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Abstract

Large sample properties are studied for a first-order autoregression (AR(1)) with a root greater than unity. It is shown that, contrary to the AR coefficient, the least-squares (LS) estimator of the intercept and its t-statistic are asymptotically normal without requiring the Gaussian error distribution, and hence an invariance principle applies. While the invariance principle does not apply to the asymptotic distribution of the LS estimator of the AR coefficient, we show explicitly how it depends on the initial condition and the intercept. Also established are the asymptotic independence between the LS estimators of the intercept and the AR coefficient and the asymptotic independence between their t-statistics. Asymptotic theory for explosive processes is compared to that for unit root AR(1) processes and stationary AR(1) processes. The coefficient based test and the t test have better power for testing the hypothesis of zero intercept in the explosive process than in the stationary process.

1 Introduction

Consider a first-order autoregression (AR(1)) defined by

$$x_t = d + \alpha x_{t-1} + u_t, \quad x_0 \sim O_p(1), \tag{1.1}$$

where $\{u_t\}$ is a sequence of independent and identically distributed (i.i.d.) random errors with $E(u_t) = 0$, $E(u_t^2) = \sigma^2 \in (0, \infty)$ (i.e., $u_t \overset{iid}{\sim} (0, \sigma^2)$). The available sample is $\{x_t\}_{t=1}^T$.

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Let $\sum$ denote $\sum_{t=1}^{T}$. If d is known apriori and assumed zero without loss of generality, based on the available sample, the least-squares (LS) estimator of α is,

$$\widehat{\alpha} = \frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}. \tag{1.2}$$

If the value of d is unknown apriori, the LS estimators of α and d are, respectively,

$$\widehat{\alpha} = \frac{\sum \left(x_t - \overline{X}\right)\left(x_{t-1} - \overline{X}_-\right)}{\sum \left(x_{t-1} - \overline{X}_-\right)^2} \quad \text{and } \widehat{d} = \overline{X} - \widehat{\alpha}\overline{X}_-, \tag{1.3}$$

where $\overline{X} = \sum x_t/T$, $\overline{X}_- = \sum x_{t-1}/T$.

The limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics have been developed in the literature in several special cases of Model (1.1). Broadly speaking, there are three cases, corresponding to the stationary model ($|\alpha| < 1$), the unit root model ($\alpha = 1$), and the explosive model ($|\alpha| > 1$) with the prior knowledge that $d = 0$.

In this paper, we extend the literature by establishing the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics for the explosive AR(1) process with an unknown intercept. We show that the asymptotic normality and, hence, an invariance principle hold true for $\widehat{d}$ and its t-statistic without assuming the Gaussian error distribution. Moreover, we obtain the limiting distributions of $\widehat{\alpha}$ and its t-statistic and show how they depends explicitly on the intercept and the initial condition. Also established are the asymptotic independence between $\widehat{d}$ and $\widehat{\alpha}$ and the asymptotic independence between their t-statistics. Finally, we compare our asymptotic theory with those for other models and the comparison leads to several interesting new observations.

The rest of the paper is organized as follows. Section 2 reviews the literature. The asymptotic theory is developed in Section 3. Section 4 compares the new limit theory with that of the stationary models and of the unit root models. Section 5 concludes and discusses how to generalize our results. All the proofs are contained in Appendix. Throughout this paper, we use the notations $\Rightarrow$, $\xrightarrow{p}$, $\overset{d}{=}$ to denote weak convergence, convergence in probability, equivalence in distribution, respectively.

2 A Literature Review

We briefly review the literature in three cases, corresponding to the stationary model ($|\alpha| < 1$), the unit root model ($\alpha = 1$), and the explosive model ($|\alpha| > 1$) with the prior knowledge of zero intercept. Table 1 summarizes the results.

In Case 1 ($|\alpha| < 1$), if d is assumed to be known apriori and equal to 0, it is known that

$$\sqrt{T}\left(\widehat{\alpha} - \alpha\right) \Rightarrow N(0, 1 - \alpha^2),$$

and the conventional t-statistic asymptotically follows a standard normal distribution as

$$t_\alpha = \frac{(\widehat{\alpha} - \alpha)\left(\sum x_{t-1}^2\right)^{1/2}}{\sqrt{\hat{\sigma}^2}} \Rightarrow N(0,1),$$

where

$$\hat{\sigma}^2 = \frac{1}{T}\sum\left(x_t - \widehat{\alpha}x_{t-1}\right)^2.$$

If d has to be estimated together with α, we have

$$\sqrt{T}\left(\widehat{\alpha} - \alpha\right) \Rightarrow \frac{1-\alpha^2}{\sigma^2}\zeta \overset{d}{=} N\left(0, 1-\alpha^2\right)$$

and

$$\sqrt{T}\left(\widehat{d} - d\right) \Rightarrow w - \frac{d(1+\alpha)}{\sigma^2}\zeta \overset{d}{=} N\left(0, \sigma^2 + \frac{d^2(1+\alpha)}{1-\alpha}\right),$$

where

$$w \overset{d}{=} N\left(0,\sigma^2\right) \quad \text{and} \quad \zeta \overset{d}{=} N\left(0, \sigma^4/\left(1-\alpha^2\right)\right)$$

are the limits of $T^{-1/2}\sum u_t$ and $T^{-1/2}\sum x_{t-1}u_t - T^{-1/2}\sum u_t d\left(1-\alpha^{t-1}\right)/(1-\alpha)$, respectively, and are independent random variables.[1] The t-statistics are

$$t_\alpha \Rightarrow N(0,1), \quad t_d \Rightarrow N(0,1).$$

The rate of the convergence in Case 1 is always $\sqrt{T}$ for both estimators. Note that the limiting distribution of $\widehat{d}$ depends on α as well as d.

To review the asymptotic results in Case 2 ($\alpha = 1$), we firstly introduce some notations. We use $W(r)$ to represent a standard Brownian motion where $r \in [0,1]$. The integral sign $\int$ denotes integration from 0 to 1. For notational convenience, we often simply write $W(r)$ as W, $\int W(r)\,dW(r)$ as $\int W dW$and $\int W^2(r)\,dr$ as $\int W^2$.

