12-2013

Minimum Investment Requirement, Financial Integration and Economic (In)stability: A Refinement to Matsuyama (2004)

Haiping ZHANG
Singapore Management University, hpzhang@smu.edu.sg

SMU ECONOMICS & STATISTICS
WORKING PAPER SERIES

SMU
SINGAPORE MANAGEMENT UNIVERSITY

Minimum Investment Requirement, Financial Integration and Economic (In)stability: A Refinement to Matsuyama (2004)

Haiping Zhang

Dec 2013

Paper No. 09 – 2013

Minimum Investment Requirement, Financial Integration and Economic (In)stability: A Refinement to Matsuyama (2004)

Haiping Zhang*

November 2013

Abstract

This note proposes a simple, more precise, necessary condition for symmetry breaking in Matsuyama (*Financial Market Globalization, Symmetry-Breaking, and Endogenous Inequality of Nations, Econometrica, 2004*), i.e., the positive interest rate response to income changes, which essentially arises from the assumptions of financial frictions and minimum investment size requirement of individual projects. This condition also holds under the more general settings. Thus, this note offers an empirically testable hypothesis, i.e., Matsuyama's symmetry breaking is more likely, if the interest rate response to income changes is positive and sufficiently large.

Keywords: financial frictions, financial market globalization, minimum investment size requirement, symmetry breaking
JEL Classification: E44, F41

*School of Economics, Singapore Management University. 90 Stamford Road, Singapore 178903. E-mail: hpzhang@smu.edu.sg

1 Introduction

Matsuyama (2004) shows that, under certain conditions, countries with identical fundamentals converge, independent of initial income levels, to the same, unique, and stable steady state under international financial autarky (IFA, hereafter), while financial market globalization (FMG, hereafter) may destabilize this symmetric steady state in the sense that countries with relatively high (low) initial income levels may converge to a new steady state with the income level higher (lower) than that under IFA. Matsuyama (2004) summarizes three general conditions for symmetry breaking in section 7:

1. For a fixed domestic interest rate, the domestic investment is an increasing function of the wealth held by the domestic entrepreneurs in the lower range.

2. Domestic investment increases the wealth held by domestic entrepreneurs (more than that of foreign entrepreneurs).

3. The domestic interest rate adjusts to balance domestic supply and domestic demand for credit in the absence of the international financial market, while it is linked to the foreign interest rate in the presence of the international financial market.

In this note, I first set up a model satisfying the three general conditions mentioned above and show that FMG does not lead to symmetry breaking there. Then, I identify a simple, necessary condition for Matsuyama's symmetry breaking from the credit market perspective, i.e., the positive interest rate response to income changes, which essentially arises from the assumptions of financial frictions and minimum investment size requirement of individual projects. As my first contribution, this positive relationship should be augmented as a more precise condition into section 7 of Matsuyama (2004).

The intuition is as follows. Suppose that the world economy consists of a continuum of countries with identical fundamentals except for the initial income level. In each country, some agents have both the technology and the funds to run the investment projects and are called entrepreneurs; without either the technology or the funds, other agents lend their net wealth to the credit market and are called households. If the interest rate is below the marginal rate of return to investment, entrepreneurs prefer to finance their investment using external funds. However, due to limited commitment, they are subject to borrowing constraints and must also put their net wealth in the project. The higher the entrepreneurial net wealth, the more they can borrow and invest.

In this model, capital accumulation and the resulting changes in aggregate income affect the interest rate through two channels. First, given that the neoclassical production function has the decreasing marginal product of capital, capital accumulation reduces the marginal rate of return to investment and the interest rate tends to fall. This is called the *neoclassical* effect. Second, capital accumulation raises the individuals' income and net wealth, triggering the credit market adjustment.

In the absence of minimum investment size requirement, capital accumulation affects the credit market on the intensive margin. The higher individual's net wealth allows entrepreneurs (households) to borrow (lend) more. The neoclassical effect reduces the pledgeable value per unit of entrepreneurial investment and dampens the expansion of their debt capacity, implying that the rise in the aggregate credit demand is dominated by that in the aggregate credit supply. Thus, capital accumulation reduces the interest rate and the interest rate is lower in the rich than in the poor country under IFA; under FMG, financial capital flows are from the rich to the poor country, narrowing the initial cross-country income gap. Eventually, countries with identical fundamentals except for the initial income converge to the same steady state as under IFA. Thus, symmetry breaking does not arise, although the three conditions mentioned above are satisfied.

In the presence of fixed or minimum investment size requirement, capital accumulation affects the credit market not only on the intensive margin and but also on the extensive margin. Take the case of fixed investment size requirement as an example. The higher individual's net wealth reduces the credit demand of individual entrepreneur as well as allows more agents to become entrepreneurs with leveraged investment. The intensive-margin (extensive-margin) effect tends to reduce (raise) the aggregate credit demand. In the net term, the size of the expansion in aggregate credit demand is identical as in the absence of minimum investment size requirement. Meanwhile, the higher individual's net wealth raises the lending of individual household and reduces the mass of lenders (households). The intensive-margin (extensive-margin) effect tends to raise (reduce) the aggregate credit supply. The intensive-margin effect is identical as in the absence of minimum investment size requirement, while the extensive-margin effect is new here. If the negative extensive-margin effect on the credit supply side dominates the negative neoclassical effect on the credit demand side, capital accumulation raises the interest rate and the interest rate is higher in the rich than in the poor country under IFA; under FMG, financial capital flows are from the poor to the rich country, widening the cross-country income gap. Eventually, countries with identical fundamentals except for the initial income level may converge to the steady states with different income levels, i.e., FMG leads to symmetry breaking. Here, fixed or minimum investment size requirement gives rise to the distinct extensive-margin effect on the credit supply side, which is key to the positive relationship between the interest rate and income changes.

Kikuchi (2008), Kikuchi and Stachurski (2009), Matsuyama (2005, 2007, 2008, 2012, 2013) apply the mechanism of symmetry breaking to the topics on endogenous fluctuations, inequality, credit traps, credit cycles, and other aggregate implications of credit market imperfections. However, it is not clear how to empirically test the theoretical conditions supporting this mechanism in these papers. As my second contribution, a straightforward, empirically testable hypothesis is proposed, i.e., Matsuyama's symmetry breaking is more likely to arise if the real interest rate response to income changes is positive and sufficiently large.[1]

[1]A comprehensive empirical investigation on the interest rate response to income changes is beyond

Matsuyama (2004) claims in subsection 7.1 and 7.2 that symmetry breaking may also exist in the presence of wealth inequality and minimum (instead of fixed) investment size requirement, while a complete characterization of multiple steady states is *"hopelessly complicated"*. As my third contribution, I incorporate wealth inequality and minimum investment size requirement in a generalized setting and provide a complete, analytical characterization of symmetry breaking, formally proving Matsuyama's conjecture.

As a side note, there is a technical error in one of the boundary conditions for symmetry breaking in figure 5 of Matsuyama (2004).

The rest of the paper is structured as follows. Section 2 sets up the basic model and compares the impacts of FMG on the model dynamics in the absence and in the presence of fixed investment size requirement, respectively. Section 3 checks the robustness in a generalized setting. The appendix collects some extensions and technical proofs.

2 The Basic Model

Consider a two-period overlapping generations model.[2] The world economy consists of a continuum of identical countries, indexed by $i \in [0,1]$. Agents live for two periods, young and old. There is no population growth and the population size of each generation is normalized at one in each country. When young, each agent is endowed with one unit of labor which is supplied inelastically to aggregate production. A final good is internationally tradable and chosen as the numeraire. It can be consumed or transformed into capital goods. Capital goods are non-tradable and used together with labor to produce final goods contemporaneously. Capital fully depreciates after production. The markets for final goods, capital goods, and labor are perfectly competitive. Y_t^i denotes aggregate output of final goods, $L = 1$ and K_t^i denote the aggregate inputs of labor and capital goods, ω_t^i and v_t^i denote the wage rate and the price of capital in country $i \in [0,1]$.

$$Y_t^i = \left(\frac{K_t^i}{\alpha}\right)^{\alpha} \left(\frac{L}{1-\alpha}\right)^{1-\alpha}, \qquad \text{where} \qquad \alpha \in (0,1), \tag{1}$$

$$v_t^i K_t^i = \alpha Y_t^i \qquad \text{and} \qquad \omega_t^i L = (1-\alpha) Y_t^i, \tag{2}$$

Agents only consume when old and they save their entire labor income when young.

In order to show the critical role of fixed investment size requirement in determining Matsuyama's symmetry breaking,[3] I compare two alternative model settings as follows.

In the first setting, a fraction $\eta \in (0,1)$ of agents in each generation are endowed with the linear project to transform final goods in period t into capital goods in period $t+1$ at the rate of R, and are called entrepreneurs. The mass of entrepreneurs η is exogenously fixed[4], while the investment size of individual project m_t^i is endogenous. With

the scope of this note and is left for future research.

[2]The model setting closely resembles that of Matsuyama (2004).

[3]Section 3 shows in a generalized model that minimum (instead of fixed) investment size requirement is an critical assumption for symmetry breaking.

[4]Appendix A.1 endogenizes the mass of entrepreneurs and the results in this setting still hold there.

no productive projects, other agents lend out their labor income and are called households. Given the fixed mass of entrepreneurs and the linearity of individual projects, aggregate investment takes place on the intensive margin, $K^i_{t+1} = Rm^i_t\eta$. With **n**o fixed investment size requirement for individual projects, it is called setting **N**.

In the second setting, each agent is endowed with an indivisible project[5] to transform m units of final goods in period t into Rm units of capital goods in period $t+1$.[6] If $\omega^i_t < m$, an agent must borrow $m-\omega^i_t$ to start its project and the aggregate resource is not sufficient to allow all agents to start the projects. According to Matsuyama (2004), random credit rationing allows a fraction $\eta^i_t \in (0,1)$ of agents to start the projects with the loans, $m-\omega^i_t$, and they are called entrepreneurs, while other agents only lend out the labor income and are called households. Different from setting **N**, the project size m is exogenously fixed, while the mass of entrepreneurs η^i_t is endogenous. Although the fixed investment size results in the non-convexity of the individual production set, Matsuyama (2007, 2008) argues that assuming a continuum of homogeneous agents convexifies the production set and aggregate investment takes place on the extensive margin, $K^i_{t+1} = Rm\eta^i_t$. With **f**ixed investment size requirement for individual projects, it is called setting **F**.

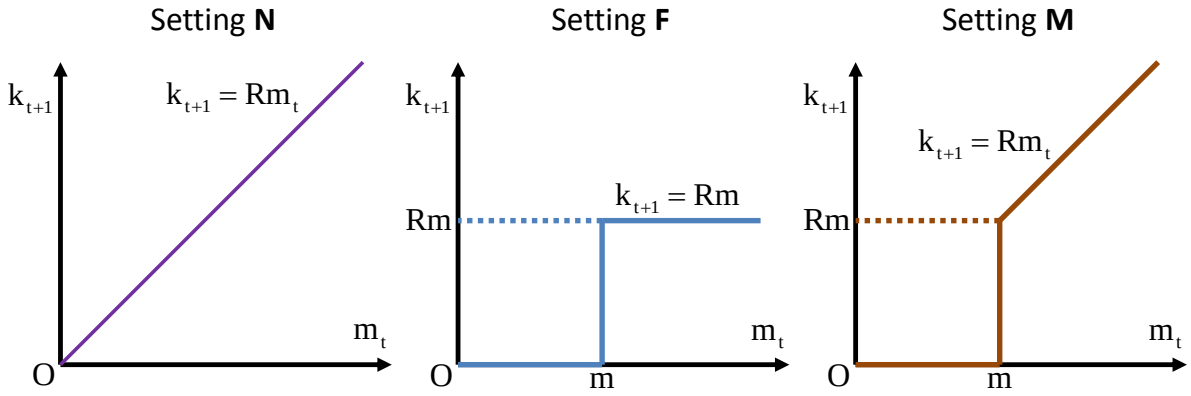

Figure 1: Projects in Various Settings

Figure 1 shows the productive function of individual project in various settings. The project output in setting **N** is linear in the input, $k_{t+1} = Rm_t$. With fixed investment size requirement, the project output is zero for the input $m_t < m$; it is constant at Rm for the input $m_t \geq m$ in setting **F**. In section 3, I analyze a setting with minimum investment size requirement (setting **M**), i.e., the project output is zero for $m_t < m$; it is linear in the input $k_{t+1} = Rm_t$ for $m_t \geq m$. I use η^i_t and m^i_t to denote the mass of entrepreneurs and the project size in the model description. Setting **N** is characterized by the fixed mass of entrepreneurs, $\eta^i_t = \eta$, while setting **F** is characterized by the fixed project size, $m^i_t = m$.

In each setting, I analyze the dynamic properties of the equilibrium allocations under two scenarios, i.e., IFA where agents can only borrow or lend domestically, international capital flows are forbidden, and the gross interest rate r^i_t clears the credit market at the

[5]Section 3 relaxes the assumption of project indivisibility and the results in this setting still hold.

[6]Matsuyama (2004) implicitly normalizes the individual project size at $m = 1$, while I allow it to be a free parameter and consider how it may affect the possibility of symmetry breaking.

country level; FMG where agents can borrow or lend domestically and aboard, there are no barriers to international borrowing/lending, and the gross interest rate $r_t^i = r_t^*$ equalizes across countries.[7] Let Φ_t^i denote the financial capital outflows from country i, with negative values indicating the financial capital inflows. The interest rate cannot exceed the marginal rate of return to investment, $r_t^i \leq v_{t+1}^i R$; otherwise, entrepreneurs would lend out their funds rather than start the project. Matsuyama (2004) calls it the entrepreneurs' profitability constraints.

