary 3.4

By Theorem 3.3 and the fact that $\tilde{D}_{\bar{E}_1} = D_{F_0} + o_p(1)$, it suffices to show that $\tilde{\Theta}_{NT} = \Theta_{NT} = o_p(1)$ for $l = 1, 2, 3, 4$, and $\tilde{\Theta}_{NT} = \Theta_{NT} = o_p(1)$.

First, we show (i) $\tilde{\Theta}_{NT} = \Theta_{NT} = o_p(1)$. Let $\tilde{\Theta}_{NT} = N^{-5/2}T^{-3/2} \sum_{i=1}^N \omega_{i1} X_i^*(\tilde{\psi} X_i^* - \Psi_{NT}) \lambda_i^* \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i$. We prove (i) by showing that (i1) $\tilde{\Theta}_{NT} = \Theta_{NT} = o_p(1)$ and (i2) $\tilde{\Theta}_{NT} = \Theta_{NT} = o_p(1)$. To show (i1), let $c_{K0} = (c_{K_01}, ..., c_{K_0K_0})'$ be an arbitrary $K_0 \times 1$ nonrandom vector such that $\|c_{K0}\| = 1$. By Lemma G.3(iii), we have

$$|c_{K0} (\tilde{\Theta}_{NT} - \Theta_{NT})|$$

$$= N^{-5/2}T^{-3/2} \sum_{k=1}^{K_0} c_{K_0k} \Psi_{NT}(\tilde{\psi} X_i^* - \Psi_{NT}) \lambda_i^* \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i \Psi_{NT} \lambda_i^* \bar{F}_i^* \tilde{f}_i \tilde{f}_i$$

$$\leq N^{-5/2}T^{-3/2} \|\lambda_i^* \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i - \Psi_{NT} \lambda_i^* \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i\| \sup_{k=1}^{K_0} \|X_k\|$$

$$= N^{-5/2}T^{-3/2} O_p \left( N^{1/2} + NT^{-1/2} \right) \sup_{k=1}^{K_0} \|X_k\|$$

To show (i2), we make the following decomposition:

$$\tilde{\Theta}_{NT} = DB_{NT,1} + DB_{NT,2} + DB_{NT,3},$$

By Lemmas A.2(i) and (v) and G.3(iii),

$$\|DB_{NT,1}\| \leq N^{-5/2}T^{-3/2} \left\| \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i \right\| \sup_{i=1}^{N} \|X_i\|$$

By Lemmas A.2(i) and (iv) and G.3(iv),

$$\|DB_{NT,2}\| \leq N^{-5/2}T^{-3/2} \left\| \bar{F}_i^* \tilde{f}_i^* \tilde{f}_i \right\| \sup_{i=1}^{N} \|X_i\|$$
and

\[ \| DB_{1NT,3} \| \leq N^{-5/2}T^{-3/2} \left\| \hat{F}_{(1)} \left( \hat{F}'_{(1)} \hat{F}_{(1)} \right) \right\| \left\| \hat{F}_{(1)} \right\|^2 \left\{ \sum_{i=1}^{N} \| X_{i(1)} \| \| \hat{\lambda}_{(1)} - \hat{\lambda}_{(i)} \| \right\} \]

\[ \leq N^{-5/2}T^{-3/2} \left\| \hat{F}_{(1)} \left( \hat{F}'_{(1)} \hat{F}_{(1)} \right) \right\| \left\| \hat{F}_{(1)} \right\|^2 \times \left\{ \sum_{i=1}^{N} \| X_{i(1)} \| \right\}^{1/2} \left\| \hat{\lambda}_{(1)} - \hat{\lambda}_{(1)} \| \right\} \]

\[ = N^{-5/2}T^{-3/2} OP \left( T^{-1/2} \right) OP \left( NT \right) OP \left( T \right) OP \left( N^{1/2}T^{1/2} \right) OP \left( N^{1/2}T^{-1/2} \right) = OP \left( N^{-1/2} \right). \]

It follows that \( \hat{B}_{1NT} - \hat{B}_{1NT} = op \left( 1 \right) \) and \( \hat{B}_{1NT} - B_{1NT} = op \left( 1 \right). \)

**Second, we prove** \( (ii) \) \( \hat{B}_{2NT} - B_{2NT} = op \left( 1 \right). \) Let \( \hat{B}_{2NT} \equiv N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} M_{F0} \Phi_{NT} \hat{F}_{(1)} \hat{\lambda}_{(i)} \).

We prove \( (ii) \) by showing that \( (ii1) \) \( \hat{B}_{2NT} - \hat{B}_{2NT} = op \left( 1 \right) \) and \( (ii2) \) \( \hat{B}_{2NT} - \hat{B}_{2NT} = op \left( 1 \right). \) Note that

\[ \hat{B}_{2NT} - \hat{B}_{2NT} = N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} \left( N^{-1} \epsilon' \epsilon - \Phi_{NT} \right) F^0 H_{(1)} \hat{\lambda}_{(i)} \]

\[ = I + II + III, \text{ say,} \]

where recall \( H_B = F^0 H_{(1)} V_{NT,11}^{-1}. \) Let \( cK_0 \equiv \left( c_1 K_0, ..., c_N K_0 \right) \) be an arbitrary \( K_0 \times 1 \) nonrandom vector with \( \| cK_0 \| = 1. \) By Lemma A.4(iii),

\[ |c'_{K_0} I| = N^{-1/2}T^{-3/2} \left\| \sum_{k=1}^{K_0} c_k K_0 \operatorname{tr} \left( X_k \left( N^{-1} \epsilon' \epsilon - \Phi_{NT} \right) F^0 \lambda^0 \right) \right\| \]

\[ \leq T^{-3/2} \sum_{k=1}^{K_0} \left\{ \| X_k \left( N^{-1} \epsilon' \epsilon - \Phi_{NT} \right) F^0 \lambda^0 \| \right\} \]

\[ = OP \left( N^{-1/2} \right) + OP \left( T^{-\left(7+4\sigma\right)/\left(16+8\sigma\right)} \right) = OP \left( 1 \right) \]