For the unit root process with the prior knowledge that $d = 0$, it is known from Phillips (1987) that

$$T(\widehat{\alpha} - 1) \Rightarrow \frac{\int W dW}{\int W^2},$$

and that

$$t_\alpha \Rightarrow \frac{\int W dW}{\left(\int W^2\right)^{1/2}}.$$

If d is zero but has to be estimated together with α, it is known from Phillips and Perron (1988) that

$$T(\widehat{\alpha} - 1) \Rightarrow \frac{\int W dW - W(1)\int W}{\int W^2 - \left[\int W\right]^2}, \quad \sqrt{T}\widehat{d} \Rightarrow \sigma\frac{W(1)\int W^2 - \int W \int W dW}{\int W^2 - \left[\int W\right]^2},$$

[1] The independence between w and ζ is proved in Appendix under a further moment condition of $E\left|u_1\right|^{2+\delta} < \infty$ for some $\delta > 0$.

and

$$t_\alpha \Rightarrow \frac{\int W dW - W(1)\int W}{\left(\int W^2 - \left[\int W\right]^2\right)^{1/2}}, \quad t_d \Rightarrow \frac{W(1)\int W^2 - \int W \int W dW}{\left\{\int W^2 \left(\int W^2 - \left[\int W\right]^2\right)\right\}^{1/2}}.$$

However, when d is not zero and has to be estimated together with α, we have

$$T^{3/2}\left(\widehat{\alpha} - 1\right) \Rightarrow N(0, 12\sigma^2/d^2), \quad \sqrt{T}\left(\widehat{d} - d\right) \Rightarrow N(0, 4\sigma^2),$$

and

$$t_\alpha \Rightarrow N(0,1), \quad t_d \Rightarrow N(0,1).$$

As it can be seen from the discussion above, the rate of the convergence in Case 2 is parameter specific and also depends on if there is a non-zero constant in the model. The rate of the convergence can be T or $T^{3/2}$ for $\widehat{\alpha}$ and $\sqrt{T}$ for $\widehat{d}$. Even when the true value of d is zero, whether or not to estimate it leads to a different limiting distribution of $\widehat{\alpha}$. The deviation of d from zero leads to a change in the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$, as well as their t-statistics.

In Case 3 ($|\alpha| > 1$), if $x_0 = 0$, $u_t \overset{iid}{\sim} N(0, \sigma^2)$, $d = 0$ and is known apriori, White (1958) showed that[2]

$$\frac{\alpha^T}{\alpha^2 - 1}\left(\widehat{\alpha} - \alpha\right) \Rightarrow Cauchy.$$

When $u_t \overset{iid}{\sim} (0, \sigma^2)$ but is not necessarily normally distributed, Anderson (1959) showed that

$$\frac{\alpha^T}{a^2 - 1}\left(\widehat{\alpha} - \alpha\right) \Rightarrow y/z,$$

where y and z are the limits of y_T and z_T defined by

$$y_T = \sum_{t=1}^{T} \alpha^{-(T-t)} u_t \ \text{ and } z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0. \tag{2.1}$$

He argued that the limiting distributions of y_T and z_T, and hence of $\widehat{\alpha}$, depend on the distribution of u's, so no central limit theorem (CLT) or invariance principle is applicable. The role played by the initial condition in the limiting distribution could be found in z.

As it can be seen from the discussion above, the rate of the convergence in Case 3 depends on both T and α. The limiting distributions of the t-statistic of α are reported in Table 1.

A closely related literature, recently developed in econometrics and applied to detect bubbles in economic and financial time series, can be found in Phillips and Magdalinos (2009), Magdalinos (2012), Phillips, Wu and Yu (2011), Phillips, Shi and Yu (2013a, b, c). In this

[2] When $x_0 = c$, a nonzero constant, by deriving the limit of the moment generating function of $\hat{\alpha}$, White (1958) showed how the initial condition affects the limiting distribution of $\hat{\alpha}$.

Table 1: Summary of the limiting distributions of $\hat{\alpha}$ and $\hat{d}$ and their t-statistics.

Model	LS Estimates	t-statistics
$x_t = \alpha x_{t-1} + u_t$ $\lvert\alpha\rvert < 1, x_0 \sim O_p(1)$	$\sqrt{T}\left(\widehat{\alpha} - \alpha\right) \Rightarrow N(0, 1-\alpha^2)$	$t_\alpha \Rightarrow N(0,1)$
$x_t = d + \alpha x_{t-1} + u_t$ $\lvert\alpha\rvert < 1, x_0 \sim O_p(1)$	$\sqrt{T}\left(\widehat{\alpha} - \alpha\right) \Rightarrow N(0, 1-\alpha^2)$ $\sqrt{T}\left(\widehat{d} - d\right) \Rightarrow N\left(0, \sigma^2 + \frac{d^2(1+\alpha)}{1-\alpha}\right)$	$t_\alpha \Rightarrow N(0,1)$ $t_d \Rightarrow N(0,1)$
$x_t = \alpha x_{t-1} + u_t$ $\alpha = 1, x_0 \sim O_p(1)$	$T\left(\widehat{\alpha} - 1\right) \Rightarrow \int W dW / \int W^2$	$t_\alpha \Rightarrow \int W dW / \left(\int W^2\right)^{1/2}$
$x_t = d + \alpha x_{t-1} + u_t$ $\alpha = 1, d = 0, x_0 \sim O_p(1)$	$T\left(\widehat{\alpha} - 1\right) \Rightarrow \frac{\int W dW - W(1) \int W}{\int W^2 - \left[\int W\right]^2}$ $\sqrt{T}\widehat{d} \Rightarrow \sigma \frac{W(1) \int W^2 - \int W \int W dW}{\int W^2 - \left[\int W\right]^2}$	$t_\alpha \Rightarrow \frac{\int W dW - W(1) \int W}{\left(\int W^2 - \left[\int W\right]^2\right)^{1/2}}$ $t_d \Rightarrow \frac{W(1) \int W^2 - \int W \int W dW}{\left\{\int W^2 \left(\int W^2 - \left[\int W\right]^2\right)\right\}^{1/2}}$
$x_t = d + \alpha x_{t-1} + u_t$ $\alpha = 1, d \neq 0, x_0 \sim O_p(1)$	$T^{3/2}\left(\widehat{\alpha} - 1\right) \Rightarrow N(0, 12\sigma^2/d^2)$ $\sqrt{T}\left(\widehat{d} - d\right) \Rightarrow N(0, 4\sigma^2)$	$t_\alpha \Rightarrow N(0,1)$ $t_d \Rightarrow N(0,1)$
$x_t = \alpha x_{t-1} + u_t,\ x_0 = 0$ $\lvert\alpha\rvert > 1, u_t \overset{iid}{\sim} N(0, \sigma^2)$	$\frac{\alpha^T}{a^2-1}\left(\widehat{\alpha} - \alpha\right) \Rightarrow Cauchy$	$t_\alpha \Rightarrow N(0,1)$
$x_t = \alpha x_{t-1} + u_t,\ x_0 = c$ $\lvert\alpha\rvert > 1, u_t \overset{iid}{\sim} (0, \sigma^2)$	$T\left(\widehat{\alpha} - \alpha\right) \Rightarrow y/z$	$t_\alpha \Rightarrow y\lvert z\rvert \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} / z$

literature, AR models are considered to model explosive dynamic behavior. The common feature shared by our model and the models considered in this new literature is that the root is large than unity. The difference is the root is assumed to be a function of T and moderately larger than unity in the new literature whereas in Model (1.1) we assume the root is larger than unity but independent of T. Also related to our study are Phillips and Magdalinos (2008), Nielsen (2010), Engsted and Nielsen (2012) where an explosive cointegrated system is analyzed.