Consider the agents of the generation born in period t. If $r_t^i < v_{t+1}^i R$, entrepreneurs prefer to fund their projects with loans, i.e., $d_t^i = m_t^i - \omega_t^i$. Due to limited commitment, they are subject to the borrowing constraint,

$$r_t^i d_t^i = r_t^i (m_t^i - \omega_t^i) \leq \lambda v_{t+1}^i R m_t^i. \tag{3}$$

$\lambda \in (0,1)$ denotes the level of financial development, which is identical for all countries.[8]

Let $\psi_t^i \equiv \frac{\omega_t^i}{m_t^i}$ denote the equity-investment ratio. The equity rate is defined as,

$$\Gamma_t^i = \frac{v_{t+1}^i R m_t^i - r_t^i d_t^i}{\omega_t^i} = v_{t+1}^i R + (v_{t+1}^i R - r_t^i)\left(\frac{1}{\psi_t^i} - 1\right). \tag{4}$$

The term $(v_{t+1}^i R - r_t^i)\left(\frac{1}{\psi_t^i} - 1\right)$ captures the excess return due to the leveraged investment. If $r_t^i < v_{t+1}^i R$, the leveraged investment leads to $\Gamma_t^i > v_{t+1}^i R > r_t^i$ so that entrepreneurs borrow up to the limit to maximize the leverage ratio $(\frac{1}{\psi_t^i} - 1)$; if $r_t^i = v_{t+1}^i R$, the zero excess return leads to $\Gamma_t^i = r_t^i$ so that entrepreneurs do not borrow to the limit. In the following, the private rates of return refer to the equity rate and the interest rate, while the social rate of return refers to the marginal rate of return to investment.

Households save the labor income when young and consume the financial return when old; entrepreneurs finance the project using the loans and their labor income when young, and then, consume the project revenue net of debt repayment when old,

$$c_{t+1}^{i,h} = r_t^i \omega_t^i, \text{ and } c_{t+1}^{i,e} = v_{t+1}^i R m_t^i - r_t^i d_t^i = \Gamma_t^i \omega_t^i. \tag{5}$$

The markets for capital goods, credit, and final goods clear simultaneously,

$$K_{t+1}^i = R m_t^i \eta_t^i, \tag{6}$$

$$\eta_t^i (m_t^i - \omega_t^i) = (1 - \eta_t^i)\omega_t^i, \tag{7}$$

$$(1 - \eta_t^i) c_t^{i,h} + \eta_t^i c_t^{i,e} + \eta_t^i m_t^i = Y_t^i. \tag{8}$$

Definition 1. *A market equilibrium under IFA is a set of allocations of households,* $\{c_t^{i,h}\}$*, entrepreneurs,* $\{m_t^i, c_t^{i,e}\}$*, and aggregate variables,* $\{Y_t^i, K_t^i, \omega_t^i, v_t^i, \Gamma_t^i, \eta_t^i, r_t^i\}$*, satisfying equations* (1)-(7)*.*
$\eta_t^i = \eta$ *is exogenously fixed in setting* $\boldsymbol{N}$*, while* $m_t^i = m$ *is exogenously fixed in setting* $\boldsymbol{F}$*.*

[7] Following Matsuyama (2004), I exclude FDI flows by assumption. von Hagen and Zhang (2011, 2014) analyze the joint determination of financial capital flows and FDI flows in setting **N**.

[8] See Matsuyama (2007, 2008) for detailed discussion on formulating the borrowing constraint in such a way. von Hagen and Zhang (2011, 2014) analyze the case where countries differ in λ.

Under FMG, the equilibrium conditions are identical as under IFA except for the credit market clearing conditions at the country and at the world level,

$$\eta_t^i(m_t^i - \omega_t^i) = (1-\eta_t^i)\omega_t^i - \Phi_t^i, \tag{9}$$

$$\int_0^1 \Phi_t^i di = 0. \tag{10}$$

Definition 2. *A market equilibrium under FMG is a set of allocations of households,* $\{c_t^{i,h}\}$*, entrepreneurs,* $\{m_t^i, c_t^{i,e}\}$*, and aggregate variables,* $\{Y_t^i, K_t^i, \omega_t^i, v_t^i, \Gamma_t^i, \eta_t^i, \Phi_t^i\}$*, satisfying equations* (1)-(6) *and* (9)*, and the world interest rate* r_t^* *is determined by the world credit market equilibrium condition* (10)*.*
$\eta_t^i = \eta$ *is exogenously fixed in setting* ***N****, while* $m_t^i = m$ *is exogenously fixed in setting* ***F****.*

2.1 Equilibrium Allocation under IFA

In setting **N**, according to equation (7) and (6), the equity-investment ratio is constant at $\psi_t^i = \eta$ and domestic investment is fully financed by domestic saving $K_{t+1}^i = R\omega_t^i$. The dynamic equation of wages[9] is

$$\omega_{t+1}^i = \frac{(1-\alpha)}{L} Y_{t+1}^i = \left(\frac{R\omega_t^i}{\rho}\right)^\alpha, \quad \text{where } \rho \equiv \frac{\alpha}{1-\alpha}. \tag{11}$$

In setting **F**, for $\omega_t^i < m$, aggregate saving is not enough to allow all agents to run their projects. According to equation (7) and (6), the mass of entrepreneurs and the equity-investment ratio are endogenous, $\eta_t^i = \psi_t^i = \frac{\omega_t^i}{m} < 1$, and domestic investment is fully financed by domestic saving. Thus, the phase diagram of wages is the same as in setting **N**. For $\omega_t^i \geq m$, all agents can self-finance their projects, $\eta_t^i = \psi_t^i = 1$. Given the fixed project size, aggregate output of capital goods is constant at $K_{t+1}^i = Rm$ and the phase diagram of wages is flat at $\omega_{t+1}^i = \left(\frac{Rm}{\rho}\right)^\alpha$.

Assumption 1. $m > \omega_{IFA}$.

Proposition 1. *In both settings, countries with identical fundamentals except for the initial income levels converge to the same steady state under IFA, which is unique and stable; for* $\lambda \in (0, 1-\psi_t^i)$*, the borrowing constraints are binding and there is a wedge between the social and private rates of return,* $\Gamma_t^i > Rv_{t+1}^i > r_t^i$*; for* $\lambda \in (1-\psi_t^i, 1)$*, the borrowing constraints are slack and* $\Gamma_t^i = r_t^i = Rv_{t+1}^i$*.*

In the following analysis, I focus on the case of the binding borrowing constraints. Use the binding borrowing constraints to rewrite the interest rate as

$$r_t^i = \frac{\lambda}{1-\psi_t^i} v_{t+1}^i R < v_{t+1}^i R. \tag{12}$$

[9]The model dynamics can also be characterized by the dynamic equation of capital, $K_{t+1}^i = R\omega_t^i = R\frac{1-\alpha}{L} Y_t^i = R\left(\frac{K_t^i}{\rho}\right)^\alpha$. For $\alpha \in (0,1)$, the phase diagram of capital is globally concave, implying the existence of a unique and stable steady state under IFA with $K_{IFA} = R\omega_{IFA}$. For notational simplicity, I use the phase diagram of wages to analyze the existence, uniqueness, and stability of the steady state.

The interest rate depends on three factors, i.e., the social rate of return, $v_{t+1}^i R$, the level of financial development λ, and the equity-investment ratio ψ_t^i.

As long as assumption 1 is satisfied, fixed investment size requirement does not matter for the dynamic properties under IFA. However, it does matter under FMG and the key is how the interest rate responds to income changes.

Figure 2 shows how a higher labor income affects the credit market equilibrium. For simplicity, I suppress the country index. The credit market equilibrium is initially at point E. A higher labor income $\tilde{\omega}_t > \omega_t$ raises the aggregate investment, leading to a lower social rate of return, $R\tilde{v}_{t+1} < Rv_{t+1}$, due to the concavity of the neoclassical aggregate production function with respect to capital. It is called the neoclassical effect.

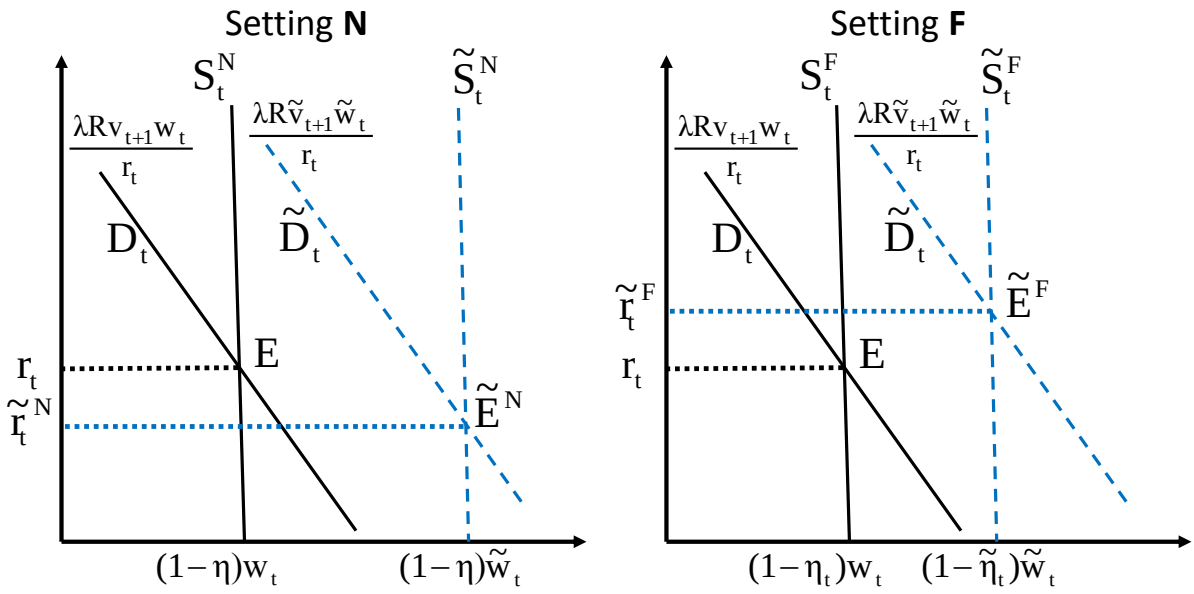

Figure 2: Income Changes and the Credit Market Adjustment

In setting **N**, the masses of households and entrepreneurs are fixed at $1-\eta$ and η, respectively. Thus, a higher labor income affects the credit market only on the intensive margin. The aggregate credit supply, $S_t = (1-\eta)\omega_t$, rises proportionally in ω_t, while the aggregate credit demand, $D_t = \eta d_t = \frac{\lambda R v_{t+1} \omega_t}{r_t}$, rises less-than-proportionally in ω_t, due to the negative neoclassical effect. Thus, the rightward shift of the credit demand curve is dominated by that of the credit supply curve and hence, the equilibrium moves from point E to point $\tilde{E}^N$ with a lower interest rate, $\tilde{r}_t^N < r_t$. See the left panel of figure 2.

Let $\Delta \ln X_t \equiv \ln \tilde{X}_t - \ln X_t$. Rewrite the credit market equilibrium condition $D_t = S_t$,

$$\ln \lambda + \ln R v_{t+1} - \ln r_t + \ln \omega_t = \ln \omega_t + \ln(1-\eta) \tag{13}$$

$$\underbrace{\Delta \ln R v_{t+1}}_{\text{the neoclassical effect}} - \Delta \ln r_t + \underbrace{\Delta \ln \omega_t}_{\text{the wealth effect}} = \underbrace{\Delta \ln \omega_t}_{\text{the wealth effect}} \tag{14}$$

$$\Delta \ln r_t = \underbrace{\Delta \ln R v_{t+1}}_{\text{the neoclassical effect}}. \tag{15}$$

With the wealth effects canceling out on both sides, the interest rate responds to income changes only through the neoclassical effect in setting **N**.

Lemma 1. *In setting **N**, the interest rate strictly decreases in ω_t^i under IFA.*

In setting **F**, given the fixed investment size at the individual level, a higher ω_t affects the credit market on the intensive and the extensive margins. First, a rise in ω_t reduces the individual entrepreneur's credit demand, $d_t = m - \omega_t$, as well as allows more agents to become entrepreneurs, $\eta_t = \frac{\omega_t}{m}$. The intensive-margin (extensive-margin) effect tends to reduce (raise) the aggregate credit demand. Overall, the aggregate credit demand curve shifts to the right at the same magnitude as in setting **N**, $D_t = \eta_t d_t = \frac{\lambda R v_{t+1} \omega_t}{r_t}$. Second, a rise in ω_t raises the individual household's saving, while it also reduces the mass of households, $1 - \eta_t$. The intensive-margin (extensive-margin) effect tends to raise (reduce) the aggregate credit supply, $S_t = (1 - \eta_t)\omega_t$. The positive intensive-margin effect is identical as in setting **N**, while the negative extensive-margin effect is new in setting **F**. Same as in setting **N**, the negative neoclassical effect dampens the expansion of the credit demand; different from setting **N**, the negative extensive-margin effect dampens the expansion of the credit supply. If the negative *extensive-margin* effect dominate the negative *neoclassical* effect, the credit market equilibrium moves from point E to point $\tilde{E}^F$ with a higher interest rate, $\tilde{r}_t^F > r_t$. See the right panel of figure 2.