It follows that \( \| I \| = op \left( 1 \right). \) Similarly, by Lemmas A.4(i) and A.2(ii), we can show that \( \| II \| = op \left( N^{1/2}T^{-1} + T^{-1/2} \right) \) and \( \| III \| = \delta^{-1}_N \| O_P \left( 1 + N^{1/2}T^{-1/2} \right) = op \left( 1 \right). \) Consequently, \( \hat{B}_{2NT} - \hat{B}_{2NT} = op \left( 1 \right) \) and \( (ii1) \) follows. Consequently, \( \hat{B}_{2NT} - \hat{B}_{2NT} = op \left( 1 \right) \) and \( (ii1) \) follows. To show \( (ii2), \) note that \( \hat{B}_{2NT} - \hat{B}_{2NT} = N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} \left( M_{F0} \hat{\Phi}_{NT} \hat{F}_{(1)} \hat{\lambda}_{(i)} - M_{F0} \Phi_{NT} \hat{F}_{(1)} \hat{\lambda}_{(i)} \right) = DB_{2NT,1} + DB_{2NT,2} + DB_{2NT,3}, \) where

\[ DB_{2NT,1} = N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} \left( M_{F0} - M_{F0} \right) \hat{\Phi}_{NT} \hat{F}_{(1)} \hat{\lambda}_{(i)} \]

\[ DB_{2NT,2} = N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} M_{F0} \left( \hat{\Phi}_{NT} - \Phi_{NT} \right) \hat{F}_{(1)} \hat{\lambda}_{(i)} \]

\[ DB_{2NT,3} = N^{-1/2}T^{-3/2} \sum_{i=1}^{N} X'_{i(1)} M_{F0} \Phi_{NT} \hat{F}_{(1)} \left( \hat{\lambda}_{(i)} - \hat{\lambda}_{(i)} \right) \]
By Lemmas A.2(ii), G.4(ii), and Theorem 3.3, we can readily show that \(\|DB_{2NT,1}\| = O_P(N^{-1/2}T^{-1/2}\delta^{-1}_{NT})\), \(\|DB_{2NT,2}\| = O_P(N^{-1/2}T^{-1/2}T^{1/(8+4\sigma)})\), and \(\|DB_{2NT,3}\| = O_P(N^{1/2}T^{-1})\). It follows that \(\hat{\mathbb{B}}_{2NT} - \mathbb{B}_{2NT} = o_P(1)\).

**Third, we prove** (iii) \(\hat{\mathbb{B}}_{3NT} - \mathbb{B}_{3NT} = o_P(1)\). Let \(\hat{\mathbb{B}}_{3NT,k} = \frac{1}{NT}\text{tr}\{[F_{(1)}F_{(1)}^*]^{-1}\hat{F}_{(1)}F^0\lambda^0\Psi_{NT}X_kF_{(1)}^*}\).

We prove (iii) by showing that (iii1) \(\hat{\mathbb{B}}_{3NT,k} - \mathbb{B}_{3NT,k} = o_P(1)\) and (iii2) \(\hat{\mathbb{B}}_{3NT,k} - \hat{\mathbb{B}}_{3NT,k} = o_P(1)\) for \(k = 1, ..., K_0\). For (iii1), we have by Lemma G.3(ii)

\[
|\mathbb{B}_{3NT,k} - \hat{\mathbb{B}}_{3NT,k}| \leq (NT)^{-1}\left|\text{tr}\left(\left(F_{(1)}^{-1}F_{(1)}^*\right)^{-1}\hat{F}_{(1)}^{-1}F^0\lambda^0\Psi_{NT}X_kF_{(1)}^*\right)\right|
\]

\[
\leq (NT)^{-1}\left\|\left(F_{(1)}^{-1}F_{(1)}^*\right)^{-1}\hat{F}_{(1)}^{-1}F^0\left\|\lambda^0\Psi_{NT}X_k\right\|F_{(1)}^*\right\|
\]

\[
= (NT)^{-1}O_P(1)O_P(N^{-1/2}(N^{-1/2} + T^{-1/2})O_P(T^{1/2}) = o_P(1).
\]

For (iii2), we decompose \(\hat{\mathbb{B}}_{3NT,k} - \hat{\mathbb{B}}_{3NT,k}\) as follows

\[
\hat{\mathbb{B}}_{3NT,k} - \mathbb{B}_{3NT,k} = (NT)^{-1}\text{tr}\left\{\left(\hat{F}_{(1)}^{-1}\hat{F}_{(1)}^*\right)^{-1}\hat{F}_{(1)}^{-1}F - \left(F_{(1)}^{-1}F_{(1)}^*\right)^{-1}\hat{F}_{(1)}^{-1}F^0\right\}\lambda^0\Psi_{NT}X_kF_{(1)}^*
\]

\[
+ (NT)^{-1}\text{tr}\left\{\left(\hat{F}_{(1)}^{-1}\hat{F}_{(1)}^*\right)^{-1}\hat{F}_{(1)}^{-1}F^0\left[\lambda^0\Psi_{NT}X_k\hat{F}_{(1)} - H^+\lambda^0\Psi_{NT}X_kF_{(1)}^*\right]\right\}
\]

\[
\equiv DB_{3NT,1} + DB_{3NT,2}.
\]

By Lemmas A.2(i) and (v), and G.3(ii), \(|DB_{3NT,1}| \leq O_P(\delta^{-2}_{NT})R_0\text{max}_{1 \leq i \leq N} \psi_iT\{N^{-1}T^{-1}\left\|H^+\lambda^0\right\|X_k\right\} \times ||F_{(1)}|| = O_P(\delta^{-2}_{NT})\), and similarly \(|DB_{3NT,2}| = O_P(\delta^{-1}_{NT})\). It follows that \(\hat{\mathbb{B}}_{3NT,k} - \mathbb{B}_{3NT,k} = o_P(1)\) and \(\hat{\mathbb{B}}_{3NT,k} - \hat{\mathbb{B}}_{3NT,k} = o_P(1)\).