3 Limit Theory for the Explosive Process with Intercept

We now focus our attention on the following explosive AR(1) process with intercept:

$$x_t = d + \alpha x_{t-1} + u_t,\ t = 1, \cdots, T,\ \ |\alpha| > 1. \tag{3.1}$$

Assume $x_0 = O_p(1)$ which is independent of $\sigma(u_1, \cdots, u_T)$, and $u_t \overset{iid}{\sim} (0, \sigma^2)$. An equivalent representation of x_t is

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + \alpha^t x_0 + \sum_{j=0}^{t-1} \alpha^j u_{t-j}. \tag{3.2}$$

Obviously, $(1 - \alpha^t) d / (1 - \alpha)$ and $\alpha^t x_0$ have the same order of $O_p(\alpha^t)$ if $d \neq 0$. It becomes clear later that $\sum_{j=0}^{t-1} \alpha^j u_{t-j}$ has the order of $O_p(\alpha^t)$ too. This is the reason why both the intercept and the initial condition play an important role in the asymptotic theory for the explosive process. The model can also be expressed as

$$x_t = \frac{1 - \alpha^t}{1 - \alpha} d + x_t^0, \tag{3.3}$$

where x_t^0 is an explosive AR(1) process with no intercept.

Denote

$$w_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t. \tag{3.4}$$

Following the Lindeberg-Feller CLT, the limiting distribution of w_T is $N(0, \sigma^2)$. Following Anderson (1959), we define y_T and z_T as in Equation (2.1). In the following lemma, we give the limits of w_T, y_T, and z_T, and show that they are independent from each other.

Lemma 3.1 *Define w_T, y_T, and z_T as in Equation (3.4) and Equation (2.1). Then we have (a) $y_T \Rightarrow y, z_T \Rightarrow z$, and y and z are independent; (b) $w_T \Rightarrow w \overset{d}{=} N(0, \sigma^2)$ and w is independent of (y, z).*

The proof of (a) was given in Anderson (1959) and the proof of (b) is in Appendix. The intuition for the independence is as following. In z_T, the first few terms, such as x_0, u_1, u_2, u_3, make non negligible contribution and hence decide the asymptotic distribution of z_T. Similarly, in y_T, while the first few terms become negligible as T goes to infinity, the last few terms, say u_T, u_{T-1}, u_{T-2}, make non negligible contribution and hence decide the asymptotic distribution of y_T. In w_T, no single term makes significant contribution and the CLT takes over. Since u_t are independently distributed and x_0 is independent of $\{u_t\}$, it is not surprising that the limits of w_T, y_T, and z_T are independent.

To obtain the limiting distribution of the LS estimator of α in the explosive AR(1) model without intercept, Anderson (1959) proved that[3]

$$\left(\alpha^{-(T-2)} \sum x_{t-1}^0 u_t, \quad \left(\alpha^2 - 1\right) \alpha^{-2(T-1)} \sum \left(x_{t-1}^0\right)^2\right) \Rightarrow \left(yz, z^2\right). \tag{3.5}$$

Using this result together with the independence of w, y, z, we obtain the following asymptotic results.

Theorem 3.2 *For Model (3.1) with $|\alpha| > 1$, we have, as $T \to \infty$,*
(a) $\alpha^{-(T-1)} x_T \Rightarrow z + \alpha d / (\alpha - 1)$;
(b) $\alpha^{-(T-2)} \sum x_{t-1} u_t \Rightarrow y \left[z + \alpha d / (\alpha - 1)\right]$;
(c) $(\alpha - 1) \alpha^{-(T-1)} \sum x_{t-1} \Rightarrow z + \alpha d / (\alpha - 1)$;
(d) $\left(\alpha^2 - 1\right) \alpha^{-2(T-1)} \sum x_{t-1}^2 \Rightarrow \left[z + \alpha d / (\alpha - 1)\right]^2$.

Since $z_T = \alpha \sum_{t=1}^{T-1} \alpha^{-t} u_t + \alpha x_0$, not surprisingly, the initial condition αx_0 appears in the limit, z. According to Theorem 3.2, the intercept term d appears in all the asymptotic distributions. In particular, the intercept and the initial condition affect the asymptotic distributions in the same manner. This observation is consistent with the one in Equation (3.2) where the three terms on the right hand side has the same order of magnitude.

The centered LS estimators of d and α and their t-statistics are given by

$$\begin{pmatrix} \widehat{d} - d \\ \widehat{\alpha} - \alpha \end{pmatrix} = \begin{pmatrix} T & \sum x_{t-1} \\ \sum x_{t-1} & \sum x_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum u_t \\ \sum x_{t-1} u_t \end{pmatrix},$$

and

$$t_d = \frac{\left(\widehat{d} - d\right) \left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}},$$

$$t_\alpha = \frac{\left(\widehat{\alpha} - \alpha\right) \left[T \sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[T \times \hat{\sigma}^2\right]^{1/2}},$$

[3] Anderson's results were derived under the condition that x_0 is a constant. However, his proofs still hold true when $x_0 = O_p(1)$ is independent of $\sigma(u_1, \cdots, u_T)$.

where $\hat{\sigma}^2 = T^{-1} \sum \left(x_t - \hat{d} - \hat{\alpha} x_{t-1} \right)^2$.

Since $\sum x_{t-1} u_t$ and $\sum x_{t-1}$ have the same rate of convergence, α^{-T}, we have

$$\begin{pmatrix} \sqrt{T}\left(\widehat{d}-d\right) \\ \alpha^T\left(\widehat{\alpha}-\alpha\right) \end{pmatrix} = \begin{pmatrix} 1 & T^{-1/2}\alpha^{-T}\sum x_{t-1} \\ T^{-1/2}\alpha^{-T}\sum x_{t-1} & \alpha^{-2T}\sum x_{t-1}^2 \end{pmatrix}^{-1} \times \begin{pmatrix} T^{-1/2}\sum u_t \\ \alpha^{-T}\sum x_{t-1}u_t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & o_p(1) \\ o_p(1) & \alpha^{-2T}\sum x_{t-1}^2 \end{pmatrix}^{-1} \times \begin{pmatrix} T^{-1/2}\sum u_t \\ \alpha^{-T}\sum x_{t-1}u_t \end{pmatrix}.$$

Consequently, we have the following theorem which extends Anderson's results to the explosive AR(1) model with intercept.