Rewrite the credit market equilibrium condition $D_t = S_t$ as

$$\ln \lambda + \ln R v_{t+1} - \ln r_t + \ln \omega_t = \ln \omega_t + \ln(1 - \eta_t) \tag{16}$$

$$\underbrace{\Delta \ln R v_{t+1}}_{\text{the neoclassical effect}} - \Delta \ln r_t + \underbrace{\Delta \ln \omega_t}_{\text{the wealth effect}} = \underbrace{\Delta \ln \omega_t}_{\text{the wealth effect}} + \underbrace{\Delta \ln(1 - \eta_t)}_{\text{the extensive-margin effect}}, \tag{17}$$

$$\Delta \ln r_t = \underbrace{\Delta \ln R v_{t+1}}_{\text{the neoclassical effect}} - \underbrace{\Delta \ln(1 - \eta_t)}_{\text{the extensive-margin effect}}. \tag{18}$$

With the wealth effects canceling out on both sides, the interest rate responds to income changes through the neoclassical effect and the extensive-margin effect in setting **F**.

Lemma 2. *In setting **F**, the equity-investment ratio $\psi_t^i = \frac{\omega_t^i}{m}$ rises in ω_t^i under IFA; given $\lambda \in (0, 1 - \tilde{\psi}_F)$, the interest rate rises in ω_t^i if $\psi_t^i \in (\tilde{\psi}_F, 1 - \lambda)$, where $\tilde{\psi}_F \equiv \frac{1-\alpha}{2-\alpha}$.*

Figure 3 shows the parameter configuration in setting **F**. According to proposition 1, for (λ, ψ_t^i) in region SD, the borrowing constraints are ***s***lack and the interest rate, coinciding with the social rate of return, ***d***eclines in ω_t^i, due to the neoclassical effect. According to lemma 2, for (λ, ψ_t^i) in region BI, the borrowing constraints are ***b***inding and the interest rate ***i***ncreases in ω_t^i, as the extensive-margin effect dominates the neoclassical effect; for (λ, ψ_t^i) in region BD, the borrowing constraints are ***b***inding and the interest rate ***d***eclines in ω_t^i, as the neoclassical effect dominates the extensive-margin effect.

Figure 4 illustrates proposition 1 and lemmas 1-2 graphically. The left panel shows that the phase diagram of wage starts from zero and is concave crosses the 45° line once from the left in the two settings. Thus, the fixed investment size requirement does not matter for the existence, uniqueness, and stability of the steady state under IFA. Given $\lambda < 1 - \eta$, the middle panel shows that the interest rate in setting **N**, proportional to the

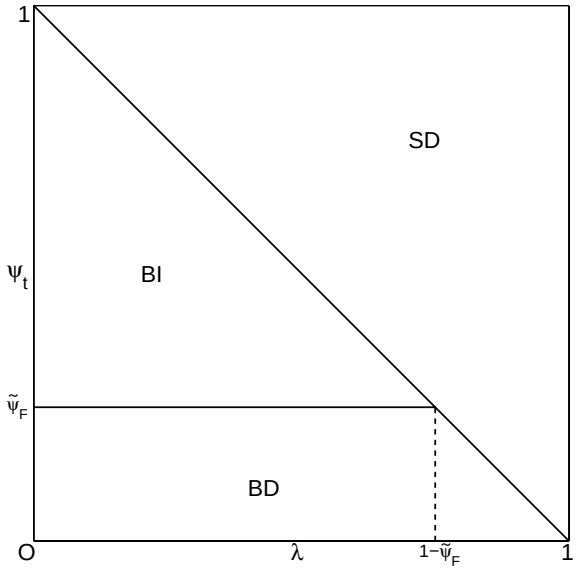

Figure 3: Parameter Configuration for the Interest Rate Patterns in Setting **F**

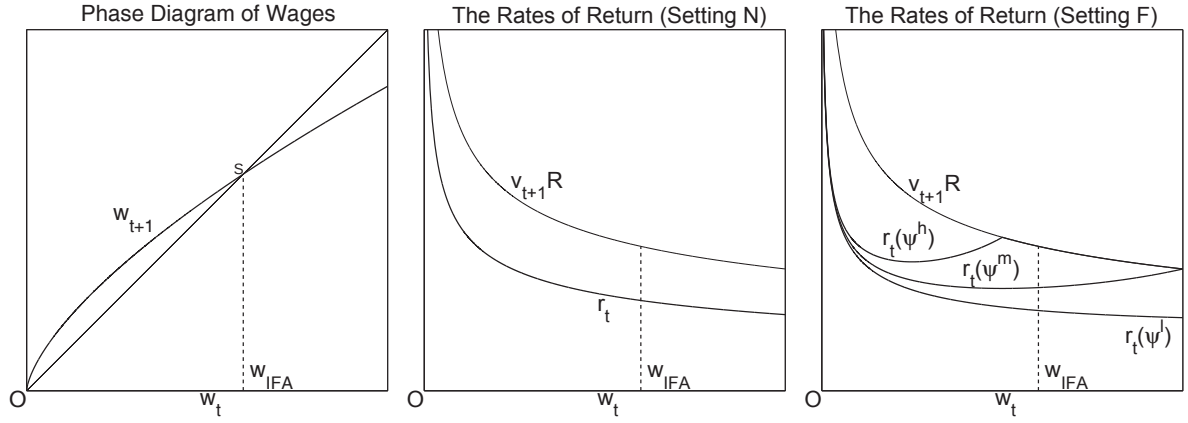

Figure 4: Patterns of Wage Rate and Interest Rates under IFA

social rate of return, declines in ω_t, due to the neoclassical effect. Given $\lambda < 1 - \tilde{\psi}_F$, the right panel shows that the interest rate in setting **F** is a non-monotonic function of ω_t, due to the interactions of the neoclassical effect and the extensive-margin effect.

Let us focus on the interest rate response to ω_t^i around the steady state. The equity-investment ratio in the steady state is $\psi_{IFA} = \frac{\omega_{IFA}}{m} = \frac{R^\rho}{m}\frac{1}{\rho^\rho}$. If $\psi_{IFA} = \psi^h \in (1-\lambda, 1)$ or $\psi_{IFA} = \psi^l \in (0, \tilde{\psi}_F)$, i.e., in region SD or BD of figure 3, the interest rate declines in ω_t around the steady state; if $\psi_{IFA} = \psi^m \in (\tilde{\psi}_F, 1-\lambda)$, i.e., in region BI of figure 3, the interest rate rises in ω_t around the steady state.[10]

In setting **F**, the non-monotonic interest rate response to income changes under IFA will lead to the non-monotonic patterns of financial capital flows, which is the key mechanism behind Matsuyama's symmetry breaking, as shown in subsection 2.2.

[10]Note that the assumption of the Cobb-Douglas production function is not essential here. The two key effects in setting **F**, i.e., the neoclassical effect and the extensive margin effect, exist as long as the aggregate production function is neoclassical, i.e., $f'(k) > 0$ and $f''(k) < 0$, where $k \equiv \frac{K}{L}$.

2.2 Equilibrium Allocation under FMG

In this subsection, I analyze whether and under what conditions FMG may destabilize the steady state under IFA. For that purpose, I assume that all countries are in their respective steady state under IFA before agents are allowed to borrow or lend globally in period $t = 0$. Agents take the world interest rate as given,

$$r_t^* = r_{IFA} = \frac{\lambda}{1-\psi_{IFA}}\rho < \rho, \text{ where } \psi_{IFA} = \eta \ (\psi_{IFA} = \frac{\omega_{IFA}}{m}) \text{ in setting } \mathbf{N} \ (\mathbf{F}).$$

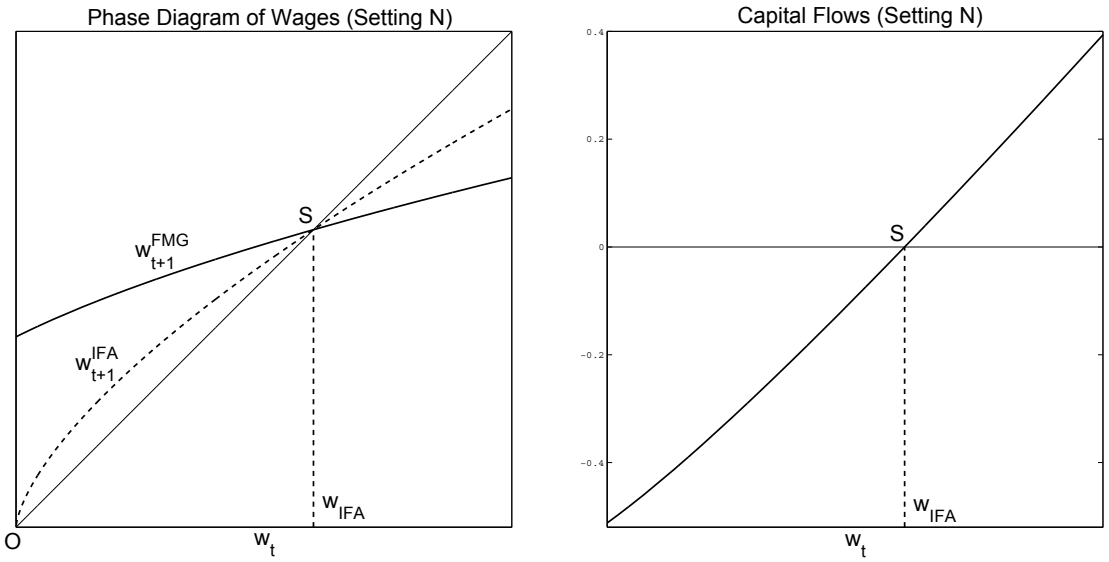

Figure 5: Phase Diagram of Wages and Capital Flows under FMG in Setting **N**

The solid and the dash-dotted curves in the left panel of figure 5 show that the phase diagrams of wage in setting **N** are concave and point S is the unique and stable steady state under IFA and under FMG. For a marginal rise (decline) in ω_t^i around point S, the interest rate tends to reduce (raise), due to the neoclassical effect, and hence, given the world interest rate constant at $r_t^* = r_{IFA}$ under FMG, financial capital flows out of (into) country i, dampening the rise (decline) in domestic investment and ω_{t+1}^i. See the right panel. Essentially, FMG makes the phase diagram of wage flatter so that countries converge to the same steady state as under IFA but faster. Although setting **N** satisfies the three conditions mentioned in section 7 of Matsuyama (2004), symmetry breaking does not arise and the key reason is the negative interest rate responses to income changes.

Proposition 2. *In setting **N**, FMG maintains the uniqueness and stability of the steady state under IFA.*

The left panel of figure 6 shows the parameter configuration for five cases in setting **F** in the (λ, ψ_{IFA}^i) space. By rescaling the vertical axis from ψ_{IFA} into $R = \rho(m\psi_{IFA})^{\frac{1}{\rho}}$, the right panel of figure 6 replicates the same result in the (λ, R) space, corresponding to figure 5 in Matsuyama (2004). For parameters in region C (A), the borrowing constraints are

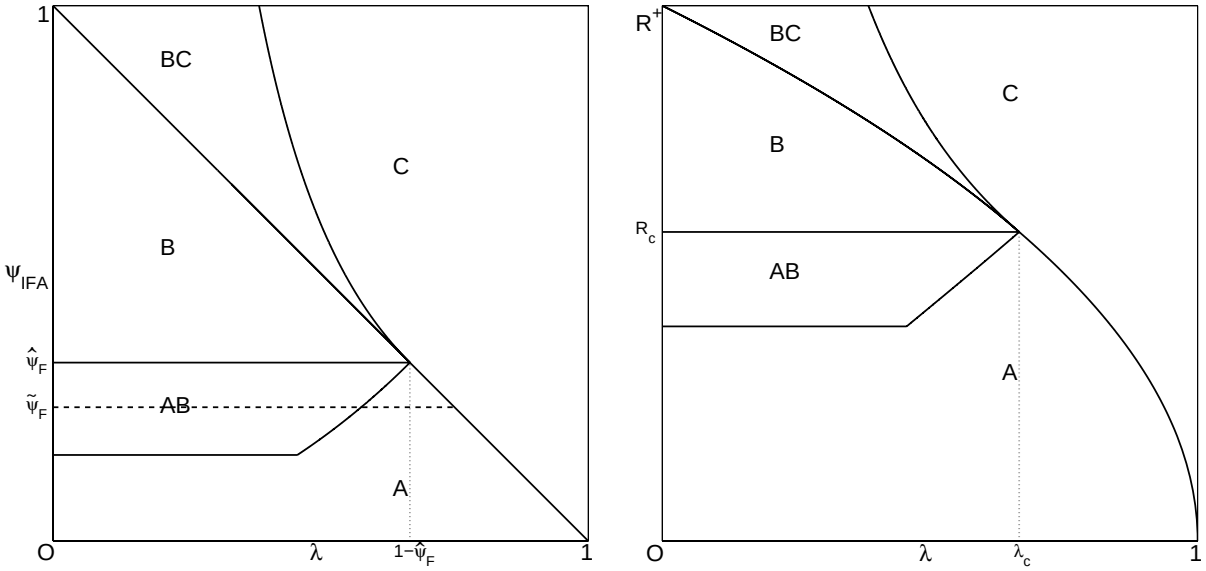

Figure 6: Parameter Configuration for Symmetry Breaking in Setting **F**

slack (binding), the steady state under FMG is identical as under IFA, which is unique and stable. In the following, I focus on region B, AB, and BC where FMG leads to multiple steady states.[11] The right panel of figure 6 is identical as figure 5 of Matsuyama (2004) except for the boundary between region AB and A. By definition, the mass of entrepreneurs cannot exceed the total mass of population in each generation, $\eta_t^i \leq 1$. Taking that into account, the boundary between region AB and A is characterized by a piecewise function with two subfunctions.[12] This result is confirmed in the generalized setting in section 3. Thus, there is a technical error in figure 5 of Matsuyama (2004).