**Fourth, we prove** (iv) \(\hat{\mathbb{B}}_{4NT} - \mathbb{B}_{4NT} = o_P(1)\) by showing that \(\hat{\mathbb{B}}_{4NT,k} - \mathbb{B}_{4NT,k} = o_P(1)\) for \(k = 1, ..., K_0\). Let \(\hat{\mathbb{B}}_{4NT,k} = E_D(\mathbb{B}_{4NT,k}) = \frac{1}{\sqrt{NT}}\text{tr}\{P_{F_0}E_D(\epsilon'X_k)\}\). We prove (iv) by showing that (iv1) \(\mathbb{R}_{4NT,k} = \mathbb{B}_{4NT,k} - \hat{\mathbb{B}}_{4NT,k} = o_P(1)\), and (iv2) \(\hat{\mathbb{B}}_{4NT,k} - \hat{\mathbb{B}}_{4NT,k} = o_P(1)\). We first show (iv1). Note that \(\mathbb{R}_{4NT,k} = (NT)^{-1/2}\text{tr}\{P_{F_0}[\epsilon'X_k - E_D(\epsilon'X_k) - \epsilon'P_{F_0}X_k]\}\). Apparently, \(E_D(\mathbb{R}_{4NT,k}) = 0\). Let \(\eta_{ik}\) be a \(k_0 \times 1\) unit vector with 1 in its \(k\)th position and zeros elsewhere. By Assumption B.2(ii) and Jensen inequality,

\[
E_D(\mathbb{R}_{4NT,k}^2) = \text{Var}_D(\mathbb{R}_{4NT,k}) = (NT)^{-1}E_D\left(\sum_{i=1}^N \eta_{ik}' \left[X_i' - E_D\left(X_i'\right)\right] P_{F_0}\epsilon_i - E_D\left(X_i'P_{F_0}\epsilon_i\right)\right)^2
\]

\[
= (NT)^{-1} \sum_{i=1}^N E_D\left[ \left[X_i' - E_D\left(X_i'\right)\right] P_{F_0}\epsilon_i - E_D\left(X_i'P_{F_0}\epsilon_i\right)\right]^2
\]

\[
\leq (NT)^{-1} \sum_{i=1}^N E_D\left[ \left[X_i' - E_D\left(X_i'\right)\right] P_{F_0}\epsilon_i\right]^2 \equiv \Xi_{NT}, \text{ say.}
\]

Let \(\eta_{it} = F^0_{it}(T^{-1}F^0_{it}F^0_{it})^{-1}F^0_{it}\) and \(X_{it,k} = X_{it,k} - E_D(X_{it,k})\) for \(k = 1, ..., K_0\). Then we can write

\[
\Xi_{NT} = \frac{1}{NT^3} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \eta_{it}\bar{X}_{it}\Xi_{it}e_{its}\Xi_{it}^2 = \frac{1}{NT^3} \sum_{i=1}^N \sum_{t=1}^T \sum_{s=1}^T \sum_{t=1}^T \sum_{s=1}^T \eta_{it}\bar{X}_{it}\Xi_{it}e_{its}E_D\left[X_{it,k}\bar{X}_{it,k}\Xi_{it}e_{its}\Xi_{it}^2\right].
\]

Let \(S \equiv \{t_1, t_2, t_3, t_4\}\). We consider three cases for the time indices in \(S\): (a) \(#S = 4\), (b) \(#S = 3\), (c) \(#S \leq 2\). We use \(E\Xi_a, E\Xi_b, \) and \(E\Xi_c\) to denote \(\Xi_{NT}\) when the time indices in the above summation are
restricted to cases \((a), (b), \) and \((c), \) respectively. It is easy to show that \(E \Xi_c = O_P(T^{-1}) = o_P(1).\) In case \((a),\) we consider two subcases : \((a1)\) for at least two \(j \in \{1, 2, 3, 4\},\) \(|t_j - k| \geq \tau_s\) for any \(k \neq j\) and \(k \in \{1, 2, 3, 4\},\) and \((a2)\) all the other remaining cases. We use \(E \Xi_{a1}\) and \(E \Xi_{a2}\) to denote \(E \Xi_a\) when the time indices in its summation are restricted to subcases \((a1)\) and \((a2)\), respectively. In subcase \((a1),\) wlog we assume that \(t_4 < t_3 < t_2 < t_1.\) (Note if either \(t_4\) or \(t_2\) is largest in \(S,\) then \(E_{\mathcal{D}} [\hat{X}_{it_1,k} \hat{X}_{it_2,k} \varepsilon_{it_2} \varepsilon_{it_4}] = 0\) by Assumption B.2(iii).) It is easy to see that either \(t_4\) or \(t_1\) (or both) has to lie at least \(\tau_s\)-apart from other time indices in \(S.\) Wlog, assume that \(t_4\) lies at least \(\tau_s\)-apart from \((t_3, t_2, t_1).\) Then by Davydov inequality

\[
|E_{\mathcal{D}} [\hat{X}_{it_1,k} \hat{X}_{it_2,k} \varepsilon_{it_2} \varepsilon_{it_4}]| \leq 8 \| \varepsilon_{it_s} \|_{8+4\sigma, \mathcal{D}} \| \hat{X}_{it_1,k} \hat{X}_{it_2,k} \varepsilon_{it_2} \|_{8+4\sigma, \mathcal{D}} \alpha_{\mathcal{D}} (\tau_s)^{(1+\sigma)/(2+\sigma)}.
\]

