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Non-normality and heteroscedasticity robust LM tests of spatial dependence

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HETEROSKEDASTICITY AND NON-NORMALITY ROBUST LM TESTS FOR SPATIAL DEPENDENCE

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Abstract

The standard LM tests for spatial dependence in linear and panel regressions are derived under the normality and homoskedasticity assumptions of the regression disturbances. Hence, they may not be robust against non-normality or heteroskedasticity of the disturbances. Following Born and Breitung (2011), we introduce general methods to modify the standard LM tests so that they become robust against heteroskedasticity and non-normality. The idea behind the robustification is to decompose the concentrated score function into a sum of uncorrelated terms so that the outer product of gradient (OPG) can be used to estimate its variance. We also provide methods for improving the finite sample performance of the proposed tests. These methods are then applied to several popular spatial models. Monte Carlo results show that they work well in finite sample.

JEL Classification: C21, C23, C5

Key Words: Centering; Heteroskedasticity; Non-normality; LM test; Panel model; Spatial dependence.

Heteroskedasticity and Non-normality Robust LM Tests for Spatial Dependence[1]

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Abstract

The standard LM tests for spatial dependence in linear and panel regressions are derived under the normality and homoskedasticity assumptions of the regression disturbances. Hence, they may not be robust against non-normality or heteroskedasticity of the disturbances. Following Born and Breitung (2011), we introduce general methods to modify the standard LM tests so that they become robust against heteroskedasticity and non-normality. The idea behind the robustification is to decompose the concentrated score function into a sum of uncorrelated terms so that the outer product of gradient (OPG) can be used to estimate its variance. We also provide methods for improving the finite sample performance of the proposed tests. These methods are then applied to several popular spatial models. Monte Carlo results show that they work well in finite sample.

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1 Introduction

Many economic processes, for example, housing decisions, technology adoption, unemployment, welfare participation, price decisions, crime rates, trade flows, etc., exhibit spatial patterns, see Anselin (1988a,b), Glaeser et al. (1996), LeSage (1999), Lin and Lee (2010), and Kelejian and Prucha (2010), to mention a few. This makes testing for the existence of spatial dependence a necessary ingredient in many empirical economic applications, see

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Anselin and Bera (1998) and Baltagi, et al. (2003), to mention a few. Among the popular tests for spatial dependence, the LM test is simple to compute as it does not require the estimation of the spatial effects. However, the standard LM tests for spatial dependence in linear and panel regressions are derived under the assumptions that the regression errors are normal and homoskedastic, and hence may not be robust against these types of misspecification. Indeed, heteroskedasticity and non-normality are the two most typical forms of misspecification commonly cited in economic applications. Hence, it is highly desirable to find ways to 'robustify' the standard LM tests so as to take advantage of their computational simplicity.

Anselin (1988b) pioneered research along these lines for spatial models, and provided a heteroskedasticity and non-normality robust test for spatial error dependence in a linear or nonlinear regression by following the methods of White (1980) and Davidson and MacKinnon (1985). Recently, Born and Breitung (2011) proposed simple regression based tests for spatial dependence in linear regression models, based on an elegant idea: decomposing the concentrated score function into a sum of uncorrelated components – making use of the fact that the diagonal elements of the spatial weight matrix are zero – so that the outer product of gradient (OPG) method can be used to estimate the variance of the score. This method leads naturally to OPG variants of the LM statistics that are robust against heteroskedasticity and non-normality. However, the finite sample performance of the OPG-based LM tests can be poor due to heavy spatial dependence, see the Monte Carlo experiments below.

This paper generalizes the idea of Born and Breitung (2011) to a more general class of LM statistics, as long as their numerator can be written as linear-quadratic forms of the error vector. Another important issue considered in this paper is the finite sample performance of the spatial LM tests. Recently, Yang (2010) and Baltagi and Yang (2013) showed that the standard LM tests for spatial regression models (linear or panel) may suffer from severe finite sample size distortion when spatial dependence is heavy. Instead, they proposed *standardized* LM tests that correct for both the mean and variance of the standard LM test statistics. While these standardized LM tests are derived under the assumption that the errors are homoskedastic, the results do show that centering and rescaling play important roles in improving the finite sample performance of these LM tests, in particular when an OPG variant of the LM test is used. However, under heteroskedasticity of the disturbances, it is more challenging to center an LM test as the analytical expression of the centering factor typically involves the unknown variances of the error terms. We propose *nearly* unbiased estimators of this centering quantity, leading to improved OPG-LM tests.