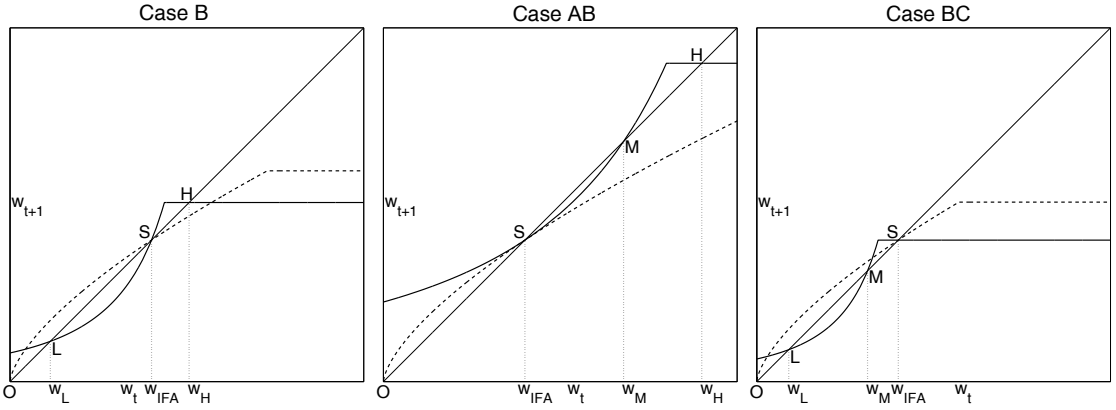

Figure 7: Phase Diagrams of Wage under FMG in Setting **F**

The solid and the dashed curves in figure 7 show the phase diagrams of wage under IFA versus under FMG in the three cases.[13] As shown in figure 3, for (λ, ψ_t) in region BI,

[11] The analysis for region A and C is in the proof of proposition 3 in the appendix.

[12] See the proof of Proposition 3 in the appendix for the explicit characterization of the two subfunctions.

[13] The three cases differ only in terms of the fixed investment size, i.e., $m_{BC} < m_B < m_{AB}$, and have the same steady state (point S) under IFA with $\omega_{IFA} = \left(\frac{R}{\rho}\right)^{\rho}$.

the interest rate rises in ω_t^i under IFA; under FMG, a marginal rise (decline) in ω_t^i leads to financial capital inflows (outflows), amplifying the change in domestic investment and ω_{t+1}^i. Thus, the phase diagram of wage is convex for ω_t^i below a threshold value.

Let $\hat{\psi}_F \equiv 1-\alpha > \tilde{\psi}_F$. Consider region B of figure 6. The amplification effect of FMG is so strong that $\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i} > 1$ at point S. As shown in the left panel of figure 7, FMG destabilizes the initial steady state and creates two new stable steady states, i.e., point H with a higher income and point L with a lower income, respectively.

Consider region AB of figure 6. According to figure 3, if $\psi_{IFA} \in (0, \tilde{\psi}_F)$, the interest rate declines in ω_t^i around point S under IFA; under FMG, financial capital flows tend to dampen the change in domestic investment, making the phase diagram of wage flatter around point S so that the initial steady state (point S) is more stable; if $\psi_{IFA} \in (\tilde{\psi}_F, \hat{\psi}_F)$, although the interest rate rises in ω_t^i around point S, the amplification effect of FMG is not strong enough, i.e., $\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i} < 1$ around point S, so that the initial steady state (point S) is still stable. However, as shown in the middle panel of figure 7, for $\omega_t^i \gg \omega_{IFA}$, ψ_t^i enters in region BI of figure 3 and FMG generates the sufficiently large amplification effect so that there exists two other steady states, i.e., point M (unstable) and point H (stable).

Consider region BC of figure 6. The borrowing constraints are slack in the steady state under IFA; FMG makes the phase diagram of wage flat around point S so that the initial steady state (point S) is still stable. However, as shown in the right panel of figure 7, for $\omega_t^i \ll \omega_{IFA}$, ω_t^i crosses region BI of figure 3 along the convergence path where the amplification effect makes the phase diagram of wage convex and there exists two other steady states, i.e., point M (unstable) and point L (stable).

Proposition 3. *In setting **F**, FMG may lead to multiple steady states.*

To sum up, although the initial income level does not matter for the steady state under IFA, it does matter under FMG. In case B, starting with the income level *slightly* higher (lower) than the steady-state one under IFA, a small open economy converges to a new, stable steady state with the income much higher (lower) than in the steady state under IFA; in case AB (BC), starting with an income *sufficiently* higher (lower) than the steady-state one under IFA, a small open economy converges to a new, stable steady state with the income much higher (lower) than in the steady state under IFA. This way, FMG amplifies the cross-country output gap. Technically, Matsuyama (2004)'s symmetry breaking arises from the convexity of the phase diagram of wage under FMG, which is a result of the positive and sufficiently large interest rate responses to income changes under IFA. Thus, one may test the empirical relevance of Matsuyama's symmetry breaking by estimating the sign and the size of interest rate responses to income changes.

3 The Generalized Model

The individual project size is exogenous in setting **F**. I extend setting **F** in two ways to allow for the endogenous individual project size and wealth heterogeneity. First, the

individual project has a minimum (instead of fixed) investment size requirement $\mathbb{m} \geq 0$, as shown in the right panel of figure 1; second, the labor endowment for agent $j \in [0,1]$ is individual specific, $l_j = \frac{\theta+1}{\theta}\frac{1}{\epsilon_j}$, where $\epsilon_j \in (1,\infty)$ follows the Pareto distribution with the cumulative distribution function $F(\epsilon_j) = 1 - \epsilon_j^{-\theta}$ and $\theta > 1$. The aggregate labor input is $L = \int_1^\infty l_j dF(\epsilon_j) = 1$. With **m**inimum investment requirement, it is called setting **M**.[14]

Given agent j's labor income $n^i_{j,t} = \omega^i_t l_j = \omega^i_t \frac{\theta+1}{\theta}\frac{1}{\epsilon_j}$ and the borrowing constraints as specified by equation (3), its maximum investment size is $m^i_{j,t} = \frac{n^i_{j,t}}{1-\lambda\frac{Rv^i_{t+1}}{r^i_t}} = \frac{\omega^i_t}{1-\lambda\frac{Rv^i_{t+1}}{r^i_t}}\frac{\theta+1}{\theta}\frac{1}{\epsilon_j}$. For a high $\mathbb{m}$ and/or a low ω^i_t, the labor income of agents with $\epsilon_j > \underline{\epsilon}^i_t$ is so low that they cannot meet the minimum investment size requirement, $m^i_{j,t} < \mathbb{m}$, and hence, they become households and lend out the labor income; agents with $\epsilon_j \in (1, \underline{\epsilon}^i_t]$ can meet the minimum investment size requirement, $m^i_{j,t} \geq \mathbb{m}$, and they become entrepreneurs. The mass of entrepreneurs is $\eta^i_t = 1 - (\underline{\epsilon}^i_t)^{-\theta}$. In equilibrium, if the cutoff value $\underline{\epsilon}^i_t$ is sufficiently low, the mass of households $1 - \eta^i_t = (\underline{\epsilon}^i_t)^{-\theta}$ is large and the relatively high aggregate credit supply depresses the interest rate, $r^i_t < Rv^i_{t+1}$. In this case, entrepreneurs borrow to the limit and the equity-investment ratio is identical among them,

$$\psi^i_{j,t} = \frac{n^i_{j,t}}{m^i_{j,t}} = 1 - \lambda\frac{Rv^i_{t+1}}{r^i_t} = \psi^i_t = \frac{\omega^i_t}{\mathbb{m}}\frac{1}{\underline{\epsilon}^i_t}\frac{\theta+1}{\theta}. \tag{19}$$

The condition for the binding borrowing constraints is the same as in setting **F**. See proposition 1. In the following, I focus on the case of the binding borrowing constraints.

Under IFA, domestic investment is financed by domestic saving,

$$K^i_{t+1} = R\int_1^{\underline{\epsilon}^i_t} m^i_{j,t} dF(\epsilon_j) = R\omega^i_t \quad \Rightarrow \quad (\underline{\epsilon}^i_t)^{-(1+\theta)} = \lambda\frac{Rv^i_{t+1}}{r^i_t}, \tag{20}$$

and the output dynamics is characterized by equation (11). The steady-state properties are independent of λ and $\mathbb{m}$. See proposition 1.

Lemma 3. *The equity-investment ratio ψ^i_t and the cutoff value $\underline{\epsilon}^i_t$ rise monotonically in the wage rate ω^i_t under IFA.*

A rise in the wage rate affects aggregate investment on the extensive and the intensive margins. First, it allows more agents to become entrepreneurs and start the projects with leveraged investment and hence, the cutoff value $\underline{\epsilon}^i_t$ is higher; second, the higher aggregate investment reduces the marginal rate of return and the decline in the mass of households tends to reduce the credit supply, which tightens the borrowing constraints for entrepreneurs and raises the equity-investment ratio. Given the mass of entrepreneurs $\eta^i_t = 1 - (\underline{\epsilon}^i_t)^{-\theta}$, equation (20) can be rewritten as,

[14] Setting **F** is a special case here, i.e., for $\theta \to \infty$, the distribution of labor endowment degenerates into a unit mass at $\epsilon_j = 1$ and hence, $l_j = 1$.

$$r_t^i = \lambda R v_{t+1}^i (\underline{\epsilon}_t^i)^{1+\theta} = \lambda R v_{t+1}^i (1-\eta_t^i)^{-\frac{1+\theta}{\theta}} \tag{21}$$

$$\ln r_t^i = \ln \lambda + \ln R v_{t+1}^i - (\frac{1}{\theta}+1)\ln(1-\eta_t^i). \tag{22}$$

$$\Delta \ln r_t^i = \underbrace{\Delta \ln R v_{t+1}^i}_{\text{the neoclassical effect}} \underbrace{-(\frac{1}{\theta}+1)\Delta \ln(1-\eta_t^i)}_{\text{the extensive-margin effect}} \tag{23}$$

Same as equation (18) in setting **F**, income changes affect the interest rate through the neoclassical effect and the extensive-margin effect. In particular, for $\theta \to \infty$, equation (23) coincides with (18). Let $\tilde{\psi}_M \equiv \frac{1-\alpha}{2-\alpha-\frac{1-\alpha}{1+\theta}} > \tilde{\psi}_F$.

Lemma 4. *Given $\lambda \in (0, 1-\tilde{\psi}_M)$, the interest rate rises in ω_t^i if $\psi_t^i \in (\tilde{\psi}_M, 1-\lambda)$.*

Figure 8 shows the region where the interest rate responds positively or negatively to income changes in the (λ, ψ_t^i) space, qualitatively identical as figure 3 for setting **F**. For $\theta \to \infty$, $\lim_{\theta\to\infty} \tilde{\psi}_M = \tilde{\psi}_F$ and lemma 4 is identical as lemma 2.

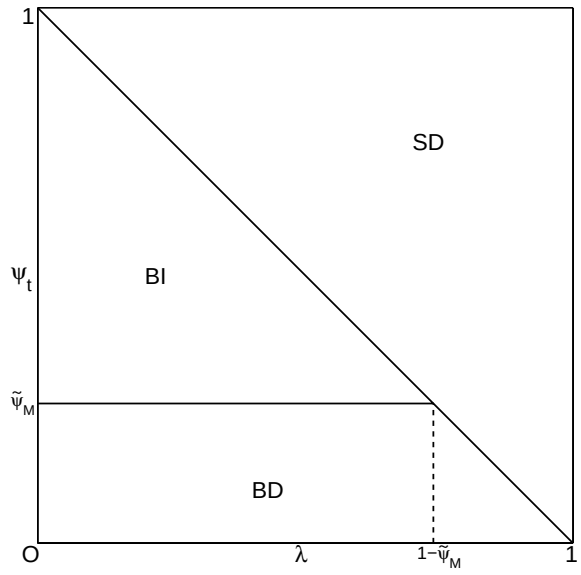

Figure 8: Parameter Configuration for the Interest Rate Patterns in Setting **M**

Agents save the labor income when young and consume the financial income when old,

$$c_{j,t+1}^{i,h} = r_t^i \omega_t^i l_j, \text{ and } c_{j,t+1}^{i,e} = \Gamma_t^i \omega_t^i l_j. \tag{24}$$

Definition 3. *A market equilibrium under IFA is a set of allocations of households, $\{c_{j,t}^{i,h}\}$, entrepreneurs, $\{m_{j,t}^i, c_{j,t}^{i,e}\}$, and aggregate variables, $\{Y_t^i, K_t^i, \omega_t^i, v_t^i, \Gamma_t^i, r_t^i, \underline{\epsilon}_t^i, \psi_t^i\}$, satisfying equations* (1)-(4), (19)-(20), *and* (24).

Under FMG, the equilibrium conditions are identical as under IFA except for the credit market clearing condition at the country and at the world level,

$$K^i_{t+1} = R\int_1^{\underline{\epsilon}^i_t} m^i_{j,t} dF(\epsilon_j) = R(\omega^i_t - \Phi^i_t), \tag{25}$$

$$\int_0^1 \Phi^i_t di = 0. \tag{26}$$

Definition 4. *A market equilibrium under FMG is a set of allocations of households,* $\{c^{i,h}_{j,t}\}$*, entrepreneurs,* $\{m^i_{j,t}, c^{i,e}_{j,t}\}$*, and aggregate variables,* $\{Y^i_t, K^i_t, \omega^i_t, v^i_t, \Gamma^i_t, r^i_t, \underline{\epsilon}^i_t, \psi^i_t, \Phi^i_t\}$*, satisfying equations* (1)-(4)*,* (19)*,* (24)*, and* (25)*. The world interest rate* r^*_t *is determined by the world credit market equilibrium condition* (26)*.*

Following the analysis in subsection 2.2, I assume that the world economy is initially in the steady state under IFA with the interest rate constant at $r^*_t = r_{IFA} = \frac{\lambda}{1-\psi_{IFA}}\rho$, before agents are allowed to borrow and lend globally in period $t = 0$.