Theorem 3.3 *For Model (3.1) with $|\alpha| > 1$, if $Pr\{z + \alpha d/(\alpha - 1) = 0\} = 0$, the following limits apply as $T \to \infty$:*

(a)

$$\sqrt{T}\left(\widehat{d}-d\right) = T^{-1/2}\sum u_t + o_p(1) \Rightarrow w \stackrel{d}{=} N\left(0,\sigma^2\right), \tag{3.6}$$

(b)

$$\frac{\alpha^T}{\alpha^2-1}\left(\widehat{\alpha}-\alpha\right) = \frac{\alpha^{-(T-2)}\sum x_{t-1}u_t}{\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2} + o_p(1) \Rightarrow \frac{y}{z+\alpha d/(\alpha-1)}, \tag{3.7}$$

(c)

$$\hat{\sigma}^2 = T^{-1}\sum\left(x_t - \hat{d} - \hat{\alpha}x_{t-1}\right)^2 \xrightarrow{p} \sigma^2, \tag{3.8}$$

(d)

$$t_d = \frac{\sqrt{T}\left(\widehat{d}-d\right)}{\left\{\hat{\sigma}^2\right\}^{1/2}} + o_p(1) \Rightarrow \frac{w}{\sigma} \stackrel{d}{=} N(0,1), \tag{3.9}$$

(e)

$$t_\alpha \Rightarrow \frac{y}{z+\alpha d/(\alpha-1)} \times \left| z + \frac{\alpha d}{\alpha - 1}\right| \times \left\{\frac{\alpha^2-1}{\alpha^2\sigma^2}\right\}^{1/2}. \tag{3.10}$$

Remark 3.4 *An invariance principle exists for $\widehat{d}$ and its t-statistic as Equations (3.6) and (3.9) hold true even when u_t is not normally distributed.*

Remark 3.5 *In Equation (3.7), if $d = 0$, the limiting distribution becomes y/z which is the same as that derived by Anderson (1959) for the model without intercept and the intercept is not estimated. It implies that when $d = 0$ the limiting distribution is the same regardless of whether or not d is estimated. This is not surprisingly as $x_t = x_t^0$ when $d = 0$. Hence, $\alpha^{-(T-2)}\sum x_{t-1}u_t = \alpha^{-(T-2)}\sum x_{t-1}^0 u_t$, $\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2 = \left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum\left(x_{t-1}^0\right)^2$, suggesting the middle term in Equation (3.7) is the same as the ratio of the two terms in Equation (3.5). This result is in sharp contrast to the unit root model reviewed in the last section.*

Remark 3.6 *With the same intuition as before, the distributions of both z and y depend on the distribution of u_t. Hence, no invariance principle applies to $\widehat{\alpha}$ and its t-statistic.*

Remark 3.7 *The independence of w, y and z suggests $\sqrt{T}\left(\widehat{d}-d\right)$ and $\alpha^T\left(\widehat{\alpha}-\alpha\right)/\left(\alpha^2-1\right)$ are asymptotically independent. Similarly, t_d and t_α are asymptotically independent.*

Remark 3.8 *As apparent in Theorem 3.3 (a) and (d), neither the initial condition (x_0) nor the intercept (d) can be found in the limiting distributions of $\sqrt{T}\left(\widehat{d}-d\right)$ and t_d. In sharp contrast, both the initial condition and the intercept appear in the limiting distributions of $\frac{\alpha^T}{\alpha^2-1}\left(\widehat{\alpha}-\alpha\right)$ and t_α. In fact, they play the same role in the limiting distributions. It is worth noting that what matters in the limiting distributions is not x_0 or d, but x_0/σ and d/σ. This point can be seen more clearly by studying a special case where $u_t \overset{iid}{\sim} N(0,\sigma^2)$. In this case, we get*

$$y = N(0,\alpha^2\sigma^2/\left(\alpha^2-1\right)) \quad and \quad z = N(0,\alpha^2\sigma^2/\left(\alpha^2-1\right)) + \alpha x_0,$$

which are independently distributed. Let

$$\xi := \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} y, \quad and \quad \eta := \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} z - \left(\frac{\alpha^2-1}{\alpha^2\sigma^2}\right)^{1/2} \alpha x_0,$$

be two independent $N\left(0,1\right)$ random variables. Then, Theorem 3.3 (b) becomes

$$\frac{\alpha^T}{\alpha^2-1}\left(\widehat{\alpha}-\alpha\right) \Rightarrow \frac{\xi}{\eta+\sqrt{\left(\alpha^2-1\right)/\alpha^2}\left[\alpha x_0/\sigma+\alpha d/\sigma\left(\alpha-1\right)\right]}.$$

It can be seen that both x_0/σ and d/σ, but not x_0 and d, determine the limiting distribution of $\widehat{\alpha}$. When $x_0 = d = 0$, we obtain the standard Cauchy limiting distribution. The dependence on the ratio of x_0/σ and d/σ was also found in the unit root and local-to-unity literature. See, for example, Phillips (1987) and Perron (1991).

Remark 3.9 *While in general the limiting distribution of t_α depends on both the initial value and the intercept as shown in Equation (3.10), the result is remarkably different when $u_t \overset{iid}{\sim} N(0,\sigma^2)$. In this case, we have*

$$t_\alpha \Rightarrow \frac{\xi}{z+\alpha d/\left(\alpha-1\right)}\left|z+\alpha d/\left(\alpha-1\right)\right|.$$

Let $P_+ = \Pr\left\{z+\alpha d/\left(\alpha-1\right)>0\right\}$ and $P_- = \Pr\left\{z+\alpha d/\left(\alpha-1\right)<0\right\}$. Then, from the independence of ξ and z, we obtain the moment generating function for the limit of t_α,

$$P_+\cdot E\left(\exp\left\{t\xi\right\}\right)+P_-\cdot E\left(\exp\left\{-t\xi\right\}\right) = P_+\cdot\exp\left\{t^2/2\right\}+P_-\cdot\exp\left\{t^2/2\right\} = \exp\left\{t^2/2\right\}.$$

Therefore, $t_\alpha \Rightarrow N\left(0,1\right)$ which does not depend on the initial condition nor the intercept.

4 Comparison with Other Models

In this section, we will compare the limit theory of the explosive AR(1) process and that of the stationary AR(1) process and the unit root AR(1) process. Some new interesting observations are discussed.