The rest of the paper is organized as follows. Section 2 presents general methods for constructing an OPG-variant of an LM test so that it becomes asymptotically robust against both heteroskedasticity and non-normality. Section 3 applies these general methods to some popular spatial models (linear and panel), to give the standard OPG-LM tests and their corresponding finite sample corrected versions. Section 4 presents some Monte Carlo results, and Section 5 concludes the paper. All proofs are given in Appendix.

2 General Methods

Consider the general model

$$q(Y_n, X_n, W_{1n} \dots W_{kn}; \theta, \lambda) = \varepsilon_n, \tag{1}$$

with a dependent variable Y_n conditional on a set of independent variables X_n, and spatial weight matrices $W_{1n} \dots W_{kn}$. In this case, θ denotes the parameters of the model considered, while λ denotes the spatial parameters. ε_n is an n-vector of model errors, independent but not (necessarily) identically distributed (inid) with mean zero and variances $\sigma_i^2, i = 1, \dots, n$. Popular spatial regression models and spatial panel data models can all be written in this form. For example, the spatial autoregressive (SAR) model, $Y_n = \lambda W_n Y_n + X_n\beta + \varepsilon_n$, can be written in the form: $B_n(\lambda)Y_n - X_n\beta = \varepsilon_n$ where $B_n = I_n - \lambda W_n$ and I_n is an $n \times n$ identity matrix. The spatial error regression model, $Y_n = X_n\beta + u_n$; with $u_n = \lambda W_n u_n + \varepsilon_n$, can be written as $B_n(\lambda)(Y_n - X_n\beta) = \varepsilon_n$. Combining these two models gives a spatial autoregressive model with spatial autoregressive error (SARAR) that can be written in the form $B_{2n}(\lambda_2)(B_{1n}(\lambda_1)Y_n - X_n\beta) = \varepsilon_n$ where $B_{rn}(\lambda_r) = I_n - \lambda_r W_{rn}, r = 1, 2$. The panel versions of these models with fixed effects can also be written in the form of (1) after a transformation to eliminate the fixed effects. Our null hypothesis corresponds to the nonexistence of spatial effects, leading to null models being typically the classical linear regression models, or the classical panel data models with fixed effects, so that the test can be carried out using only the OLS estimates and residuals. See Anselin (1988b) for a comprehensive coverage of the popular spatial regression models, and Baltagi, et al. (2003) for the LM tests in the spatial panel data regression models. While our discussion focuses on spatial models, the methods presented below apply to more general econometric models.

2.1 One-directional test

Consider the case where $k = 1$, i.e., λ is a *scalar*. Suppose that the numerator of the LM test statistic for testing $H_0 : \lambda = 0$, derived under normality and homoskedasticity, can be written as a linear-quadratic form in ε_n:

$$Q_n(\varepsilon_n) = \varepsilon_n' A_n \varepsilon_n + b_n' \varepsilon_n, \tag{2}$$

where A_n is an $n \times n$ non-stochastic matrix that may involve X_n and W_n, and b_n is an $n \times 1$ non-stochastic vector that may involve X_n and some model parameters contained in θ. This holds if the null model is a linear regression model or a panel data model with fixed effects. Clearly, the matrix A_n is crucial in the application of the OPG method for variance estimation. For example, for the SAR model described above we have $A_n = M_n W_n$ where $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$. For the spatial error components (SEC) model introduced by Kelejian and Robinson (1995) we have $A_n = M_n[W_n W_n' - \frac{1}{n}tr(W_n W_n')I_n]M_n$.