With this, one can readily show that \(|E \Xi_{a1}| \leq O_P(T \alpha_{\mathcal{D}} (\tau_s)^{(1+\sigma)/(2+\sigma)}) = o_P(1).\) In subcase \((a2),\) noting that the total number of terms in the summation of \(E \Xi_{a2}\) is of order \(O(NT^2 \tau_s^2),\) we can readily show that \(|E \Xi_{a2}| = O_P(T^{-1} \tau_s^2) = o_P(1).\) Consequently, \(E \Xi_a = o_P(1).\) Analogously, we can show that \(E \Xi_b = o_P(1).\) Consequently, we have \(E_{\mathcal{D}} \left( \mathbb{R}^2_{4NT,k} \right) = o_P(1)\) and \(E_{\mathcal{D}} \left( \mathbb{R}^2_{4NT,k} \right) = o_P(1)\) by Chebyshev inequality. Then \((iv1)\) follows.

Now we show \((iv2).\) We make the following decomposition:

\[
\begin{align*}
\hat{B}_{4NT,k} - \overline{B}_{4NT,k} &= (NT)^{-1/2} \text{tr} \left[ P_{F(1)} (\hat{\varepsilon}' X_k)^{\text{trunc}} - P_{F_0} E_{\mathcal{D}} (\varepsilon' X_k) \right] \\
&= (NT)^{-1/2} \text{tr} \left[ (P_{F(1)} - P_{F_0}) (\hat{\varepsilon}' X_k)^{\text{trunc}} \right] + (NT)^{-1/2} \text{tr} \left[ P_{F_0} (\hat{\varepsilon}' X_k)^{\text{trunc}} - E_{\mathcal{D}} (\varepsilon' X_k) \right] \\
&= (NT)^{-1/2} \text{tr} \left[ (P_{F(1)} - P_{F_0}) (\hat{\varepsilon}' X_k)^{\text{trunc}} \right] + (NT)^{-1/2} \text{tr} \left[ P_{F_0} (\hat{\varepsilon}' X_k)^{\text{trunc}} - E_{\mathcal{D}} (\varepsilon' X_k) \right] \\
&= (NT)^{-1/2} \text{tr} \left[ P_{F_0} [\hat{\varepsilon}' X_k - E_{\mathcal{D}} (\varepsilon' X_k)]^{\text{trunc}} \right] + (NT)^{-1/2} \text{tr} \left[ P_{F_0} (\hat{\varepsilon}' X_k - \varepsilon' X_k)^{\text{trunc}} \right].
\end{align*}
\]

Recall that \(\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Nt})'\) and \(\hat{\varepsilon}_t = (\hat{\varepsilon}_{1t}, \ldots, \hat{\varepsilon}_{Nt})'.\) Let \(X_{s,k}\) denote the \(s\)th columns of the \(N \times T\) matrices \(X_k.\) Then \(\max_s \| X_{s,k} \| = O_P \left( N^{1/2} \right) \) by Lemma G.5(iv). Then by Lemmas G.5(ii), (iv), and (ix), we have

\[
\left\| (\hat{\varepsilon}' X_k)^{\text{trunc}} \right\|_{sp} \leq M \max_{t,s} \| \varepsilon_t X_{s,k} \| \leq M \max_t \| \varepsilon_t \| \max_s \| X_{s,k} \|
\leq M \left\{ \max_t \| \varepsilon_t \| + \max_t \| \varepsilon_t - \varepsilon_t \| \right\} \max_s \| X_{s,k} \|
= M \left\{ O_P \left( N^{1/2} \right) + O_P \left( N^{1/2} \right) \right\} O_P \left( N^{1/2} \right) = O_P \left( NM \right).
\]

It follows that by Lemma A.2(iv) and Assumption B.2(iv)

\[
|D_{B_{41}}| \leq R_0 (NT)^{-1/2} \left\| P_{F(1)} - P_{F_0} \right\|_{sp} \left\| (\hat{\varepsilon}' X_k)^{\text{trunc}} \right\|_{sp} = (NT)^{-1/2} O_P (\delta_{NT}^{-1}) O_P \left( NM \right) = O_P \left( MN^{1/2} T^{-1/2} \delta_{NT}^{-1} \right) = o_P(1).
\]

Following Moon and Weidner (2010b), let \(A = \varepsilon' X_k\) and \(B = A - A^{\text{trunc}}.\) Let \(A_{ts}\) and \(B_{ts}\) denote the \((t,s)\)th elements of \(A\) and \(B,\) respectively. Then \(B_{ts} = 0\) for \(t < s \leq t + M\) and \(B_{ts} = A_{ts}\) otherwise. By construction, \(A_{ts} = 0\) for \(t \geq s.\) When \(t < s,\) we apply Davydov inequality for conditional strong mixing processes to obtain

\[
|A_{ts}| = \left| \sum_{i=1}^N E_{\mathcal{D}} (\varepsilon_{it} X_{is,k}) \right| \leq \sum_{i=1}^N \left| E_{\mathcal{D}} (\varepsilon_{it} X_{is,k}) \right| \leq \sum_{i=1}^N \| \varepsilon_{it} \|_{8+4\sigma, \mathcal{D}} \| X_{is,k} \|_{8+4\sigma, \mathcal{D}} \alpha_{\mathcal{D}} (s - t)^{(3+2\sigma)/(4+2\sigma)}.
\]
For an \( m \times n \) matrix \( E = (E_{ij}) \), define \( \|E\|_1 = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |E_{ij}| \) and \( \|E\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |E_{ij}| \).