4.1 Explosive root versus unit root

The limit theory for the explosive model and that for the unit root model are distinctively different. First, for a unit root model, we have

$$x_t = x_0 + \sum_{j=0}^{t-1} u_{t-j} = O_p\left(\sqrt{t}\right), \text{ when } d = 0,$$

and

$$x_t = dt + x_0 + \sum_{j=0}^{t-1} u_{t-j} = O_p(t), \text{ when } d \neq 0.$$

Obviously, the presence of a nonzero intercept changes the asymptotic property of x_t, and consequently, leads to a change in the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics, as reported in Table 1. The discontinuity of the limiting distributions of $\widehat{d}$ and t_d at $d = 0$ makes it hard to analyze the local power behavior when they are used to test $d = 0$. To analyze the local power, we often use the limit theory for the unit root model with an intercept dependent on T:

$$x_t = d_T + \alpha x_{t-1} + u_t \text{ with } d_T = d/\sqrt{T},\ \alpha = 1.$$

In contrast, for the explosive process, the limiting distributions of $\widehat{\alpha}$ and $\widehat{d}$ and their t-statistics become continuous at the point $d = 0$ as we have shown in the Theorem 3.3. Hence, the local power can be obtained directly.

Second, for the explosive process, we have shown in Theorem 3.3 that $\widehat{d} - d$ and $\widehat{\alpha} - \alpha$ are asymptotically independent, and that t_d and t_α are also asymptotically independent, regardless of the value of d. In contrast, when a unit root process is considered, the asymptotic distributions of $\widehat{d} - d$ and $\widehat{\alpha} - \alpha$ (as well as t_d and t_α) are always correlated, and the strength of the correlation varies as the value of d changes.

For the explosive process, the comparison of the limit theory between Anderson (1959) and Theorem 3.3 reveals that, when the intercept is zero, the limiting distribution of $\widehat{\alpha}$ is the same regardless of whether or not the intercept is estimated. On the other hand, for unit root process, the estimation of the intercept changes the limiting distribution of $\widehat{\alpha}$.

4.2 Explosive root versus stationary root

The differences between the explosive process and the stationary model are more subtle and have important implications. First, for the stationary AR(1) process, the limiting distribution of $\sqrt{T}\left(\widehat{d}-d\right)$ is a linear combination of the limiting distribution of $T^{-1/2}\sum u_t$ and that of $T^{-1/2}\sum x_{t-1}u_t$. As a result, the asymptotic variance of $\sqrt{T}\left(\widehat{d}-d\right)$ is $\sigma^2+d^2\left(1+\alpha\right)/\left(1-\alpha\right)$ which depends on d. For the explosive AR(1) process, the limiting distribution of $\sqrt{T}\left(\widehat{d}-d\right)$ is dominated by $T^{-1/2}\sum u_t$, the asymptotic distribution of which, as shown in Equation (3.6), is $N\left(0,\sigma^2\right)$ whose variance is independent of d. This distinction sheds insights on the differences in the finite sample power behavior of the test of the null hypothesis $H_0 : d = 0$ in the context of the explosive process and the stationary AR(1) process. Under the null,

$$\sqrt{T}\widehat{d} \Rightarrow w \overset{d}{=} N\left(0,\sigma^2\right),$$

for both the explosive and the stationary models. Under the alternative hypothesis $H_1 : d \neq 0$, the finite sample distribution of $\sqrt{T}\widehat{d}$ can be approximated by

$$\sqrt{T}\widehat{d} \overset{d}{\approx} w + \sqrt{T}d \overset{d}{=} N\left(\sqrt{T}d,\ \sigma^2\right), \quad \text{if } |\alpha| > 1,$$

and by

$$\sqrt{T}\widehat{d} \overset{d}{\approx} w - \frac{d\left(1+\alpha\right)}{\sigma^2}\zeta + \sqrt{T}d \overset{d}{=} N\left(\sqrt{T}d,\ \sigma^2 + \frac{d^2\left(1+\alpha\right)}{1-\alpha}\right), \quad \text{if } |\alpha| < 1.$$

Note that the shift of the mean is the same in both cases. However, when $|\alpha| < 1$, the variance of the finite sample distribution increases with $|d|$ whereas when $|\alpha| > 1$, the variance of the finite sample distribution remains unchanged. Therefore, we expect the test to have a better power for the explosive model than for the stationary model.

A similar observation applies to the t test. Under the null hypothesis $H_0 : d = 0$, for both the explosive process and the stationary process, we have

$$\tilde{t}_d = \frac{\widehat{d}\left[T\sum x_{t-1}^2 - \left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \Rightarrow N\left(0,1\right).$$

Under the alternative hypothesis that $H_1 : d \neq 0$, Theorem 3.3 (d) gives us an approximation of the finite sample distribution of $\tilde{t}_d$ for the explosive case:

$$\begin{aligned}
\tilde{t}_d &= t_d + \frac{\sqrt{T}d\left[\sum x_{t-1}^2 - T^{-1}\left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \\
&= \frac{\sqrt{T}\left(\widehat{d}-d\right)}{\left\{\hat{\sigma}^2\right\}^{1/2}} + \frac{\sqrt{T}d}{\left\{\hat{\sigma}^2\right\}^{1/2}} + o_p\left(1\right) \overset{d}{\approx} \frac{w}{\sigma} + \frac{\sqrt{T}d}{\sigma} \overset{d}{=} N\left(\frac{\sqrt{T}d}{\sigma}, 1\right).
\end{aligned}$$

For the stationary case, the approximation of the finite sample distribution of $\tilde{t}_d$ is given by

$$\begin{aligned}\tilde{t}_d &= t_d + \frac{\sqrt{T}d\left[\sum x_{t-1}^2 - T^{-1}\left(\sum x_{t-1}\right)^2\right]^{1/2}}{\left[\sum x_{t-1}^2 \times \hat{\sigma}^2\right]^{1/2}} \\ &\overset{d}{\approx} N\left(\sqrt{T}d\left[\frac{1}{\sigma^2 + d^2\left(1+\alpha\right)/\left(1-\alpha\right)}\right]^{1/2}, 1\right).\end{aligned}$$

Note that in both cases, the variance of the approximate finite sample distribution is the same but the means are different. Since $\sigma^2 + d^2\left(1+\alpha\right)/\left(1-\alpha\right) > \sigma^2$ when $|\alpha| < 1$, we have

$$\sqrt{T}d\left[\frac{1}{\sigma^2 + d^2\left(1+\alpha\right)/\left(1-\alpha\right)}\right]^{1/2} < \frac{\sqrt{T}d}{\sigma},$$

the shift of the mean of $\tilde{t}_d$ from H_0 to H_1 under the explosive model is greater than that under the stationary model. Therefore, the t test is expected to have better power for the explosive process than for the stationary process.