Proposition 4. *In setting* ***M****, FCM may lead to multiple steady states.*

Figure 9 shows the parameter configuration for five cases in the (λ, ψ^i_{IFA}) space. As shown analytically in the proof of proposition 4 in the appendix, for $\theta \to \infty$, the respective boundaries of the five regions converge to those in the left panel of figure 6.

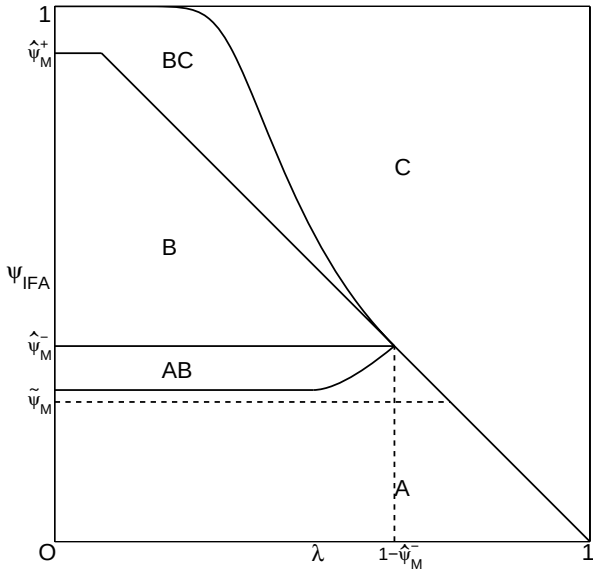

Figure 9: Parameter Configuration for Symmetry Breaking in Setting **M**

Figure 10 shows the phase diagrams of wage under IFA versus under FMG in the three cases of symmetry breaking, qualitatively the same as figure 7 in setting **F**. As shown in the proof of proposition 3, the phase diagram of wage in setting **F** consists of two subfunctions, i.e., a convex part due to the positive interest rate response to income changes, and a flat part due to the profitability constraints $Rv^i_{r+1} \geq r^*_t$ or the upper limit for the mass of entrepreneurs $\eta^i_t \leq 1$. As a result, Matsuyama (2004) shows that in the case of symmetry breaking, the borrowing constraints are strictly binding (slack) in the poor

(rich) country so that, under FMG, the equity premium is always zero, $\Gamma_t^i = r_t^i = Rv_{t+1}^i$, in the rich country. As shown in the proof of proposition 4, the phase diagram of wage in setting **M** may consist of three subfunctions, i.e., a convex part, a concave part and a flat part. Thus, in the case of symmetry breaking, the borrowing constraints can be strictly binding in the poor and in the rich country so that, under FMG, the equity premium can still be positive $\Gamma_t^i > Rv_{t+1}^i > r_t^i$, in the rich country but smaller than in the poor country. Thus, one can test empirically the tightness of the borrowing constraints by estimating the spread between the equity rate and the interest rate across countries.

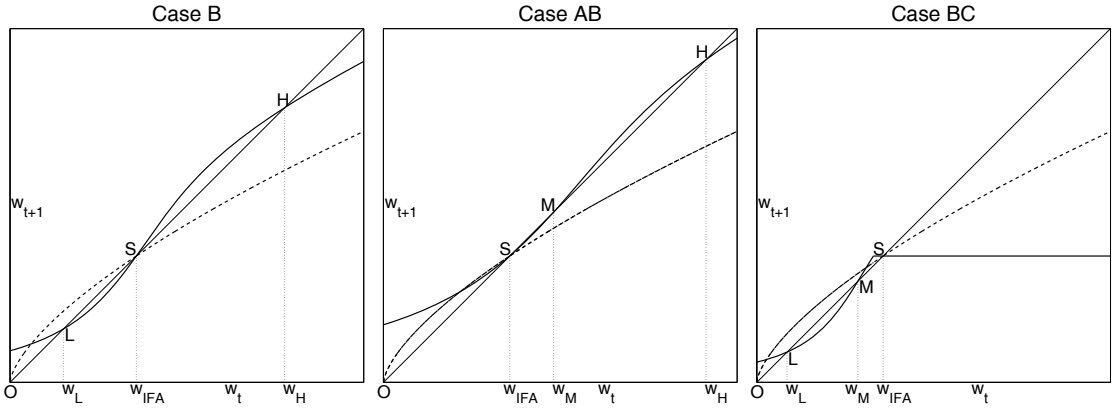

Figure 10: Phase Diagrams of Wage under FMG in Setting **M**

To sum up, financial frictions and minimum (instead of fixed) investment size requirement are key to the positive interest rate response to income changes which essentially underpins Matsuyama's symmetry breaking. Furthermore, I prove formally that the symmetry breaking results still hold in the presence of wealth inequality and endogenous project size, confirming Matsuyama's conjectures in subsection 7.1 and 7.2.

References

Kikuchi, T. (2008): "International asset market, nonconvergence, and endogenous fluctuations," *Journal of Economic Theory*, 139(1), 310–334.

Kikuchi, T., and J. Stachurski (2009): "Endogenous inequality and fluctuations in a two-country model," *Journal of Economic Theory*, 144(4), 1560–1571.

Matsuyama, K. (2004): "Financial Market Globalization, Symmetry-Breaking, and Endogenous Inequality of Nations," *Econometrica*, 72(3), 853–884.

——— (2005): "Credit Market Imperfections and Patterns of International Trade and Capital Flows," *Journal of the European Economic Association*, 3(2-3), 714–723.

——— (2007): "Credit Traps and Credit Cycles," *American Economic Review*, 97(1), 503–516.

——— (2008): "Aggregate Implications of Credit Market Imperfections," in *NBER Macroeconomics Annual 2007*, ed. by D. Acemoglu, K. Rogoff, and M. Woodford, vol. 22. MIT Press.

——— (2012): "Institution-Induced Productivity Differences and Patterns of International Capital Flows," *Journal of the European Economic Association*, forthcoming.

——— (2013): "The good, the bad, and the ugly: An inquiry into the causes and nature of credit cycles," *Theoretical Economics*, 8(3), 623–651.

von Hagen, J., and H. Zhang (2011): "International Capital Flows with Limited Commitment and Incomplete Markets," CEPR Discussion Paper No. 8750.

——— (2014): "Financial Development, International Capital Flows, and Aggregate Output," *Journal of Development Economics*, 106, 66–77.

A Appendix

A.1 Endogenize the Mass of Entrepreneurs in Setting N

The mass of entrepreneurs is exogenous in setting **N**. Here, I endogenize it by assuming that all agents can produce capital goods using final goods and the productivity for agent $j \in [0,1]$ is individual specific,

$$k^i_{j,t+1} = R_j m^i_{j,t}, \quad \text{and} \quad R_j = \frac{\theta+1}{\theta}\frac{R}{\epsilon_j} \tag{27}$$

where $\epsilon_j \in (1,\infty)$ follows the Pareto distribution with the cumulative distribution function $F(\epsilon_j) = 1-\epsilon_j^{-\theta}$ and $\theta > 1$. $R_j \in (0,\bar{R})$ has the mean $E(R) = \int_1^\infty R_j dF(\epsilon_j) = R$ and the upperbound $\bar{R} \equiv \frac{1+\theta}{\theta}R$. With **p**roductivity heterogeneity, it is called setting **P**.[15]

Let $\underline{\epsilon}^i_t \equiv \frac{\theta+1}{\theta}\frac{Rv^i_{t+1}}{r^i_t}$. The profitability constraint, $R_j v^i_{t+1} \geq r^i_t$, implies that the agents with $\epsilon_j \in (1,\underline{\epsilon}^i_t]$ choose to become entrepreneurs and finance their projects with loans, while the agents with $\epsilon_j > \underline{\epsilon}^i_t$ choose to become households and lend out the labor income. This way, the mass of entrepreneurs $\eta^i_t = 1-(\underline{\epsilon}^i_t)^{-\theta}$ is endogenized.

Suppose that the borrowing constraints are binding for all entrepreneurs. The project investment size rises in the individual-specific productivity, $m^i_{j,t} = \frac{\omega^i_t}{1-\lambda\frac{R_j v^i_{t+1}}{r^i_t}} = \frac{\omega^i_t}{1-\lambda\frac{\underline{\epsilon}^i_t}{\epsilon_j}}$. Under IFA, aggregate investment is financed purely by domestic saving,

$$\int_1^{\underline{\epsilon}^i_t} m^i_{j,t} dF(\epsilon_j) = \omega_t \int_1^{\underline{\epsilon}^i_t} \frac{1}{1-\lambda\frac{\underline{\epsilon}^i_t}{\epsilon_j}} dF(\epsilon_j) = \omega^i_t \;\Rightarrow\; \int_1^{\underline{\epsilon}^i_t} \frac{1}{1-\lambda\frac{\underline{\epsilon}^i_t}{\epsilon_j}} dF(\epsilon_j) = 1, \tag{28}$$

implying that the cutoff value is time invariant $\underline{\epsilon}^i_t = \underline{\epsilon}_{IFA}$ and depends only on the level of financial development λ and the distribution function $F(\epsilon_j)$.

Aggregate output of capital goods is

$$K^i_{t+1} = \int_1^{\underline{\epsilon}^i_t} R_j m^i_{j,t} dF(\epsilon_j) = \omega_t \mathbb{R}_{IFA}, \;\text{ where }\; \mathbb{R}_{IFA} \equiv \int_1^{\underline{\epsilon}^i_t} \frac{\frac{\theta+1}{\theta}\frac{1}{\epsilon_j}}{1-\lambda\frac{\underline{\epsilon}^i_t}{\epsilon_j}} dF(\epsilon_j). \tag{29}$$

[15] Setting **N** is a special case here, i.e., for $\theta \to \infty$, the distribution of ϵ_j degenerates into a unit mass at $\epsilon_j = 1$ and hence, $R_j = 1$.

$\mathbb{R}_{IFA} \in (R, \bar{R})$ measures the aggregate productivity under IFA and is time invariant. The dynamics of aggregate output can be characterized by

$$\omega_{t+1}^i = \frac{(1-\alpha)}{L} Y_{t+1}^i = \left(\frac{\mathbb{R}_{IFA}\omega_t^i}{\rho}\right)^{\alpha}, \tag{30}$$

and there exists a unique and stable steady state with $\omega_{IFA} = \left(\frac{\mathbb{R}_{IFA}}{\rho}\right)^{\rho}$, similar as equation (11) in setting **N**.

Agent j can get the loan $\lambda \frac{R_j v_{t+1}^i}{r_t^i} = \lambda \frac{\epsilon_{IFA}}{\epsilon_j}$ per unit of its project investment. As long as $\lambda \underline{\epsilon}_{IFA} < 1$, even the most productive agents $\epsilon_j = 1$ cannot finance their entire project investment by loans and hence, the borrowing constraints are binding for all entrepreneurs.

In equilibrium, the interest rate is determined by the rate of return of the marginal entrepreneurs with $\epsilon_j = \underline{\epsilon}_{IFA}$,

$$r_t^i = \underline{R} v_{t+1}^i = \frac{\theta+1}{\theta} \frac{R v_{t+1}^i}{\underline{\epsilon}_{IFA}} = \frac{\theta+1}{\theta} \frac{R}{\underline{\epsilon}_{IFA}} \left(\frac{\rho}{\mathbb{R}_{IFA}\omega_t^i}\right)^{1-\alpha}. \tag{31}$$

Lemma 5. *For $\lambda \in (0, \frac{1}{\underline{\epsilon}_{IFA}})$, the borrowing constraints are binding. The interest rate, proportional to the price of capital goods, decreases in ω_t^i.*

Intuitively, although the mass of entrepreneurs $\eta_t^i = 1 - (\underline{\epsilon}_t^i)^{-\theta}$ is endogenously determined, the time-invariant cutoff value $\underline{\epsilon}_t^i = \underline{\epsilon}_{IFA}$ implies the time-invariant mass of entrepreneurs in equilibrium under IFA. Thus, income changes only affect the credit market on the intensive margin, same as in setting **N**. Due to the absence of extensive-margin effect, the neoclassical effect leads to the negative interest rate response to income changes under IFA; FMG makes the phase diagram of wage flatter and the initial steady state under IFA is still the unique and stable steady state under FMG.

To sum up, with no fixed or minimum investment size requirements, FMG does not lead to Matsuyama's symmetry breaking in setting **P** as well as in setting **N**.

A.2 Proofs

Proof of Proposition 1

Proof. According to equation (11), the phase diagram of wage in setting **N** starts from zero and is strictly concave, given $\alpha \in (0,1)$. As shown in the left panel of figure 4, it crosses the 45° line once and only once from the left with the wage at $\omega_{IFA} = \left(\frac{R}{\rho}\right)^{\rho}$. Given assumption 1, the phase diagram of wage in setting **F** is identical as in setting **N**, except for a kink at $\omega_t = m$. Thus, there exists a unique and stable steady state under IFA in both settings.