Then

\[
\left\| E_D (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k) \right\|_1 = \max_{1 \leq t \leq T-M-1} \sum_{s=t+M+1}^{T} |A_{ts}|
\leq \max_{1 \leq t \leq T-M-1} \frac{N}{2} \sum_{i=1}^{T} \left( \|\varepsilon_{it}\|_{s+4T,D} + \|X_{it,k}\|_{s+4T,D} \right) \alpha_D \left( s - t \right)^{(3+2\sigma)/(4+2\sigma)}
\leq \max_{i} \frac{T}{2} \sum_{t=1}^{N} \left( \|\varepsilon_{it}\|_{s+4T,D}^2 + \|X_{it,k}\|_{s+4T,D}^2 \right) \alpha_D \left( M + 1 \right)^{(3+2\sigma)/(4+2\sigma)}
= O_P \left( NT \alpha_D \left( M + 1 \right)^{(3+2\sigma)/(4+2\sigma)} \right).
\]

Similarly, we can show that \( \left\| E_D (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k) \right\|_\infty = O_P \left( NT \alpha_D \left( M + 1 \right)^{(3+2\sigma)/(4+2\sigma)} \right) \).

By the inequality \( \|E\|_{sp} \leq \|E\|_1 \|E\|_\infty \) and Assumption B.2(iv),

\[
|DB_{42}|^2 \leq \left\{ R_0 \left( NT \right)^{-1/2} \left\| E_D (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k) \right\|_{sp} \right\}^2
\leq R_0^2 \left( NT \right)^{-1} \left\| E_D (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k) \right\|_1 \left\| E_D (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k) \right\|_\infty
\leq O_P \left( NT \alpha_D \left( M + 1 \right)^{(3+2\sigma)/(4+2\sigma)} \right) = o_P \left( 1 \right).
\]

In addition, by Lemmas G.5(iv)-(v) and (vii) and Assumption B.2(iv)

\[
|DB_{43}| \leq R_0 \left( NT \right)^{-1/2} \left\| \left[ (\varepsilon' X_k)^{\text{trunc}} - E_D (\varepsilon' X_k)^{\text{trunc}} \right] \right\|_{sp}
\leq R_0 \left( NT \right)^{-1/2} M \max_{t} \max_{s \leq t+M} \left| \varepsilon_{t+s} X_{s,k} - E_D (\varepsilon_{t+s} X_{s,k}) \right|
= R_0 \left( NT \right)^{-1/2} M O_P \left( N \alpha_{NT} \right) = O_P \left( N^{1/2} T^{-1/2} M \alpha_{NT} \right) = o_P \left( 1 \right),
\]

and

\[
|DB_{44}| \leq R_0 \left( NT \right)^{-1/2} \left\| \left[ (\varepsilon' X_k)^{\text{trunc}} - (\varepsilon' X_k)^{\text{trunc}} \right] \right\|_{sp}
\leq R_0 \left( NT \right)^{-1/2} M \max_{t} \max_{s \leq t+M} \left| \varepsilon_{t+s} X_{s,k} \right|
\leq R_0 \left( NT \right)^{-1/2} M \max_{t} \max_{s} \left| \varepsilon_{it} - \varepsilon_{it} \right| X_{s,k}
= R_0 \left( NT \right)^{-1/2} M O_P \left( N^{1/2} \right) = o_P \left( T^{-1/2} M \right) = o_P \left( 1 \right).
\]

Consequently, we have shown that \( \tilde{D}_{4NT,k} - \tilde{D}_{4NT,k} = o_P \left( 1 \right) \) for \( k = 1, ..., K_0 \) and (iv) follows.

**Finally, we prove** (v) \( \hat{\Theta}_{NT} - \Theta_{NT} = o_P \left( 1 \right) \). We decompose \( \hat{\Theta}_{NT} - \Theta_{NT} \) as follows:

\[
\hat{\Theta}_{NT} - \Theta_{NT} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{z}_{it}^2 - \varepsilon_{it}^2) \tilde{z}_{it} \tilde{z}_{it}' + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it}^2 (\tilde{z}_{it} \tilde{z}_{it}' - \tilde{z}_{it} \tilde{z}_{it}')
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\varepsilon_{it}^2 - \varepsilon_{it}^2) (\tilde{z}_{it} \tilde{z}_{it}' - \tilde{z}_{it} \tilde{z}_{it}') \equiv \Theta_{NT,1} + \Theta_{NT,2} + \Theta_{NT,3}.
\]
It suffices to prove $\Theta_{NT,s} = o_P(1)$ for $s = 1, 2, 3$. By the triangle inequality, the fact that $a^2 - b^2 = (a - b)^2 + 2(a - b)b$, and Lemma G.5(viii),

$$
\|\Theta_{NT,1}\| \leq \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\hat{\varepsilon}_{it} - \varepsilon_{it}\| \|\tilde{Z}_{it}\|^2
$$

$$
\leq \max_{1 \leq i \leq N} \max_{1 \leq t \leq T} (\hat{\varepsilon}_{it} - \varepsilon_{it})^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|\tilde{Z}_{it}\|^2 + 2 \max_{1 \leq i \leq N} \max_{1 \leq t \leq T} |\varepsilon_{it} - \varepsilon_{it}| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \|\tilde{Z}_{it}\|^2
$$

$$
= o_P(1)O_P(1) + o_P(1)O_P(1) = o_P(1)
$$

as we can readily demonstrate $\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \|\tilde{Z}_{it}\|^2 = O_P(1)$ for $s = 0, 1$. To prove that $\Theta_{NT,s} = o_P(1)$ for $s = 2, 3$, we argue that it suffices to show that for arbitrary $K_0 \times 1$ nonrandom vectors $c_{K_0}$ and $\tilde{c}_{K_0}$ with $\|c_{K_0}\| = 1$ and $\|\tilde{c}_{K_0}\| = 1$, we have

$$
DZ_{NT}^{(r)} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \left| c'_{K_0} \left( \tilde{Z}_{it} \tilde{Z}_{it}' - \hat{Z}_{it} \hat{Z}_{it}' \right) \tilde{c}_{K_0} \right| = o_P(1) \text{ for } r = 0, 1, 2. \quad (H.1)
$$