Kelejian and Prucha (2001) presented a central limit theorem (CLT) for the above linear-quadratic (LQ) forms, which we will use to prove most of our theorems. However, simple methods for estimating the variance of $Q_n(\varepsilon_n)$ were not given. Clearly, $Q(\varepsilon_n)$ is not a sum of uncorrelated components and hence the OPG method cannot be (directly) applied to estimate the variance of $Q_n(\varepsilon_n)$. Inspired by Born and Breitung (2011), we write

$$A_n = A_n^u + A_n^l + A_n^d, \tag{3}$$

where the three $n \times n$ matrices on the right hand side of (3) are, respectively, the upper triangular, the lower triangular and the diagonal matrices of A_n. Define $\zeta_n = (A_n^{u\prime} + A_n^l)\varepsilon_n$. Let $a_n = \text{diag}(A_n)$ be the vector formed by the diagonal elements $\{a_{n,ii}\}$ of A_n. We have,

$$\begin{aligned} Q_n(\varepsilon_n) &= \varepsilon_n' A_n \varepsilon_n + b_n'\varepsilon_n \\ &= \varepsilon_n'(A_n^u + A_n^l)\varepsilon_n + a_n'\varepsilon_n^2 + b_n'\varepsilon_n \\ &= \varepsilon_n'(A_n^{u\prime} + A_n^l)\varepsilon_n + a_n'\varepsilon_n^2 + b_n'\varepsilon_n \\ &= \varepsilon_n'\zeta_n + a_n'\varepsilon_n^2 + b_n'\varepsilon_n \\ &= \textstyle\sum_{i=1}^n \varepsilon_{n,i}(\zeta_{n,i} + a_{n,ii}\varepsilon_{n,i} + b_{n,i}), \end{aligned}$$

where $\varepsilon_n^2 = \{\varepsilon_{n,i}^2\}_{n\times 1}$, and $\varepsilon_{n,i}, \zeta_{n,i}, a_{n,ii}$ and $b_{n,i}$ are, respectively, the elements of $\varepsilon_n, \zeta_n, a_n$ and b_n. It can easily be seen that the elements $\varepsilon_{n,i}(\zeta_{n,i} + a_{n,ii}\varepsilon_{n,i} + b_{n,i})$ in the above summation are uncorrelated, and thus $Q_n(\varepsilon)$ is decomposed into a sum of n uncorrelated terms. It follows that the variance of $Q_n(\varepsilon_n)$ can be estimated by the following OPG form:

$$\sum_{i=1}^n \left(\varepsilon_{n,i}(\zeta_{n,i} + a_{n,ii}\varepsilon_{n,i} + b_{n,i})\right)^2.$$

With this variance estimator, the CLT for LQ forms of Kelejian and Prucha (2001) is made feasible provided that $\text{E}[Q_n(\varepsilon_n)] = \sum_{i=1}^n a_{n,ii}\sigma_i^2$ is 'negligible', i.e., $\frac{1}{\sqrt{n}}\sum_{i=1}^n a_{n,ii}\sigma_i^2 = o(1)$. Clearly, this is true if $a_{n,ii} = o(n^{-1/2})$ for all i and σ_i^2 are finite constants. For all the three tests considered in Born and Breitung (2011) and the tests for fixed effects panel models considered in this paper, we have $a_{n,ii} = O(n^{-1})$. In general, as $Q_n(\varepsilon_n)$ corresponds to the concentrated score of λ (at $\lambda = 0$) derived under normality and homoskedasticity, it is typical that $\frac{1}{\sqrt{n}}\sum_{i=1}^n a_{n,ii} = o(1)$ if homoskedasticity holds. With this, it can be seen that $\frac{1}{\sqrt{n}}\sum_{i=1}^n a_{n,ii}\sigma_i^2 = o(1)$ holds as long as $\{a_{n,ii}\}$ and $\{\sigma_i^2\}$ are weakly correlated (see Theorem 1 below). The following set of assumptions are needed:

Assumption 1. *The errors $\{\varepsilon_{n,i}\}$ are independent such that $E(\varepsilon_{n,i}) = 0$, $Var(\varepsilon_{n,i}) = \sigma_i^2$, and $\sup_{1\le i\le n} E(|\varepsilon_{n,i}|^{4+\delta}) < \infty$ for some $\delta > 0$.*

Assumption 2. *The elements* $\{a_{n,ij}\}$ *of* A_n *satisfy* $\sup_{1\le j\le n}\sum_{i=1}^{n}|a_{n,ij}|<\infty$ *for all* n*. The elements* $\{b_{n,i}\}$ *of* b_n *satisfy* $\sup_n n^{-1}|b_{n,i}|^{2+\eta}<\infty$ *for some* $\eta>0$.