Rewrite the binding borrowing constraints (12) as

$$\frac{r_t^i}{v_{t+1}^i R} = \frac{\lambda}{1-\psi_t^i}. \tag{32}$$

According to equation (4), entrepreneurs prefer to finance their investment using loans iff $r_t^i < v_{t+1}^i R$, or equivalent, $\lambda < 1 - \psi_t^i$. In setting **N**, the credit market clearing equation (7) implies that $\psi_t^i = \frac{\omega_t^i}{m_t^i} = \eta$. Thus, for $\lambda \in (0, 1-\eta)$, the private rates of return are proportional to the social rate of return, $r_t^i = \frac{\lambda}{1-\eta} R v_{t+1}^i < R v_{t+1}^i < \Gamma_t^i = \frac{1-\lambda}{\eta} R v_{t+1}^i$; for $\lambda = 1 - \eta$, the private rates of return coincide with the social rate of return, $r_t^i = \Gamma_t^i = R v_{t+1}^i$, and the borrowing constraints are weakly binding; for $\lambda \in (1-\eta, 1)$, the borrowing constraints are slack and entrepreneurs do not borrow to the limit, because $r_t^i = \Gamma_t^i = R v_{t+1}^i$. Similar analysis applies to setting **F**. □

Proof of Lemma 1

Proof. According to equation (1)-(2),

$$v_{t+1}^i = \left(\frac{K_{t+1}^i}{\rho}\right)^{\alpha-1} = \left(\frac{R\omega_t^i}{\rho}\right)^{\alpha-1} = \frac{\rho}{R}\left(\frac{\omega_{IFA}}{\omega_t^i}\right)^{1-\alpha}. \tag{33}$$

According to the proof of proposition 1, for $\lambda \in (0, 1-\eta)$, the borrowing constraints are binding and $r_t^i = \frac{\lambda}{1-\eta} R v_{t+1}^i = \frac{\lambda\rho}{1-\eta}\left(\frac{\omega_{IFA}}{\omega_t^i}\right)^{1-\alpha}$; for $\lambda \in (1-\eta, 1)$, the borrowing constraints are slack and $r_t^i = R v_{t+1}^i = \rho\left(\frac{\omega_{IFA}}{\omega_t^i}\right)^{1-\alpha}$ under IFA. Due to the neoclassical effect, the social rate of return $R v_{t+1}^i$ declines in ω_t^i and so does the interest rate, $\frac{\partial r_t^i}{\partial \omega_t^i} < 0$. □

Proof of Lemma 2

Proof. According to proposition 1, for $\lambda \in (0, 1-\psi_t^i)$, the borrowing constraints are binding under IFA. Combine the borrowing constraints (3) and equation (33) to get

$$r_t^i = \frac{\lambda}{1-\psi_t^i} R v_{t+1}^i = \frac{\lambda\rho}{1-\psi_t^i}\left(\frac{\omega_{IFA}}{\psi_t^i m}\right)^{1-\alpha} \tag{34}$$

$$\frac{\partial r_t^i}{\partial \omega_t^i} = \frac{\partial \ln r_t^i}{\partial \psi_t^i}\frac{\partial \psi_t^i}{\partial \omega_t^i} r_t^i = \left(\frac{1}{1-\psi_t^i} - \frac{1-\alpha}{\psi_t^i}\right)\frac{r_t^i}{m} > 0, \quad \text{iff } \psi_t^i \in (\tilde{\psi}_F, 1-\lambda). \tag{35}$$

□

Proof of Proposition 2

Proof. Combining equations (1)-(2) with the binding borrowing constraints (3), I derive the phase diagram of wage in setting **N** and its properties are as follows,

$$r_t^*(m_t^i - \omega_t^i) = \lambda v_{t+1}^i R m_t^i \;\Rightarrow\; r_t^*\left[\frac{\rho}{R}(\omega_{t+1}^i)^{\frac{1}{\alpha}} - \eta\omega_t^i\right] = \lambda\rho\omega_{t+1}^i, \tag{36}$$

$$\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i} = \frac{\eta}{\rho}\left[\frac{(\omega_{t+1}^i)^{\frac{1}{\rho}}}{\alpha R} - \frac{\lambda}{r_t^*}\right]^{-1} = \frac{\eta v_{t+1}^i R}{1 + \frac{\rho\omega_t^i}{m_t^i}} > 0, \tag{37}$$

$$\frac{\partial^2 \omega_{t+1}^i}{\partial (\omega_t^i)^2} = -\left(\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i}\right)^3 \frac{(\omega_{t+1}^i)^{\frac{1}{\rho}-1}}{\eta\alpha R} < 0. \tag{38}$$

Given the world interest rate r_t^*, equation (36) implies that, for $\omega_t^i \to 0$, the phase diagram of wage has a positive intercept on the vertical axis at $\omega_{t+1}^i = \left(\frac{R\lambda}{r_t^*}\right)^\rho$. Define a threshold value $\bar{\omega}_t^N = \frac{\rho}{r_t^*}\frac{1-\lambda}{\eta}(\frac{R}{r_t^*})^\rho$. For $\omega_t^i \in (0, \bar{\omega}_t^N)$, the borrowing constraints are binding and the phase diagram of wage is increasing and concave, according to equations (37)-(38). For $\omega_t^i > \bar{\omega}_t^N$, aggregate saving and investment are so high that the social rate of return is equal to the world interest rate, $Rv_{t+1}^i = r_t^*$ and the borrowing constraints are slack. The phase diagram is flat at $\omega_{t+1}^i = \bar{\omega}_{t+1}^N = (\frac{R}{r_t^*})^\rho$. Given $r_t^* < \rho$ and $(1-\lambda) > \eta$, $\bar{\omega}_{t+1}^N < \bar{\omega}_t^N$ so that the kink point on the phase diagram is below the 45° line. Graphically, the phase diagram of wage crosses the 45° line once and only once from the left, and the intersection is in its concave part. Given $r_t^* = r_{IFA}$, the steady state coincides with the one under IFA at $\omega_{FMG} = \omega_{IFA} = \left(\frac{R}{\rho}\right)^\rho$. See the left panel of figure 5.

Use equation (11) and (37) to evaluate the slope of the phase diagram at the steady state under IFA and under FMG, $\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i}|_{FMG} = \frac{\alpha}{1+\frac{(1-\alpha)(1-\eta)}{\eta}} < \alpha = \frac{\partial\omega_{t+1}^i}{\partial\omega_t^i}|_{IFA}$. Thus, FMG makes the phase diagram of wage flatter than under IFA, which speeds up the convergence to the same steady state as under IFA. □

Proof of Proposition 3

Proof. I first prove the shape of the phase diagram of wage and then describe the conditions for symmetry breaking.

For $Rv_{t+1}^i > r_t^*$ or equivalently $\psi_t^i < 1-\lambda$, the borrowing constraints are binding. Use equations (1)-(2) to rewrite the binding borrowing constraints (3) as

$$1 - \frac{\omega_t^i}{m} = \lambda\frac{v_{t+1}^i R}{r_t^*} = \frac{\lambda\rho}{r_t^*}\left(\frac{\omega_{IFA}}{\omega_{t+1}^i}\right)^{\frac{1}{\rho}}, \tag{39}$$

$$\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i} = \frac{\rho}{\frac{1}{\psi_t^i}-1}\frac{\omega_{t+1}^i}{\omega_t^i} > 0, \text{ and } \frac{\partial^2\omega_{t+1}^i}{\partial(\omega_t^i)^2} = \left(\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i}\right)^2\frac{1}{\alpha\omega_{t+1}^i\omega_t^i} > 0. \tag{40}$$

Combine equation (39) with (1)-(2) and then compute the mass of entrepreneurs,

$$\omega_{t+1}^i = \omega_{IFA}\left[\frac{\lambda\rho}{r_t^*(1-\psi_t^i)}\right]^\rho \Rightarrow \eta_t^i = \frac{K_{t+1}^i}{Rm} = \frac{\rho(\omega_{t+1}^i)^{\frac{1}{\alpha}}}{Rm} = \psi_{IFA}\left[\frac{\lambda\rho}{r_t^*(1-\psi_t^i)}\right]^{\frac{1}{1-\alpha}}. \tag{41}$$

The mass of entrepreneurs cannot exceed the population size of each generation, $\eta_t^i \leq 1$. For $\psi_{IFA} \in (0, 1-\lambda)$, the borrowing constraints are binding in the steady state under IFA and $r_t^* = \frac{\lambda\rho}{1-\psi_{IFA}} < \rho$; under FMG, according to equation (41), $\eta_t^i \leq 1$ implies that $\psi_t^i \leq \check{\psi}_t^F \equiv 1 - \psi_{IFA}^{1-\alpha}(1-\psi_{IFA})$. For $\psi_{IFA} > 1-\lambda$, the borrowing constraints are slack in the steady state under IFA and $r_t^* = \rho$; under FMG, according to equation (41), $\eta_t^i \leq 1$ implies that $\psi_t^i \leq 1 - \lambda\psi_{IFA}^{1-\alpha}$. Thus, the phase diagram of wage under FMG is a piecewise function with two subfunctions and there are two cases.

- Case 1: if $\check{\psi}_t^F > 1-\lambda$,
  For $\psi_t^i \in (0, 1-\lambda)$, the borrowing constraints are binding, some agents become entrepreneurs, $\eta_t^i < 1$, and the phase diagram of wage is convex, $\omega_{t+1}^i = \omega_{IFA}\left[\frac{1-\psi_{IFA}}{1-\frac{\omega_t^i}{m}}\right]^\rho$;

for $\psi_t^i > 1-\lambda$, the borrowing constraints are slack, some agents become entrepreneurs, $\eta_t^i < 1$, and the phase diagram of wage is flat at $\omega_{t+1}^i = \omega_{IFA}\left[\frac{1-\psi_{IFA}}{\lambda}\right]^\rho$.

- Case 2: if $\check{\psi}_t^F < 1-\lambda$,
  For $\psi_t^i \in (0, \check{\psi}_t^F)$, the borrowing constraints are binding, some agents become entrepreneurs, $\eta_t^i < 1$, and the phase diagram of wage is convex, $\omega_{t+1}^i = \omega_{IFA}\left[\frac{1-\psi_{IFA}}{1-\frac{\omega_t^i}{m}}\right]^\rho$;
  for $\psi_t^i > \check{\psi}_t^F$, the borrowing constraints are binding, all agents become entrepreneurs, $\eta_t^i = 1$, and the phase diagram of wage is flat at $\omega_{t+1}^i = \left(\frac{Rm}{\rho}\right)^\alpha = \frac{\omega_{IFA}}{\psi_{IFA}^\alpha}$.

According to equation (41), for $\omega_t^i \to 0$, $\psi_t^i \to 0$ and the phase diagram has a positive intercept on the vertical axis at $\omega_{t+1}^i = \omega_{IFA}\left(\frac{\lambda\rho}{r_t^*}\right)^\rho$. The convex part of the phase diagram creates the possibility of multiple steady states.

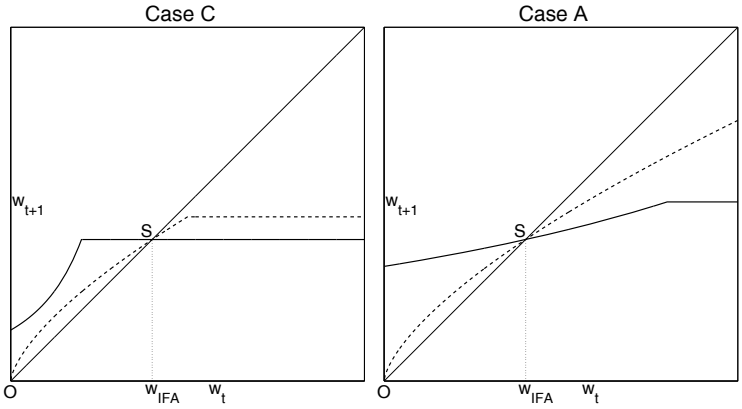

Figure 11: Phase Diagrams of Wage under FMG in Setting **F**

Figure 6 shows the parameter configuration of five regions in the $\{\lambda, \psi_{IFA}\}$ space. Besides the three symmetry-breaking cases shown in figure 7, figure 11 shows two cases where the steady state under IFA is still the unique, stable steady state under FMG.

In the following, I derive the boundary conditions for the five regions in figure 6. Given $r_t^* = r_{IFA}$, the steady state under IFA is still a steady state under FMG, though it may not be stable or unique. For the parameters in the lower-left (upper-right) triangle of figure 6, the borrowing constraints are binding (slack) around the steady state under IFA.

Start with the upper-right triangle of figure 6, i.e., $\psi_{IFA} \in (1-\lambda, 1)$. Compare the right panel of figure 7 and the left panel of figure 11. Given $r_t^* = r_{IFA} = \rho$, the phase diagram of wage under FMG is flat at the initial steady state (point S); the boundary between region BC and C is defined as the case where the convex part of the phase diagram of wage is tangent with the 45° line, i.e., $\omega_t^i = \omega_{t+1}^i = \omega^F < \omega_{IFA}$. Rewrite equations (39) and (40) at the tangent point,

$$1 - \frac{\omega^F}{m} = \lambda(\omega^F)^{-\frac{1}{\rho}}\frac{R}{\rho}, \;\Rightarrow\; \left(1-\frac{\omega^F}{m}\right)\left(\frac{\omega^F}{m}\right)^{\frac{1}{\rho}} = \psi_{IFA}^{\frac{1}{\rho}}\lambda \tag{42}$$

$$\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i} = \frac{\rho\psi^F}{1-\psi^F} = \frac{\rho\rho}{\lambda Rm}(w^F)^{\frac{1}{\alpha}} = 1, \;\Rightarrow\; \left(\frac{\omega^F}{m}\right)^{\frac{1}{\alpha}} = \frac{\lambda}{\rho}\psi_{IFA}^{\frac{1}{\rho}}. \tag{43}$$

Combine them to get

$$\frac{\omega^F}{m} = 1-\alpha \text{ and } \psi_{IFA} = (1-\alpha)\left(\frac{\alpha}{\lambda}\right)^{\rho}, \tag{44}$$

$$\omega^F < \omega_{IFA} \Rightarrow \frac{\omega^F}{m} < \psi_{IFA} \text{ and } \lambda < \alpha. \tag{45}$$

Equations (44)-(45) jointly define the boundary between region BC and C.