Note that (H.1) implies that $c'_{K_0} \Theta_{NT,2} \tilde{c}_{K_0} = o_P(1)$ by taking $r = 2$ and hence $\Theta_{NT,2} = o_P(1)$. In addition,

$$
|c'_{K_0} \Theta_{NT,3} \tilde{c}_{K_0}| \leq \frac{1}{NT} \left| \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{\varepsilon}_{it} - \varepsilon_{it})^2 c'_{K_0} \left( \tilde{Z}_{it} \tilde{Z}_{it}' - \hat{Z}_{it} \hat{Z}_{it}' \right) \tilde{c}_{K_0} \right|
$$

$$
+ \frac{2}{NT} \left| \sum_{i=1}^{N} \sum_{t=1}^{T} (\tilde{\varepsilon}_{it} - \varepsilon_{it}) \varepsilon_{it} c'_{K_0} \left( \tilde{Z}_{it} \tilde{Z}_{it}' - \hat{Z}_{it} \hat{Z}_{it}' \right) \tilde{c}_{K_0} \right|
$$

$$
\leq \max_{1 \leq t} (\tilde{\varepsilon}_{it} - \varepsilon_{it})^2 \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left| c'_{K_0} \left( \tilde{Z}_{it} \tilde{Z}_{it}' - \hat{Z}_{it} \hat{Z}_{it}' \right) \tilde{c}_{K_0} \right|
$$

$$
+ 2 \max_{1 \leq t} \tilde{\varepsilon}_{it} - \varepsilon_{it} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} \left| c'_{K_0} \left( \tilde{Z}_{it} \tilde{Z}_{it}' - \hat{Z}_{it} \hat{Z}_{it}' \right) \tilde{c}_{K_0} \right|
$$

$$
= o_P(1)O_P(1) + o_P(1)O_P(1) = o_P(1)
$$

by Lemma G.5(viii) and by taking $r = 0$ and 1 in (H.1). It follows that $\Theta_{NT,3} = o_P(1)$.

Now we show (H.1). We observe that

$$
DZ_{NT}^{(r)} \leq \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \left| c'_{K_0} \left( \hat{Z}_{it} - \tilde{Z}_{it} \right) \left( \tilde{Z}_{it} - \hat{Z}_{it} \right)' \tilde{c}_{K_0} \right|
$$

$$
+ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \left| c'_{K_0} \left( \tilde{Z}_{it} - \hat{Z}_{it} \right) \tilde{Z}_{it}' \tilde{c}_{K_0} \right| + \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} |\varepsilon_{it}| \left| c'_{K_0} \tilde{Z}_{it}' \left( \tilde{Z}_{it} - \hat{Z}_{it} \right)' \tilde{c}_{K_0} \right|
$$

$$
\equiv \varpi_{1NT}^{(r)} + \varpi_{2NT}^{(r)} + \varpi_{3NT}^{(r)}, \text{ say.}
$$

Notice that

$$
\tilde{Z}_{i} - \hat{Z}_{i} = - \left[ P_{F_i(1)} X_{i(1)} - P_{F_0} E_D \left( X_{i(1)} \right) \right] - \left[ M_{F_i(1)} \hat{X}_{i1NT} - M_{F_0} \hat{X}_{i1NT} \right]
$$

$$
+ X_{i} \left( \hat{W}_{NT}^{-1} \hat{C}_{NT} - W_{0}^{-1} C_{0} \right) - \left[ P_{F_i(1)} X_{i(1)} \hat{W}_{NT}^{-1} \hat{C}_{NT} - P_{F_0} E_D(X_i) W_{0}^{-1} C_{0} \right]
$$

$$
- \left[ M_{F_i(1)} \hat{X}_{i2NT} \hat{W}_{NT}^{-1} \hat{C}_{NT} - M_{F_0} \hat{X}_{i2NT} W_{0}^{-1} C_{0} \right]
$$

$$
\equiv - \chi_{1i} - \chi_{2i} - \chi_{3i} - \chi_{4i} - \chi_{5i}, \text{ say.}
$$
Let $\chi_{l,t}^r$ denote the $t$th row of $\chi_l$ for $l = 1, 2, \ldots, 5$, and $t = 1, 2, \ldots, T$. By Cauchy-Schwarz inequality, $\varpi_{1 NT}^{(r)} \leq \{\varpi_{1 NT}^{(r)} (c_K)\}^{1/2} \{\varpi_{1 NT}^{(r)} (\bar{c}_K)\}^{1/2}$ where $\varpi_{1 NT}^{(r)} (c_K) = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}| \left|\alpha_{K_0}^{(r)} (Z_{it} - \bar{Z}_{it}) (\bar{Z}_{it} - \bar{Z}_{it}) (\bar{c}_K) \right|$. So it suffices to prove that $\varpi_{1 NT}^{(r)} = o_P (1)$ for any $c_K$ and $\bar{c}_K$ with $\|c_K\| = 1$ and $\|\bar{c}_K\| = 1$ by showing that $\varpi_{1 NT}^{(r)} (c_K) = o_P (1)$ for any $c_K$ with $\|c_K\| = 1$. Note that $\varpi_{1 NT}^{(r)} (c_K) \leq \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}| \varepsilon_{it} \cdots c_K \chi_{l,t}^r c_K = \frac{5}{NT} \sum_{i=1}^N \sum_{t=1}^T \varepsilon_{it} \cdots c_K \chi_{l,t}^r c_K = \frac{5}{NT} \sum_{i=1}^N (\chi_{l,t}^r)^2$, say. We show that $\chi_{l,N,T}^{(r)} = o_P (1)$ for $l = 1, 2, \ldots, 5$. Let $t_{1T}$ denote a $T \times 1$ unit vector with one in its $t$th position and zeros elsewhere. Noting that

$$\chi_{1,t}^r = \left[ X_{i(1)}^r P_{\hat{F}_1} - E_D \left( X_{i(1)}^r \right) P_F \right] t_{1T}$$

we have

$$\chi_{1,N,T}^{(r)} \leq \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left\| X_{i(1)}^r - E_D \left( X_{i(1)}^r \right) \right\|^2 \left( \left\| P_{\hat{F}_1} - P_F \right\|^2 \right)$$