These are essentially the same set of assumptions maintained by Kelejian and Prucha (2001) for their central limit theorem for a linear quadratic form. The following theorem provides a feasible OPG variant of this central limit theorem:

Theorem 1. *If Assumptions 1 and 2 hold, and if* $\mathrm{Cov}(a_n,\varsigma_n^2)=o(n^{-1/2})$*, then for testing* $H_0:\lambda=0$*, we have the following OPG-variant of the LM test:*

$$\mathrm{LM}_{\texttt{OPG}}=\frac{\varepsilon_n' A_n\varepsilon_n+b_n'\varepsilon_n}{\sqrt{\sum_{i=1}^{n}(\varepsilon_{n,i}\xi_{n,i})^2}},\tag{4}$$

where $\xi_{n,i}=\zeta_{n,i}+a_{n,ii}\varepsilon_{n,i}+b_{n,i}$, $\varsigma_n^2=(\sigma_1^2,\ldots,\sigma_n^2)$, and $\mathrm{Cov}(a_n,\varsigma_n^2)$ is the (sample) covariance between a_n and ς_n^2. Under H_0, $\mathrm{LM}_{\texttt{OPG}}\xrightarrow{D}N(0,1)$.

In empirical applications, $\varepsilon_{n,i}$ are replaced by the restricted residuals and $b_{n,i}$ by their restricted estimates (under H_0). The above theorem directly extends the results of Born and Breitung (2011) which require $Q_n(\varepsilon_n)$ to be of the form: $\varepsilon_n' W_n\varepsilon_n+b_n\varepsilon_n$. It leads naturally to OPG variants of the LM tests that are robust to heteroskedasticity and non-normality for a more general class of LM tests. All popular one-directional LM tests of spatial dependence can be robustified using Theorem 1 such as the LM test for spatial error dependence in linear regression; the LM test for spatial lag dependence; the LM test for spatial error components, etc. The OPG LM statistics derived this way differ from those suggested by Born and Breitung (2011) in that they take into account the estimation of the 'other' parameters such as the regression coefficients and the scale parameter in the linear spatial regression model. It should be stressed that the results of Theorem 1 can be applied to any one-directional LM test to provide an OPG variant that is robust against misspecification in normality and homoskedasticity, as long as the numerator of the test can be written in the form of (2).

While the $\mathrm{LM}_{\texttt{OPG}}$ statistic given in Theorem 1 is robust asymptotically against heteroskedasticity and non-normality, its finite sample performance may not be satisfactory, simply because $\mathrm{E}[Q_n(\varepsilon_n)]=\sum_{i=1}^{n}a_{n,ii}\sigma_i^2\neq 0$ unless $a_{n,ii}$ are all zero. Furthermore, the condition $\mathrm{Cov}(a_n,\varsigma_n^2)=o(n^{-1/2})$ may not be satisfied by all LM tests including non-spatial LM tests. This motivates us to work with

$$Q_n^0(\varepsilon_n)=\varepsilon_n' A_n^0\varepsilon_n+b_n'\varepsilon_n,\tag{5}$$

where $A_n^0=A_n-A_n^d$. Clearly, $\mathrm{E}[Q_n^0(\varepsilon_n)]=0$. We have the following result:

Corollary 1. *If Assumptions 1 and 2 hold, then for testing* $H_0:\lambda=0$*, we have the following OPG-variant of the LM test with finite sample corrections:*

$$\mathrm{LM}_{\texttt{OPG}}^0=\frac{\varepsilon_n' A_n^0\varepsilon_n+b_n'\varepsilon_n}{\sqrt{\sum_{i=1}^{n}(\varepsilon_{n,i}\xi_{n,i}^0)^2}},\tag{6}$$

where $\xi^0_{n,i} = \zeta_{n,i} + b_{n,i}$. Under H_0, $\mathrm{LM}^0_{\mathtt{OPG}} \xrightarrow{D} N(0,1)$.