Second, for the explosive process, the results in Theorem 3.3 suggest that, regardless of the value of d, $\widehat{d} - d$ and $\widehat{\alpha} - \alpha$ are asymptotically independent, and t_d and t_α are also asymptotically independent. On the contrary, for the stationary process, the asymptotic independence between $\widehat{d} - d$ and $\widehat{\alpha} - \alpha$ and that between t_d and t_α can only be guaranteed by the condition of $d = 0$.

Third, for the explosive process with intercept, the value of d affects the limiting distributions of $\sum x_{t-1}u_t$ and $\sum x_{t-1}^2$, and, hence, the limiting distribution of $\widehat{\alpha} - \alpha$, as shown in Theorem 3.3. The value of d has no impact on the limiting distribution of $\widehat{d} - d$ because it is decided by the unique dominating term, $T^{-1/2}\sum u_t$. On the contrary, for the stationary process with an intercept, the magnitude of d does not change the limiting distribution of $\widehat{\alpha} - \alpha$, but only affect the limiting distribution of $\widehat{d} - d$.

5 Discussions and Conclusions

In this paper the asymptotic theory is developed for the explosive AR(1) process with intercept. The results extend the literature in several directions. First, it is proved that an invariance principle applies to the intercept and its t-statistic while it continues to fail to apply to the AR coefficient. Second, the asymptotic independence between LS estimators of the intercept and the AR coefficient and the asymptotic independence between their t-statistics are established. Third, the comparison conducted in the paper reveals that the coefficient based test and the t test have better power for testing $H_0 : d = 0$ under the explosive process than under the stationary process.

It may be interesting to consider the case where the condition $u_t = i.i.d.(0, \sigma^2)$ is replaced by $u_t = i.ni.d.(0, \sigma_t^2)$. With the assumption of heteroskedasticity, we need a condition such as $E\left(u_t^2\right) = \sigma_t^2 < M$ for some $M < \infty$. In this case, the asymptotic results have to be modified by replacing $\alpha^2\sigma^2/\left(\alpha^2 - 1\right)$ with

$$\sum_{s=0}^{\infty} \alpha^{-2s}\sigma_{s+1}^2,$$

which exists as it is less than $\alpha^2 M/\left(\alpha^2 - 1\right)$. It is worth noting that when u_t are not identically distributed, the limit of y_T may not exist as u_T always plays non negligible effect. Therefore, extra assumptions that ensure the existence of the limit of y_T are required. Whereas, it is easy to show that z_T is still a Cauchy sequence in the L^2 space, and hence, $z_T \Rightarrow z$ remains true.

To obtain the limiting distribution of $\hat{\alpha}$, the restriction that $Pr\{z + \alpha d/\left(\alpha - 1\right) = 0\} = 0$ is needed. It is possible that this restriction is violated. For example, if $d = 0$, $x_0 = 0$, and u_t follows a Poisson distribution taking 0 with probability of $\exp\{-\left(1/2\right)^t\}$, then, the probability of $z = 0$ is $\exp\{-\sum_{t=1}^{\infty}\left(1/2\right)^t\} = 1/e > 0$. As a result, $Pr\{z + \alpha d/\left(\alpha - 1\right) = 0\} > 0$ and hence the limiting distribution of $\hat{\alpha}$ is not well defined.

APPENDIX

Proof of the independence between w and ζ. Note that the AR(1) process as in (1.1) can be equivalently expressed as

$$x_t = \frac{1-\alpha^t}{1-\alpha}d + \alpha^t x_0 + \sum_{j=0}^{t-1}\alpha^j u_{t-j} = \frac{1-\alpha^{t-1}}{1-\alpha}d + x_t^0$$

where $x_t^0 = \alpha x_{t-1}^0 + u_t$ is an AR(1) process with no intercept. As a result,

$$T^{-1/2}\sum x_{t-1}u_t - T^{-1/2}\sum u_t d\left(1-\alpha^{t-1}\right)/\left(1-\alpha\right) = T^{-1/2}\sum x_{t-1}^0 u_t.$$

Then, by the Cramér-Wold device (e.g. Kallenberg, 2002, Corollary 5.5), it is sufficient to show that

$$a \cdot T^{-1/2}\sum u_t + b \cdot T^{-1/2}\sum x_{t-1}^0 u_t \Rightarrow a \cdot w + b \cdot \zeta, \quad \text{for all } a, b \in \mathbb{R},$$

where $w = N\left(0, \sigma^2\right)$ and $\zeta = N\left(0, \sigma^4/\left(1-\alpha^2\right)\right)$ are independent random variables. If Y is an $N\left(0, a^2\sigma^2 + b^2\sigma^4/\left(1-\alpha^2\right)\right)$ random variable, $a \cdot w + b \cdot \zeta \overset{d}{=} Y$, so $a \cdot T^{-1/2}\sum u_t + b \cdot T^{-1/2}\sum x_{t-1}^0 u_t \Rightarrow Y$, for all $a, b \in \mathbb{R}$, is sufficient to show the asymptotic independence. We can write

$$a \cdot T^{-1/2}\sum u_t + b \cdot T^{-1/2}\sum x_{t-1}^0 u_t = \sum \frac{au_t + bx_{t-1}^0 u_t}{\sqrt{T}} = \sum Y_{Tt},$$

where Y_{Tt} is a martingale difference sequence. Hence, week convergence to a Gaussian random variable can be derived as a consequence of the CLT for the martingale difference sequence (e.g. Hall and Heyde, 1980, Corollary 3.1).

The conditional variance is given by

$$\begin{aligned} V_{TT}^2 &= \sum_{t=1}^{T} E\left(Y_{Tt}^2 \middle| \mathfrak{F}_{T(t-1)}\right) = \frac{1}{T}\sum_{t=1}^{T}\left(a + bx_{t-1}^0\right)^2 E\left(u_t^2\right) \\ &= \frac{\sigma^2}{T}\left(\sum a^2 + \sum \left(bx_{t-1}^0\right)^2 + \sum 2abx_{t-1}^0\right) \\ &= \sigma^2 a^2 + \frac{\sigma^2 b^2}{T}\sum \left(x_{t-1}^0\right)^2 + o_p(1) \Rightarrow a^2\sigma^2 + \frac{b^2\sigma^4}{1-\alpha^2}, \end{aligned}$$

where the fourth equation and the final asymptotic result come from

$$T^{-1}\sum x_{t-1}^0 \xrightarrow{p} 0, \quad \text{as } T \to \infty,$$

and

$$T^{-1}\sum_{t=1}^{\lfloor Ts \rfloor}\left[x_{t-1}^0\right]^2 \xrightarrow{p} s\sigma^2/\left(1-\alpha^2\right) \quad \text{for any } s \in [0,1],$$

respectively, which are standard asymptotic results for stationary process x_t^0.