Consider the lower-left triangular of figure 6, i.e., $\psi_{IFA} \in (0, 1-\lambda)$. Case B arises if the slope of the phase diagram of wage under FMG is larger than unity at the initial steady state,

$$\frac{\partial \omega^i_{t+1}}{\partial \omega^i_t}|_{FMG} = \frac{\rho}{\frac{1}{\psi_{IFA}}-1} > 1 \quad \Rightarrow \quad \psi_{IFA} > \hat{\psi}_F \equiv 1-\alpha, \tag{46}$$

which specifies the boundary between region B and AB.

If $\frac{\partial \omega^i_{t+1}}{\partial \omega^i_t}|_{FMG} < 1$, the initial steady state is locally stable. However, FMG may still generate a multiple steady-state equilibrium if the kink point of the phase diagram of wage is above the 45° line, i.e., $\bar{\omega}^F_{t+1} > \bar{\omega}^F_t$. There are two cases.

- Case 1: if $\check{\psi}^F_t > 1-\lambda$, the kink point is at $\bar{\omega}^F_t = (1-\lambda)m$ and $\bar{\omega}^F_{t+1} = \omega_{IFA}\left(\frac{1-\psi_{IFA}}{\lambda}\right)^{\rho}$.

$$\bar{\omega}^F_{t+1} > \bar{\omega}^F_t, \Leftrightarrow (1-\psi_{IFA})^{\rho}\psi_{IFA} > (1-\lambda)\lambda^{\rho}. \tag{47}$$

- Case 2: if $\check{\psi}^F_t < 1-\lambda$, the kink point is at $\bar{\omega}^F_t = [1-(1-\psi_{IFA})\psi^{1-\alpha}_{IFA}]m$ and $\bar{\omega}^F_{t+1} = \frac{\omega_{IFA}}{\psi^{\alpha}_{IFA}}$.

$$\bar{\omega}^F_{t+1} > \bar{\omega}^F_t, \Leftrightarrow \psi^{1-\alpha}_{IFA}(2-\psi_{IFA}) \geq 1. \tag{48}$$

Equations (47) and (48) define the boundary conditions between AB and A. □

Proof of Lemma 3

Proof. Combining equations (19) and (20), the cutoff value $\underline{\epsilon}^i_t$ is the solution to equation (49) and the equity-investment ratio ψ^i_t is an increasing function of the cutoff value,

$$\underline{\epsilon}^i_t - \frac{1}{(\underline{\epsilon}^i_t)^{\theta}} = \frac{\omega^i_t}{\mathbb{m}}\frac{\theta+1}{\theta}, \quad \Rightarrow \quad \frac{\partial \ln \underline{\epsilon}^i_t}{\partial \ln \omega^i_t} = \frac{1-(\underline{\epsilon}^i_t)^{-(1+\theta)}}{1+\theta(\underline{\epsilon}^i_t)^{-(1+\theta)}} = 1-\frac{1+\theta}{\frac{1}{1-\psi^i_t}+\theta} \in (0,1), \tag{49}$$

$$\psi^i_t = 1-(\underline{\epsilon}^i_t)^{-(1+\theta)}, \quad \Rightarrow \quad \frac{\partial \ln \psi^i_t}{\partial \ln \omega^i_t} = 1-\frac{\partial \ln \underline{\epsilon}^i_t}{\partial \ln \omega^i_t} = \frac{1+\theta}{\frac{1}{1-\psi^i_t}+\theta} \in (0,1). \tag{50}$$

□

Proof of Lemma 4

Proof. Combine equations (1) and (2) with equation (20) to rewrite the interest rate as

$$r_t^i = \lambda R v_{t+1}^i (\underline{\epsilon}_t^i)^{1+\theta} = \lambda\rho \left(\frac{\omega_{IFA}^i}{\omega_t^i}\right)^{1-\alpha} (\underline{\epsilon}_t^i)^{1+\theta}, \tag{51}$$

$$\ln r_t^i = \ln \lambda\rho + (1-\alpha)\ln \omega_{IFA}^i - (1-\alpha)\ln \omega_t^i + (1+\theta)\ln \underline{\epsilon}^i, \tag{52}$$

$$\frac{\partial \ln r_t^i}{\partial \ln \omega_t^i} = -(1-\alpha) + (1+\theta)\frac{\partial \ln \underline{\epsilon}_t^i}{\partial \ln \omega_t^i} = -(1-\alpha) + (1+\theta)\frac{\psi_t^i}{1+\theta(1-\psi_t^i)}. \tag{53}$$

For $\lambda \in (0, 1-\tilde{\psi}_M)$, the interest rate rises in ω_t^i, $\frac{\partial \ln r_t^i}{\partial \ln \omega_t^i} > 0$, if $\psi_t^i \in (\tilde{\psi}_M, 1-\lambda)$. □

Proof of Proposition 4

Proof. The structure of the proof resembles that of Proposition 3. I first prove the shape of the phase diagram of wage and then describe the conditions for symmetry breaking.

For $Rv_{t+1}^i > r_t^*$ or equivalently $\psi_t^i < 1-\lambda$, the borrowing constraints are binding and the model dynamics under FMG are featured by a recursive equation system of $\{\omega_t^i, \psi_t^i, \underline{\epsilon}_t^i\}$,

$$\frac{\psi_t^i \underline{\epsilon}_t^i}{\omega_t^i} = \frac{1+\theta}{\theta \mathbb{m}} = \frac{\psi_{IFA}\underline{\epsilon}_{IFA}}{\omega_{IFA}}, \quad \Rightarrow \quad \omega_t^i = \omega_{IFA}\frac{\psi_t^i \underline{\epsilon}_t^i}{\psi_{IFA}\underline{\epsilon}_{IFA}}, \tag{54}$$

$$1-\psi_t^i = \lambda\frac{R v_{t+1}^i}{r_t^*}, \quad \Rightarrow \quad \omega_{t+1}^i = \omega_{IFA}\left[\frac{\lambda\rho}{(1-\psi_t^i)r_t^*}\right]^{\rho}. \tag{55}$$

$$\frac{\omega_t^i[1-(\underline{\epsilon}_t^i)^{-(1+\theta)}]}{\psi_t^i} = \frac{K_{t+1}^i}{R}, \quad \Rightarrow \quad \frac{\underline{\epsilon}_t^i[1-(\underline{\epsilon}_t^i)^{-(\theta+1)}]}{\psi_{IFA}\underline{\epsilon}_{IFA}} = \left[\frac{\lambda\rho}{(1-\psi_t^i)r_t^*}\right]^{\frac{1}{1-\alpha}}, \tag{56}$$

Equation (54) specifies the equity-investment ratio, as equation (19); equation (55) features the binding borrowing constraints, as equation (12); equation (56) shows that entrepreneurs produce capital goods with leveraged investment, as equation (25). Use equations (54)-(56) to derive the dynamic property of the phase diagram of wage under FMG,

$$\frac{\partial \ln \underline{\epsilon}_t^i}{\partial \ln \psi_t^i} = \frac{1}{1-\alpha}\frac{\psi_t^i}{1-\psi_t^i}\frac{1-(\underline{\epsilon}_t^i)^{-(1+\theta)}}{1+\theta(\underline{\epsilon}_t^i)^{-(1+\theta)}} > 0, \tag{57}$$

$$\frac{\partial \ln \omega_t^i}{\partial \ln \psi_t^i} = \frac{\partial \ln \underline{\epsilon}_t^i}{\partial \ln \psi_t^i} + 1 = \frac{1}{1-\alpha}\frac{\psi_t^i}{1-\psi_t^i}\frac{1-(\underline{\epsilon}_t^i)^{-(1+\theta)}}{1+\theta(\underline{\epsilon}_t^i)^{-(1+\theta)}} + 1 > 1, \tag{58}$$

$$\frac{\partial \ln \omega_{t+1}^i}{\partial \ln \psi_t^i} = \frac{\alpha}{1-\alpha}\frac{\psi_t^i}{1-\psi_t^i} > 0 \tag{59}$$

$$\Rightarrow \quad \frac{\partial \omega_{t+1}^i}{\partial \omega_t^i} = \frac{\omega_{t+1}^i}{\omega_t^i}\frac{\frac{\partial \ln \omega_{t+1}^i}{\partial \ln \psi_t^i}}{\frac{\partial \ln \omega_t^i}{\partial \ln \psi_t^i}} = \frac{\omega_{t+1}^i}{\omega_t^i}\frac{\alpha}{\frac{1-(\underline{\epsilon}_t^i)^{-(1+\theta)}}{1+\theta(\underline{\epsilon}_t^i)^{-(1+\theta)}} + (1-\alpha)\frac{1-\psi_t^i}{\psi_t^i}} > 0. \tag{60}$$

Let $\mathbb{A}_t \equiv \frac{1-(\underline{\epsilon}_t^i)^{-(1+\theta)}}{1+\theta(\underline{\epsilon}_t^i)^{-(1+\theta)}}$ and $\mathbb{Z}_t \equiv 1-\psi_t^i - \frac{\mathbb{A}_t}{(1-\mathbb{A}_t)} - \frac{\psi_t^i\theta}{(1-\alpha)}\mathbb{A}_t^2$

$$\frac{\partial \mathbb{A}_t}{\partial \psi_t^i} = \left[\frac{1+\theta}{1+\theta(\underline{\epsilon}_t^i)^{-(1+\theta)}}\right]^2 \frac{(\underline{\epsilon}_t^i)^{-(1+\theta)}}{\psi_t^i}\frac{\partial \ln \underline{\epsilon}_t^i}{\partial \ln \psi_t^i} > 0 \tag{61}$$

$$\frac{\partial \mathbb{Z}_t}{\partial \psi_t^i} = -1 - \frac{\theta \mathbb{A}_t^2}{1-\alpha} - \frac{\partial \mathbb{A}_t^i}{\partial \psi_t^i}\left[\frac{1}{(1-\mathbb{A})^2} + \frac{2\theta\psi_t^i\mathbb{A}_t}{1-\alpha}\right] < 0, \tag{62}$$

$$\frac{\partial^2 \omega_{t+1}^i}{(\partial \omega_t^i)^2} = \mathbb{Z}_t \frac{1}{\psi_t^i \alpha \omega_{t+1}^i}\frac{1-\mathbb{A}_t}{\frac{\mathbb{A}_t}{1-\alpha}+\frac{1-\psi_t^i}{\psi_t^i}}\left(\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i}\right)^2 \quad \Rightarrow \quad sgn\left(\frac{\partial \omega_{t+1}^i}{\partial^2 \omega_t^i}\right) = sgn(\mathbb{Z}_t). \tag{63}$$

It is trivial to prove that for $\psi_t^i \to 0$, $\mathbb{Z}_t > 0$ and the phase diagram of wage is convex. According to equation (62), $\mathbb{Z}_t$ declines in ψ_t^i and hence, it is possible that $\mathbb{Z}_t < 0$ and the phase diagram of wage becomes concave. Let $\check{\psi}_t^M$ define the threshold value of ψ_t^i such that $\mathbb{Z}_t = 0$, i.e., the inflection point of the phase diagram of wage. There are two cases.

- Case 1: if $\check{\psi}_t^M > 1-\lambda$, the phase diagram of wage is a piecewise function with two subfunctions:
  for $\psi_t^i \in (0, 1-\lambda)$, the borrowing constraints are binding, the mass of entrepreneurs is significantly smaller than one, and the phase diagram of wage is convex;
  for $\psi_t^i \in (1-\lambda, 1)$, the borrowing constraints are slack, the mass of entrepreneurs is significantly smaller than one, and the phase diagram of wage is flat.

- Case 2: if $\check{\psi}_t^M < 1-\lambda$, the phase diagram of wage is a piecewise function with three subfunctions:
  for $\psi_t^i \in (0, \check{\psi}_t^M)$, the borrowing constraints are binding, the mass of entrepreneurs is significantly smaller than one, and the phase diagram of wage is convex;
  for $\psi_t^i \in (\check{\psi}_t^M, 1-\lambda)$, the borrowing constraints are binding, the mass of entrepreneurs is close to one, and the phase diagram of wage is concave;
  for $\psi_t^i \in (1-\lambda, 1)$, the borrowing constraints are slack, the mass of entrepreneurs is close to one, and the phase diagram of wage is flat.

Given the world interest rate r_t^*, equation (56) implies that, for $\omega_t^i \to 0$ or equivalently $\psi_t^i \to 0$, the phase diagram has a positive intercept on the vertical axis at $\omega_{t+1}^i = \omega_{IFA}\left(\frac{\lambda\rho}{r_t^*}\right)^\rho$; for $\psi_t^i = 1-\lambda$, the phase diagram of wage has a kink point with $\omega_{t+1}^i = \omega_{IFA}\left(\frac{\rho}{r_t^*}\right)^\rho$. The convex/concave part of the phase diagram of wage creates the possibility of multiple steady states.[16]

[16]Although the shape of the phase diagram of wage under FMG in setting **M** may differ from that in setting **F**, they are fundamentally identical. In setting **M**, for a sufficiently low level of income, the equity-investment ratio is low and so is the cutoff value $\underline{\epsilon}_t^i$, according to equations (57)-(58). Thus, the mass of entrepreneurs $\eta_t^i = 1-(\underline{\epsilon}_t^i)^{-\theta}$ is very small. Capital accumulation raises the wage rate and allows more individuals to become entrepreneurs. The extensive-margin effect amplifies the rise in domestic investment and income, which makes the phase diagram of wage convex under FMG. For a sufficiently high level of income, the mass of entrepreneurs is close to one and a marginal rise in the wage rate

Figure 9 shows the parameter configuration of five regions in the $\{\lambda, \psi_{IFA}\}$ space. Besides the three symmetry-breaking cases shown in figure 10, figure 12 shows two cases where the steady state under IFA is still the unique, stable steady state under FMG.