Theorem 1 offers one-step finite sample improvement over the results of Born and Breitung (2011) by taking into account the estimation of the regression coefficients. Corollary 1 offers further improvement by centering the numerator of the test statistic, and it removes the condition imposed on the mean of $Q_n(\varepsilon_n)$. In practical applications, however, $Q^0_n(\varepsilon_n)$ has to be replaced by its feasible version $Q^0_n(\tilde{\varepsilon}_n)$. However, $\mathrm{E}[Q^0_n(\tilde{\varepsilon}_n)]$ may not be zero, and further corrections may be necessary (see Section 3).

2.2 Multi-directional test

We now consider the case where $k \geq 2$, e.g., the spatial dependence appears both as a spatial lag and as a spatial error in the linear regression model. Suppose that the numerators of the elements of the score vector which forms the LM test statistic for testing $H_0 : \lambda = 0$ can be written in linear-quadratic forms in ε_n:

$$Q_n(\varepsilon_n) = \begin{cases} \varepsilon'_n A_{1n}\varepsilon_n + b'_{1n}\varepsilon_n \\ \vdots \\ \varepsilon'_n A_{kn}\varepsilon_n + b'_{kn}\varepsilon_n \end{cases}$$

where for $r = 1, \ldots, k$, $\{A_{rn}\}$ are $n \times n$ non-stochastic matrices that may involve X_n and $\{W_{rn}\}$. While $\{b_{rn}\}$ are $n \times 1$ non-stochastic vectors that may involve X_n and some model parameters contained in θ. Kelejian and Prucha (2010) extend Kelejian and Prucha (2001) to give a CLT for a vector of linear quadratic forms, upon which our theorem is based.

Decomposing each A_{rn} in the same manner as in (3), i.e.,

$$A_{rn} = A^u_{rn} + A^l_{rn} + A^d_{rn}, \quad r = 1 \ldots, k$$

and defining $a_{rn} = \mathrm{diag}(A_{rn})$, and $\zeta_{rn} = (A^{u\prime}_{rn} + A^l_{rn})\varepsilon_n$, $r = 1 \ldots, k$, we have the following theorem which requires the extended assumption given below.

Assumption 2′. *The elements of A_{rn} satisfy $\sup_{1\leq j\leq n} \sup^n_{i=1} |a_{rn,ij}| < \infty$ for all n, and the elements of b_{rn} satisfy $\sup_n n^{-1}|b_{rn,i}|^{2+\eta_r} < \infty$ for some $\eta_r > 0$, $r = 1, \ldots, k$.*

Theorem 2. *If Assumptions 1 and 2′ hold, and if $\mathrm{Cov}(a_{rn}, \varsigma^2_n) = o(n^{-1/2})$, $r = 1, \ldots, k$, then for testing $H_0 : \lambda = 0$, we have the following OPG-variant of the joint LM test:*

$$\mathrm{LM}^{\mathtt{J}}_{\mathtt{OPG}} = \left(\sum_{i=1}^n \varepsilon_{n,i}\Upsilon_{n,i}\right)' \left(\sum_{i=1}^n \varepsilon^2_{n,i}\Upsilon_{n,i}\Upsilon'_{n,i}\right)^{-1} \left(\sum_{i=1}^n \varepsilon_{n,i}\Upsilon_{n,i}\right), \tag{7}$$

where $\Upsilon_{n,i} = \{\zeta_{rn,i} + a_{rn,ii}\varepsilon_{n,i} + b_{rn,i}, r = 1, \ldots, k\}'$, with $\lim_{n\to\infty} \frac{1}{n}\sum^n_{i=1} \mathrm{Var}(\varepsilon_{n,i}\Upsilon_{n,i})$ being finite and positive definite. Under H_0, $\mathrm{LM}_{\mathtt{OPG}} \xrightarrow{D} \chi^2_k$.