To prove the conditional Lindeberg condition, we first get, for any $\varepsilon > 0$,

$$\begin{aligned} &\sum_{t=1}^{T} E\left(Y_{Tt}^2 1\left\{\left|Y_{Tt}\right| > \varepsilon\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \\ = \; & \frac{1}{T}\sum_{t=1}^{T}\left(a + bx_{t-1}^0\right)^2 E\left(u_t^2 1\left\{\left|\frac{au_t + bx_{t-1}^0 u_t}{\sqrt{T}}\right| > \varepsilon\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \\ \leq \; & V_{TT}^2 \cdot \max_{1\leq t\leq T} E\left(u_t^2 1\left\{\left|\frac{au_t + bx_{t-1}^0 u_t}{\sqrt{T}}\right| > \varepsilon\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \\ \leq \; & V_{TT}^2 \cdot \max_{1\leq t\leq T} E\left(u_t^2 1\left\{\left|\frac{au_t}{\sqrt{T}}\right| + \left|\frac{bx_{t-1}^0 u_t}{\sqrt{T}}\right| > \varepsilon\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \\ \leq \; & V_{TT}^2 \cdot \max_{1\leq t\leq T} E\left(u_t^2 1\left\{\left|\frac{bx_{t-1}^0 u_t}{\sqrt{T}}\right| > \frac{\varepsilon}{2}\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \text{ when } T \text{ is large,} \end{aligned}$$

where the last inequality is based on the fact that $u_t/\sqrt{T} \to 0$ almost everywhere as $T \to \infty$. Hence, the conditional Lindeberg condition will be satisfied if

$$\max_{1\leq t\leq T} E\left(u_t^2 1\left\{\left|x_{t-1}^0 u_t\right| > \frac{\sqrt{T}\varepsilon}{2\left|b\right|}\right\} \middle| \mathfrak{F}_{T(t-1)}\right) \xrightarrow{p} 0 \text{ as } T \to \infty.$$

Applying the Hölder and Chebyshev inequalities, we obtain, for some $\delta > 0$

$$\begin{aligned}
& E_{\mathfrak{F}_{T(t-1)}}\left(u_t^2 1\left\{\left|x_{t-1}^0 u_t\right| > \frac{\sqrt{T}\varepsilon}{2|b|}\right\}\right) \\
\leq \ & E_{\mathfrak{F}_{T(t-1)}}^{2/(2+\delta)}\left(|u_t|^{2+\delta}\right) \cdot P_{\mathfrak{F}_{T(t-1)}}^{\delta/(2+\delta)}\left\{\left|x_{t-1}^0 u_t\right| > \frac{\sqrt{T}\varepsilon}{2|b|}\right\} \\
\leq \ & \left(E|u_t|^{2+\delta}\right)^{2/(2+\delta)} \cdot \left(\frac{E_{\mathfrak{F}_{T(t-1)}}\left\{\left|x_{t-1}^0 u_t\right|^2\right\}}{T\varepsilon^2/2b^2}\right)^{\delta/(2+\delta)} \\
= \ & \left(E|u_1|^{2+\delta}\right)^{2/(2+\delta)} \cdot \left(\frac{\left(x_{t-1}^0\right)^2}{T}\right)^{\delta/(2+\delta)} \cdot \left(\frac{\sigma^2}{\varepsilon^2/2b^2}\right)^{\delta/(2+\delta)}, \ \text{ for each } t \in \{1, \cdots, T\}.
\end{aligned}$$

Assume $E|u_1|^{2+\delta} < \infty$, then

$$\max_{1\leq t\leq T} \frac{\left(x_{t-1}^0\right)^2}{T} \xrightarrow{p} 0 \ \text{ as } \ T \to \infty$$

is sufficient for satisfaction of the conditional Lindeberg condition. For $m \in \{1, \cdots, T\}$, define the sets

$$B_{T,m} := \bigcap_{j=1}^{m}\left\{\omega : \left|\frac{1}{T}\sum_{t=1}^{\lfloor T(j/m)\rfloor}\left[x_{t-1}^0(\omega)\right]^2 - \frac{j}{m}\frac{\sigma^2}{1-\alpha^2}\right| \leq \frac{1}{m}\right\}.$$

As $T^{-1}\sum_{t=1}^{\lfloor Ts\rfloor}\left[x_{t-1}^0\right]^2 - s\sigma^2/(1-\alpha^2)$ for each $s \in [0,1]$, we have $P(B_{T,m}) \to 1$ as $T \to \infty$. Next, note that

$$\max_{1\leq t\leq T} \frac{\left(x_{t-1}^0\right)^2}{T} \leq \frac{1}{T}\sup_{s\in[0,1]}\sum_{t=\lfloor Ts\rfloor+1}^{\lfloor T(s+1/m)\rfloor}\left[x_{t-1}^0\right]^2.$$

For given $s \in [0,1]$ choose $j \in \{1, \cdots, m\}$ so that $s \in [(j-1)/m, j/m]$. Then, for each $s \in [0,1]$, $\omega \in B_{T,m}$ implies

$$\begin{aligned}
\frac{1}{T}\sum_{t=\lfloor Ts\rfloor+1}^{\lfloor T(s+1/m)\rfloor}\left[x_{t-1}^0\right]^2 \ & \leq \ \frac{1}{T}\sum_{t=\lfloor T(j-1)/m\rfloor+1}^{\lfloor T(j+1)/m\rfloor}\left[x_{t-1}^0\right]^2 \\
& = \ \left(\frac{1}{T}\sum_{t=1}^{\lfloor T(j+1)/m\rfloor}\left[x_{t-1}^0\right]^2 - \frac{j+1}{m}\frac{\sigma^2}{1-\alpha^2}\right) \\
& \quad - \left(\frac{1}{T}\sum_{t=1}^{t=\lfloor T(j-1)/m\rfloor}\left[x_{t-1}^0\right]^2 - \frac{j-1}{m}\frac{\sigma^2}{1-\alpha^2}\right) + \frac{2\sigma^2}{m(1-\alpha^2)} \\
& \leq \ \frac{2}{m} + \frac{2\sigma^2}{m(1-\alpha^2)} = \frac{2}{m}\left(1 + \frac{\sigma^2}{1-\alpha^2}\right).
\end{aligned}$$