For $\chi_{1,N,T1}^{(r)}$, we have by Lemma G.2(vii)

$$\chi_{1,N,T1}^{(r)} \leq \| (T^{-1} F^{(r)} F^{(r)})^{-1} \|^2 \left\| T^{-1} \left[ X_{i(1)}^r - E_D \left( X_{i(1)}^r \right) \right] F^{(r)} \right\|^2 \left\| \left\| \varepsilon_{it} \right\| F^{(r)} \right\|^2$$

For $\chi_{1,N,T2}^{(r)}$, we have by Lemmas A.2(i) and (v) and Lemma G.5(vi)

$$\chi_{1,N,T2}^{(r)} = \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left\| X_{i(1)}^r - E_D \left( X_{i(1)}^r \right) \hat{F}^{(r)} (\hat{F}^{(r)} F^{(r)})^{-1} F^{(r)} (\hat{F}^{(r)} F^{(r)})^{-1} F^{(r)} \right\|^2$$

For $\chi_{1,N,T3}^{(r)}$, we have by Lemma G.2(vii)

$$\chi_{1,N,T3}^{(r)} \leq \| (T^{-1} F^{(r)} F^{(r)})^{-1} \|^2 \left\| T^{-1} \left[ X_{i(1)}^r - E_D \left( X_{i(1)}^r \right) \right] F^{(r)} \right\|^2 \left\| \left\| \varepsilon_{it} \right\| F^{(r)} \right\|^2$$
\[
\begin{align*}
&\left\{ T \left\| \hat{F}_{(1)} \left( \hat{F}_{(1)}^T \hat{F}_{(1)} \right)^{-1} \right\|^2 \right\} \max_t \left\| \hat{F}_{(1)} - F^*_{(1)} \right\|^2 \left\{ \frac{2}{NT^2} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| E_D \left( X_{(1)} \right) \right\|^2 \right\} \\
&= O_P \left( \delta^{-1}_{NT} \right) O_P \left( 1 \right) + O_P \left( 1 \right) O_P \left( \eta_{1NT} \right) O_P \left( 1 \right) = o_P \left( 1 \right).
\end{align*}
\]

Analogously we can show that \( \chi^{(r)}_{1,NT3} = o_P \left( 1 \right) \). It follows that \( \chi^{(r)}_{1,NT} = o_P \left( 1 \right) \).

Next, noting that \( \chi_{2t} = M_{F^0} \left( \hat{X}_{1NT} - \bar{X}_{1NT} \right) + (M_{F^1(1)} - M_{F^0}) \hat{X}_{1NT} + (M_{F^1(1)} - M_{F^0}) \left( \hat{X}_{1NT} - \bar{X}_{1NT} \right) \), we have
\[
\chi^{(r)}_{2,NT} \leq \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \| \chi_{2it} \|^2 \leq \frac{3}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \left( \hat{X}_{1NT} - \bar{X}_{1NT} \right)' M_{F^0} uT \right\|^2 \\
&\quad + \frac{3}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \hat{X}'_{1NT} \left( M_{F^1(1)} - M_{F^0} \right) uT \right\|^2 \\
&\quad + \frac{3}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \left( \hat{X}_{1NT} - \bar{X}_{1NT} \right)' \left( M_{F^1(1)} - M_{F^0} \right) uT \right\|^2 \\
&\equiv 3\chi^{(r)}_{2,NT1} + 3\chi^{(r)}_{2,NT2} + 3\chi^{(r)}_{2,NT3}, \text{ say.}
\]

Noting that
\[
\hat{X}_{1NT} - \bar{X}_{1NT} = \frac{1}{NT} \sum_{j=1}^N \left[ \hat{\lambda}_{i(1)} F_{(1)} \hat{F}_{(1)} \hat{\lambda}_{j(1)} - \lambda^0_{i(1)} F^0_{(1)} \lambda^0_{j(1)} \right] X_{j(1)} + \frac{1}{NT} \sum_{j=1}^N \lambda^0_{i(1)} F^0_{(1)} \lambda^0_{j(1)} \left[ X_{j(1)} - E_D \left( X_{j(1)} \right) \right] \\
= \left( \hat{\lambda}_{i(1)} - \lambda^*_{i(1)} \right)' \left( T^{-1} F^0_{(1)} \right) \frac{1}{N} \sum_{j=1}^N \lambda^0_{j(1)} X_{j(1)} + \hat{\lambda}'_{i(1)} T^{-1} \left( \hat{F}_{(1)} \hat{F}_{(1)}' - H_{(1)} F^0_{(1)} H_{(1)} \right) \frac{1}{N} \sum_{j=1}^N \lambda^0_{j(1)} X_{j(1)} \\
+ \hat{\lambda}'_{i(1)} T^{-1} \hat{F}_{(1)} \hat{F}_{(1)}' \frac{1}{N} \sum_{j=1}^N \left( \hat{\lambda}_{j(1)} - \hat{\lambda}_{j(1)} \right) X_{j(1)} + \frac{1}{NT} \sum_{j=1}^N \lambda^0_{i(1)} F^0_{(1)} \lambda^0_{j(1)} \left[ X_{j(1)} - E_D \left( X_{j(1)} \right) \right],
\]
we have
\[
\chi_{2,NT1} \leq 4 \left\| T^{-1} F^0_{(1)} \right\|^2 \frac{1}{N} \sum_{i=1}^N \left\| \hat{\lambda}_{i(1)} - \lambda^*_{i(1)} \right\| \frac{1}{T} \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \frac{1}{N} \sum_{j=1}^N X_{j(1)} \right\|^2 \\
+ 4 \left\| T^{-1} \left( \hat{F}_{(1)} \hat{F}_{(1)}' - H_{(1)} F^0_{(1)} H_{(1)} \right) \right\|^2 \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \hat{\lambda}'_{i(1)} \right\|^2 \left\| \frac{1}{N} \sum_{j=1}^N \left( \hat{\lambda}_{j(1)} - \hat{\lambda}_{j(1)} \right) X_{j(1)} \right\|^2 \\
+ 4 \left\| T^{-1} \hat{F}_{(1)} \hat{F}_{(1)}' \right\|^2 \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \hat{\lambda}'_{i(1)} \right\|^2 \left\| \frac{1}{N} \sum_{j=1}^N \left( \hat{\lambda}_{j(1)} - \hat{\lambda}_{j(1)} \right) X_{j(1)} \right\|^2 \\
+ 4 \left\| \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \left| \varepsilon_{it} \right|^r \left\| \frac{1}{NT} \sum_{j=1}^N \lambda^0_{i(1)} F^0_{(1)} \lambda^0_{j(1)} \left[ X_{j(1)} - E_D \left( X_{j(1)} \right) \right] \right\|^2.
\]