The results of Theorem 2 extend those of Born and Breitung (2011) by allowing A_{rn} to be arbitrary matrices rather than W_{rn}. This means that it can be applied to LM tests that do not have matrices of zero diagonal elements in the quadratic form. It also allows the test to be of a general k-dimension rather than 2. Similar to the case of one-directional tests, the test given in Theorem 2 can be standardized for better finite sample performance. With this, the condition imposed on the mean of $Q_n(\varepsilon_n)$ is also removed.

Corollary 2. *If Assumptions 1 and 2′ hold, then for testing $H_0 : \lambda = 0$, we have the following OPG-variant of the LM test with finite sample corrections:*

$$\mathrm{LM}_{\mathtt{OPG}}^{\mathtt{JO}} = \left(\sum_{i=1}^{n} \varepsilon_{n,i} \Upsilon_{n,i}^{0}\right)' \left(\sum_{i=1}^{n} \varepsilon_{n,i}^{2} \Upsilon_{n,i}^{0} \Upsilon_{n,i}^{0\prime}\right)^{-1} \left(\sum_{i=1}^{n} \varepsilon_{n,i} \Upsilon_{n,i}^{0}\right), \tag{8}$$

where $\Upsilon_{n,i}^{0} = \{\zeta_{rn,i} + b_{rn,i}, r = 1, \ldots, k\}'$. Under H_0, $\mathrm{LM}_{\mathtt{OPG}}^{0} \xrightarrow{D} \chi_k^2$.

3 Robust Spatial LM Tests with Finite Sample Corrections

In this section, we apply the results of Section 2 to several popular spatial regression models. Due to the existence of spatial dependence, the finite sample performance of the $\mathrm{LM}_{\mathtt{OPG}}$ tests defined in Theorems 1 and 2 may not be satisfactory. Thus, further finite sample corrections may be necessary. In sum, Corollaries 1 and 2 point to general directions for finite sample corrections. For a specific spatial model, however, a finer correction may be possible. The key idea for improving the finite sample performance of an LM test is centering, arising from the fact that the expectation of the concentrated score (from which the LM statistic is derived) is not zero. In Theorem 1 above, for example, $E[Q_n(\varepsilon_n)] = \sum_{i=1}^{n} a_{n,i}\sigma_i^2$ which is not necessarily zero. As a result, the finite sample mean of the LM test may be far from its nominal value, and the finite sample size of the test severely distorted; see Baltagi and Yang (2013) for the case of a linear regression with spatial error dependence. Our idea is to obtain a feasible version of $E[Q(\varepsilon_n)]$, and then subtract this feasible version from $Q(\varepsilon_n)$. There are two *complications* in centering. The first is that a feasible version may not be readily available and some approximation may be necessary, and the second is that the variance estimator may need to be adjusted after centering.

3.1 Linear regression with SARAR(1,1) dependence

Consider the popular SARAR(1,1) model, i.e., the spatial autoregressive model with spatial autoregressive errors of the form

$$Y_n = \lambda_1 W_{1n} Y_n + X_n \beta + u_n; \quad u_n = \lambda_2 W_{2n} u_n + \varepsilon_n. \tag{9}$$

The two sub-models with $\lambda_2 = 0$ or $\lambda_1 = 0$ are called SAR (spatial autoregressive) model and SED (spatial error dependence) models, respectively. The three null hypotheses considered are: $H_0^a : \lambda_1 = 0$ for the SAR model; $H_0^b : \lambda_2 = 0$ for the SED model; and $H_0^c : \lambda_1 = 0, \lambda_2 = 0$ for the SARAR model. The corresponding LM tests (existing and new) are discussed next.