Thus, for any $m \in N$,

$$\lim_{T\to\infty} P\left\{\max_{1\le t\le T} \frac{\left(x_{t-1}^0\right)^2}{T} \le \frac{2}{m}\left(1+\frac{\sigma^2}{1-\alpha^2}\right)\right\} \ge \lim_{T\to\infty} P\left(B_{T,m}\right) = 1.$$

Therefore, the conditional Lindeberg condition is satisfied, and the independence between w and ζ follows. ■

Proof of Lemma 3.1. (b): The fact of $w_T \Rightarrow w \stackrel{d}{=} N\left(0,\sigma^2\right)$ simply follows the Lindeberg-Feller CLT. To prove the independence between w and z, let

$$z_T^* = \alpha x_0 + \alpha \sum_{s=1}^{\lfloor\sqrt{T}\rfloor} \alpha^{-s} u_s, \quad \tilde{z}_T = \alpha \sum_{s=\lfloor\sqrt{T}\rfloor+1}^{T-1} \alpha^{-s} u_s,$$

and

$$w_T^* = \frac{1}{\sqrt{T}} \sum_{s=\lfloor\sqrt{T}\rfloor+1}^{T-1} u_s, \quad \tilde{w}_T = \frac{1}{\sqrt{T}} \sum_{s=1}^{\lfloor\sqrt{T}\rfloor} u_s,$$

where $\left\lfloor\sqrt{T}\right\rfloor$ is the largest integer not greater than $\sqrt{T}$. Then z_T^* and w_T^* are independently distributed because they involve disjoint sets of u's. As T goes to infinity, we have

$$\begin{aligned} E\left(z_T - z_T^*\right)^2 &= E\left(\tilde{z}_T\right)^2 = \left(\alpha^2 \sum_{s=\lfloor\sqrt{T}\rfloor+1}^{T-1} \alpha^{-2s}\right)\sigma^2 \\ &= \frac{\alpha^{-2\left(\lfloor\sqrt{T}\rfloor-1\right)}\left(1-\alpha^{-2\left(T-\lfloor\sqrt{T}\rfloor-1\right)}\right)}{\alpha^2-1}\sigma^2 \to 0, \end{aligned}$$

and

$$E\left(w_T - w_T^*\right)^2 = E\left(\tilde{w}_T\right)^2 = \frac{\left\lfloor\sqrt{T}\right\rfloor}{T}\sigma^2 \to 0.$$

Then, $z_T - z_T^*$ and $w_T - w_T^*$ converge with probability 1 to 0, therefore, the asymptotic independence between z_T and w_T follows. The independence between w and y can be proved in a similar way. ■

Proof of Theorem 3.2. (a): Starting from Equation (3.3), we have

$$\begin{aligned} \alpha^{-(T-1)} x_T &= \alpha^{-(T-1)}\left(\frac{1-\alpha^T}{1-\alpha} d + x_T^0\right) = \alpha^{-(T-1)}\left(\frac{1-\alpha^T}{1-\alpha} d + \alpha x_{T-1}^0 + u_T\right) \\ &= \frac{\alpha d}{\alpha-1} + \alpha^{-(T-2)} x_{T-1}^0 + o_p(1) \\ &= \frac{\alpha d}{\alpha-1} + z_T + o_p(1) \Rightarrow z + \frac{\alpha d}{\alpha-1}, \end{aligned}$$

where the fourth equality comes from the definition of z_T in (2.1), and the final limit is a result of Lemma 3.1.

(b): Again, starting from Equation (3.3), it can be obtained that

$$\begin{aligned}\alpha^{-(T-2)}\sum x_{t-1}u_t &= \alpha^{-(T-2)}\sum x_{t-1}^0u_t+\alpha^{-(T-2)}\sum\frac{\left(1-\alpha^{t-1}\right)d}{1-\alpha}u_t\\ &= \alpha^{-(T-2)}\sum x_{t-1}^0u_t-\frac{\alpha d}{1-\alpha}\sum\alpha^{-(T-t)}u_t+\frac{d}{1-\alpha}\alpha^{-(T-2)}\sum u_t\\ &= \alpha^{-(T-2)}\sum x_{t-1}^0u_t+\frac{\alpha d}{\alpha-1}y_T+o_p\left(1\right)\\ &\Rightarrow yz+\frac{\alpha d}{\alpha-1}y=y\left(z+\frac{\alpha d}{\alpha-1}\right),\end{aligned}$$

where the third equality comes from the definition of y_T in (2.1), and the combination of the results in Lemma 3.1 and Equation (3.5) leads to the final limit.

(c): From Model (3.1) it is easy to get $x_t-x_{t-1}=d+\left(\alpha-1\right)x_{t-1}+u_t$. Then,

$$\left(\alpha-1\right)\sum x_{t-1}=x_T-x_0-Td-\sum u_t.$$

Hence, based on the limiting distribution derived in (a), we have

$$\begin{aligned}\left(\alpha-1\right)\alpha^{-(T-1)}\sum x_{t-1} &= \alpha^{-(T-1)}\left(x_T-x_0\right)-\alpha^{-(T-1)}Td-\alpha^{-(T-1)}\sum u_t\\ &= \alpha^{-(T-1)}x_T+o_p\left(1\right)\Rightarrow z+\frac{\alpha d}{\alpha-1}.\end{aligned}$$

(d): Squaring both sides of Model (3.1), we get

$$x_t^2=\alpha^2x_{t-1}^2+2\alpha dx_{t-1}+2\alpha x_{t-1}u_t+d^2+u_t^2+2du_t.$$

Therefore, $x_t^2-x_{t-1}^2=\left(\alpha^2-1\right)x_{t-1}^2+2\alpha dx_{t-1}+2\alpha x_{t-1}u_t+d^2+u_t^2+2du_t$, which leads to

$$\left(\alpha^2-1\right)\sum x_{t-1}^2=x_T^2-x_0^2-2\alpha d\sum x_{t-1}-2\alpha\sum x_{t-1}u_t-Td^2-\sum u_t^2-2d\sum u_t.$$

Based on the results reported in (a), (b), (c) and the assumption that $x_0=O_p\left(1\right)$, it is straightforward to get

$$\left(\alpha^2-1\right)\alpha^{-2(T-1)}\sum x_{t-1}^2=\alpha^{-2(T-1)}x_T^2+o_p\left(1\right)\Rightarrow\left(z+\frac{\alpha d}{\alpha-1}\right)^2.$$

■

Proof of Theorem 3.3. The results come immediately from the Lemma 3.1 and Theorem 3.2, hence the proofs are omitted. ■

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