One can readily show that each term on the right hand side of the last expression is \( o_P \left( 1 \right) \). For example,
the first term is $o_P(1)$ because it is bounded above by
\[
4 \left\| T^{-1} F^0 F^0 \right\|^2 \left\{ \frac{1}{N} \left\| \lambda_1 - \lambda_1^* \right\|^2 \right\}^{1/2} \left\{ \frac{1}{N} \sum_{i=1}^N \left[ \frac{1}{T} \sum_{t=1}^T |\varepsilon_{it}|^r \right] \right\}^{1/2} \left\{ \frac{1}{N} \sum_{j=1}^N \lambda_j^0 X_j(1) \right\}^2 \]
\[= o_P(1) O_P \left( T^{-1/2} \right) O_P(1) = o_P(1). \]

To see why the last term is $o_P(1)$, we first notice that for any nonrandom $K_0 \times 1$ vector $c_{K_0}$ with $\|c_{K_0}\| = 1,$
\[
\left\| \frac{1}{NT} \sum_{j=1}^N \lambda_i^0 F^0 F^0 \lambda_j^0 c_{K_0} \left[ X_{j(1)} - E_D(X_{j(1)}) \right] \right\| \leq \left\| \lambda_i^0 \right\| \left\| (T^{-1} F^0 F^0) \right\| \max_{1 \leq t \leq T} \left\| \frac{1}{N} \sum_{j=1}^N \lambda_j^0 c_{K_0} \left[ X_{j(1)} - E_D(X_{j(1)}) \right] \right\| \leq \|\lambda_i^0\| O_P(\alpha_{NT}) \right\text{ by Lemma G.5 (v).}
\]

It follows that
\[
\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}|^r \frac{1}{NT} \sum_{j=1}^N \lambda_i^0 F^0 F^0 \lambda_j^0 \left[ X_{j(1)} - E_D(X_{j(1)}) \right] \leq O_P(\alpha_{NT}^2) \left\{ \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}|^r \right\} \left\{ \frac{1}{N} \sum_{j=1}^N \|\lambda_j^0\|^2 \right\}^2 = O_P(\alpha_{NT}^2) = o_P(1).
\]

Consequently, $\chi_{2,NT}^{(r)} = o_P(1).$ Analogously, using Lemma A.2(vi) we can show that $\chi_{s,NT}^{(r)} = o_P(1)$ for $s = 2, 3.$ Hence $\chi_{2,NT}^{(r)} = o_P(1).$

Next, we can readily show that $\hat{W}_{NT} = W_0 + o_P(1)$ and $\hat{C}_{NT} = C_0 + o_P(1).$ It follows that
\[
\chi_{3,NT}^{(r)} \leq \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}|^r \|\chi_{3it}\|^2 \leq \left\| \hat{W}_{NT}^{-1} \hat{C}_{NT} - W_0^{-1} C_0 \right\|^2 \frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T |\varepsilon_{it}|^r \|X_{it}\|^2 \leq o_P(1) O_P(1) = o_P(1).
\]

Analogously, we can show that $\chi_{4,NT}^{(r)} = o_P(1)$ and $\chi_{5,NT}^{(r)} = o_P(1).$ It follows that $\varpi_{1NT}^{(r)}(c_{K_0}) = o_P(1)$ and thus $\varpi_{1NT}^{(r)} = o_P(1).$

By Cauchy-Schwarz inequality, we have
\[
\varpi_{2NT}^{(r)} \leq \left\{ \varpi_{1NT}^{(r)}(c_{K_0}) \right\}^{1/2} \left\{ \varpi_{2NT}^{(r)}(\hat{c}_{K_0}) \right\}^{1/2} = o_P(1) O_P(1) = o_P(1)
\]
where we apply the fact that $\varpi_{NT}^{(r)}(\hat{c}_{K_0}) = \frac{1}{N} \sum_{t=1}^T \sum_{i=1}^N |\varepsilon_{it}|^r \hat{c}_{K_0}^t \hat{Z}_t^t \hat{c}_{K_0} = O_P(1).$ By the same token, $\varpi_{3NT}^{(r)} = o_P(1).$ Consequently we have $DZ_{NT}^{(r)} = o_P(1)$ for $r = 0, 1, 2.$ This completes the proof of the corollary. \(\blacksquare\)
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