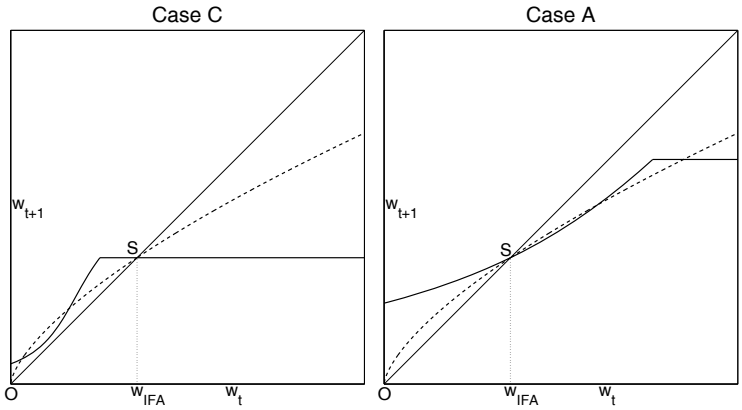

Figure 12: Phase Diagrams of Wage under FMG in Setting **M**

In the following, I derive the boundary conditions for the five regions in figure 9. Given $r_t^* = r_{IFA}$, the steady state under IFA is still a steady state under FMG, though it may not be stable and unique. For the parameters in the lower-left (upper-right) triangle of figure 9, the borrowing constraints are binding (slack) around the steady state of IFA.

Start with the upper-right triangle of figure 9, i.e., $\psi_{IFA} \in (1-\lambda, 1)$. Compare the right panel of figure 10 and the left panel of figure 12. Given $r_t^* = r_{IFA} = \rho$, the phase diagram of wage under FMG is flat at the initial steady state (point S); the boundary between region BC and C is defined as the case where the convex part of the phase diagram of wage is tangent with the 45° line, i.e., $\omega_t^i = \omega_{t+1}^i = \omega^M < \omega_{IFA}$. Combine equations (54)-(56) and evaluate equation (60) at the tangent point with $r_t^* = r_{IFA} = \rho$

$$\left\{\frac{\underline{\epsilon}_t^i[1-(\underline{\epsilon}_t^i)^{-(1+\theta)}]}{\underline{\epsilon}_{IFA}\psi_{IFA}}\right\}^{1-\alpha} = \frac{\lambda}{1-\psi_t} = \left(\frac{\psi_t^i \underline{\epsilon}_t^i}{\psi_{IFA}\underline{\epsilon}_{IFA}}\right)^{\frac{1-\alpha}{\alpha}}, \tag{64}$$

$$\Rightarrow \quad 1-(\underline{\epsilon}_t^i)^{-(1+\theta)} = \mathbb{D}_t \equiv \lambda\frac{\psi_t^i}{1-\psi_t^i}, \text{ and } 0 < \mathbb{D}_t < \psi_t^i < \psi_{IFA} < 1, \tag{65}$$

$$\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i} = \frac{\omega_{t+1}^i}{\omega_t^i}\frac{\alpha}{\frac{\mathbb{D}_t}{1+\theta(1-\mathbb{D}_t)}+(1-\alpha)\frac{\lambda}{\mathbb{D}_t}} = 1 \tag{66}$$

$$\Rightarrow \quad (\frac{1}{\alpha\theta}+1)\mathbb{D}_t^2 - [\frac{\lambda}{\rho}+(\frac{1}{\theta}+1)]\mathbb{D}_t + \frac{\lambda}{\rho}(1+\frac{1}{\theta}) = 0. \tag{67}$$

and the labor income cannot raise the mass of entrepreneurs very much so that the neoclassical effect dominates the extensive-margin effect and hence, the phase diagram is concave. In setting **F**, for the sufficiently low level of income, the small mass of entrepreneurs allows for the strong extensive-margin effect, explaining the convexity of the phase diagram of wage; for a sufficiently high level of income, the mass of entrepreneurs reaches one and, due to the fixed investment size requirement of individual projects, any further rise in income does not raise domestic investment and future income so that the phase diagram of wage becomes flat. In this sense, the threshold value $\check{\psi}_t^M$ in setting **M** corresponds to $\check{\psi}_t^F$ in setting **F**.

As a root of equation (67), $\mathbb{D}_t$ is a function of λ. Combine it with equation (65) to solve for ψ_t^i and $\underline{\epsilon}_t^i$. Then, plug ψ_t^i and $\underline{\epsilon}_t^i$ in equation (64) to solve for ψ_{IFA} as the function of λ, which defines the boundary between region BC and C.[17]

For $\theta \to \infty$, equation (67) has two roots, i.e., $\mathbb{D}_t = 1$ and $\mathbb{D}_t = \frac{\lambda}{\rho}$. As $\mathbb{D}_t = 1$ violates the condition of $\mathbb{D}_t < \psi_t^i < 1$, the only solution is $\mathbb{D}_t = \frac{\lambda}{\rho}$. Use equation (65) to get $\psi_t^i = 1 - \alpha$. For $\theta \to \infty$, $\underline{\epsilon}_t^i = (1 - \mathbb{D}_t)^{-\frac{1}{1+\theta}} \to 1$ and $\underline{\epsilon}_{IFA} = (1 - \psi_{IFA})^{-\frac{1}{1+\theta}} \to 1$. Inserting them in equation (64), one get the result identical as equation (44) for setting **F**.

Now, consider the lower-left triangular of figure 9, i.e., $\psi_{IFA} \in (0, 1 - \lambda)$. According to equation (50), $\underline{\epsilon}_{IFA}^{-(1+\theta)} = 1 - \psi_{IFA}$. Case B arises if the slope of the phase diagram of wage is larger than unity under FMG at the initial steady state, $\omega_{t+1}^i = \omega_t^i = \omega_{IFA}$. Rewrite equation (60) as

$$\frac{\partial \omega_{t+1}^i}{\partial \omega_t^i} \mid_{FMG} = \frac{\alpha}{\frac{\psi_{IFA}}{1+\theta(1-\psi_{IFA})} + (1-\alpha)\frac{1-\psi_{IFA}}{\psi_{IFA}}} > 1. \tag{68}$$

$$\Rightarrow \quad (1 + \frac{1}{\theta})\psi_{IFA}^2 - [(2-\alpha) + \frac{1}{\theta}]\psi_{IFA} + (1-\alpha)(1 + \frac{1}{\theta}) < 0, \tag{69}$$

$$\Rightarrow \quad \psi_{IFA} \in (\hat{\psi}_{IFA}^-, \hat{\psi}_{IFA}^+), \quad \text{where} \tag{70}$$

$$\hat{\psi}_{IFA}^- = \frac{(2-\alpha) + \frac{1}{\theta} - \sqrt{[(2-\alpha) + \frac{1}{\theta}]^2 - 4(1-\alpha)(1+\frac{1}{\theta})^2}}{2(1+\frac{1}{\theta})}, \quad \text{and} \tag{71}$$

$$\hat{\psi}_{IFA}^+ = \frac{(2-\alpha) + \frac{1}{\theta} + \sqrt{[(2-\alpha) + \frac{1}{\theta}]^2 - 4(1-\alpha)(1+\frac{1}{\theta})^2}}{2(1+\frac{1}{\theta})}. \tag{72}$$

Equations (70)-(72) and $\lambda \in (0, 1 - \psi_{IFA})$ define the boundary of region B. For $\theta \to \infty$, $\lim_{\theta\to\infty} \hat{\psi}_{IFA}^+ = 1$ and $\lim_{\theta\to\infty} \hat{\psi}_{IFA}^- = 1 - \alpha$ coincide with equation (46) for setting **F**.

Consider the region of $\psi_{IFA} < \hat{\psi}_{IFA}^-$. As the slope of the phase diagram of wage at the initial steady state is smaller than one under FMG, the initial steady state under IFA is still a stable steady state under FMG. However, FMG may create multiple steady states in either one of the two cases as follows.

- Case 1: the kink point of the phase diagram of wage is above the 45° line, i.e., given $r_t^* = \frac{\lambda\rho}{1-\psi_{IFA}}$ and $\psi_t^i = 1 - \lambda$, $\bar{\omega}_{t+1}^i > \bar{\omega}_t^i$. According to equations (54)-(56),

$$\frac{\underline{\epsilon}_t^i[1 - (\underline{\epsilon}_t^i)^{-(\theta+1)}]}{\psi_{IFA}\underline{\epsilon}_{IFA}} = \left(\frac{1-\psi_{IFA}}{\lambda}\right)^{\frac{1}{1-\alpha}}, \qquad \frac{(1-\lambda)\underline{\epsilon}_t^i}{\psi_{IFA}\underline{\epsilon}_{IFA}} < \left(\frac{1-\psi_{IFA}}{\lambda}\right)^{\rho}. \tag{73}$$

$$\Rightarrow \quad \underline{\epsilon}_t^i < \left[1 - \frac{(1-\psi_{IFA})(1-\lambda)}{\lambda}\right]^{-\frac{1}{1+\theta}}. \tag{74}$$

  Combine equations (73)-(74) to get,

$$(1-\psi_{IFA})^{\rho}\psi_{IFA}\left(\frac{1}{1-\psi_{IFA}} - \frac{1-\lambda}{\lambda}\right)^{\frac{1}{1+\theta}} > (1-\lambda)\lambda^{\rho}, \tag{75}$$

[17] Equation (67) is a quadratic function of $\mathbb{D}_t$ and there are two roots for $\mathbb{D}_t$. However, only one root satisfies the condition of $\mathbb{D}_t < \psi_t^i < \psi_{IFA}$.

which specifies ψ_{IFA} as a function of λ and is the upward-sloping part of the boundary between region AB and A. For $\theta \to \infty$, equation (75) degenerates into the condition same as equation (47) for setting **F**.

- Case 2: the concave part of the phase diagram of wage is tangent with the 45° line,[18] i.e., given $r_t^* = r_{IFA} = \frac{\lambda\rho}{1-\psi_{IFA}}$, $\omega_t^i = \omega_{t+1}^i = \omega^M > \omega_{IFA}$ and $\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i} = 1$. Combine these conditions with equations (54)-(56) to get

$$\left\{\frac{\underline{\epsilon}_t^i[1-(\underline{\epsilon}_t^i)^{-(1+\theta)}]}{\underline{\epsilon}_{IFA}\psi_{IFA}}\right\}^{1-\alpha} = \frac{1-\psi_{IFA}}{1-\psi_t} = \left(\frac{\psi_t^i\underline{\epsilon}_t^i}{\psi_{IFA}\underline{\epsilon}_{IFA}}\right)^{\frac{1-\alpha}{\alpha}}, \tag{76}$$

$$\Rightarrow \quad 1-(\underline{\epsilon}_t^i)^{-(1+\theta)} = \mathbb{D}_t \equiv (1-\psi_{IFA})\frac{\psi_t^i}{1-\psi_t^i}, \text{ and } \psi_{IFA} < \psi_t^i < \mathbb{D}_t \leq 1, \tag{77}$$

$$\frac{\partial\omega_{t+1}^i}{\partial\omega_t^i} = \frac{\alpha}{\frac{\mathbb{D}_t}{1+\theta(1-\mathbb{D}_t)} + (1-\alpha)\frac{1-\psi_{IFA}}{\mathbb{D}_t}} = 1 \tag{78}$$

$$\Rightarrow \quad (\frac{1}{\alpha\theta}+1)\mathbb{D}_t^2 - [\frac{1-\psi_{IFA}}{\rho} + (\frac{1}{\theta}+1)]\mathbb{D} + \frac{1-\psi_{IFA}}{\rho}(1+\frac{1}{\theta}) = 0. \tag{79}$$

  As a root of equation (79), $\mathbb{D}_t$ is independent of λ. Combine it with equation (77) to solve for ψ_t^i and $\underline{\epsilon}_t^i$. Then, plug ψ_t^i and $\underline{\epsilon}_t^i$ in equation (76) to solve for ψ_{IFA}. Independent of λ, the threshold value ψ_{IFA} is the flat part of the boundary between region AB and A.[19]

  For $\theta \to \infty$, equation (79) has two roots, i.e., $\mathbb{D}_t = 1$ and $\mathbb{D}_t = \frac{1-\psi_{IFA}}{\rho}$. Combine $\mathbb{D}_t = \frac{1-\psi_{IFA}}{\rho}$ with equation (77) to get $\psi_t^i = 1-\alpha$. Then, plug it back in equation (76) to get $\psi_{IFA} = 1-\alpha$, which violates the condition of $\psi_{IFA} < \psi_t^i$. Thus, the solution should be $\mathbb{D}_t = 1$. Combine it with equation (77) to get $\psi_t^i = \frac{1}{2-\psi_{IFA}}$. For $\theta \to \infty$, $\underline{\epsilon}_t^i = (1-\mathbb{D}_t)^{-\frac{1}{1+\theta}} \to 1$ and $\underline{\epsilon}_{IFA} = (1-\psi_{IFA})^{-\frac{1}{1+\theta}} \to 1$. Inserting them in equation (76), one get the result identical as equation (44) for setting **F**.

To sum up, the boundary conditions for the five regions of figure 9 in setting **M** converge to those of figure 6 in setting **F**, if $\theta \to \infty$. □

[18] The analysis is almost the same as deriving the boundary condition of region BC and C, except for $r_t^* = r_{IFA} = \frac{\lambda\rho}{1-\psi_{IFA}}$

[19] Equation (79) is a quadratic function of $\mathbb{D}_t$ and there are two roots for $\mathbb{D}_t$. However, only one root satisfies the condition of $\psi_t^i < \mathbb{D}_t \leq 1$.