Let $\tilde{\varepsilon}_n$ be the OLS residuals from regressing Y_n on X_n, $\tilde{\beta}_n$ and $\tilde{\sigma}_n^2$ the OLS estimators of β and σ^2, respectively, $T_{rn} = \text{tr}[(W_{rn} + W'_{rn})W_{rn}]$, $r = 1, 2$, $T_{3n} = \text{tr}[(W_{2n} + W'_{2n})W_{1n}]$, $M_n = I_n - X_n(X'_nX_n)^{-1}X'_n$, and I_n is an $n \times n$ identity matrix. The LM test of $H_0^a : \lambda_1 = 0$, given in Anselin (1988a,b), takes the form:

$$\text{LM}_{\texttt{SAR}} = \frac{\tilde{\varepsilon}'_n W_{1n} Y_n}{\tilde{\sigma}_n^2\, (\tilde{D}_n + T_{1n})^{\frac{1}{2}}}, \tag{10}$$

where $\tilde{D}_n = \tilde{\sigma}_n^{-2}(W_nX_n\tilde{\beta}_n)'M_nW_nX_n\tilde{\beta}_n$. The LM test of H_0^b given in Burridge (1980) is:

$$\text{LM}_{\texttt{SED}} = \frac{\tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n}{\tilde{\sigma}_n^2\, T_{2n}^{\frac{1}{2}}}. \tag{11}$$

The joint LM test of H_0^c given in Anselin (1988a,b) has the form:

$$\text{LM}_{\texttt{SARAR}} = \frac{1}{\tilde{\sigma}_n^4} \begin{pmatrix} \tilde{\varepsilon}'_n W_{1n} Y_n \\ \tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n \end{pmatrix}' \begin{pmatrix} T_{1n} + \tilde{D}_n & T_{3n} \\ T_{3n} & T_{2n} \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varepsilon}'_n W_{1n} Y_n \\ \tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n \end{pmatrix}. \tag{12}$$

Born and Breitung (2011) proposed OPG variants of the above LM tests which are robust against heteroskedasticity and non-normality, making use of the fact that the diagonal elements of the spatial weight matrices are zero. Let W_{rn}^u and W_{rn}^l be the upper and lower triangular matrices of W_{rn}, $r = 1, 2$. Define $\tilde{\xi}_{1n} = (W_{1n}^{u\prime} + W_{1n}^l)\tilde{\varepsilon}_n + M_nW_nX_n\tilde{\beta}_n$ and $\tilde{\xi}_{2n} = (W_{2n}^{u\prime} + W_{2n}^l)\tilde{\varepsilon}_n$. The three OPG variants of the LM tests of Born and Breitung (2011) are as follows:

$$\begin{aligned}
\text{LM}_{\texttt{SAR}}^{\texttt{OPG}} &= \frac{\tilde{\varepsilon}'_n W_{1n} Y_n}{(\tilde{\varepsilon}_n^{2\prime}\, \tilde{\xi}_{1n}^2)^{\frac{1}{2}}}, & (13)\\
\text{LM}_{\texttt{SED}}^{\texttt{OPG}} &= \frac{\tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\prime}\, \tilde{\xi}_{2n}^2)^{\frac{1}{2}}}, \text{ and} & (14)\\
\text{LM}_{\texttt{SARAR}}^{\texttt{OPG}} &= \begin{pmatrix} \tilde{\varepsilon}'_n W_{1n} Y_n \\ \tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n \end{pmatrix}' \begin{pmatrix} \tilde{\varepsilon}_n^{2\prime}\tilde{\xi}_{1n}^2 & \tilde{\varepsilon}_n^{2\prime}(\tilde{\xi}_{1n} \otimes \tilde{\xi}_{2n}) \\ \tilde{\varepsilon}_n^{2\prime}(\tilde{\xi}_{1n} \otimes \tilde{\xi}_{2n}) & \tilde{\varepsilon}_n^{2\prime}\tilde{\xi}_{2n}^2 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varepsilon}'_n W_{1n} Y_n \\ \tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n \end{pmatrix}, & (15)
\end{aligned}$$

where $\otimes$ denotes the Hadamard product, and the *square* of a vector, e.g., $\tilde{\varepsilon}_n^2 = \tilde{\varepsilon}_n \otimes \tilde{\varepsilon}_n$.

The OPG variants of the LM tests considered by Born and Breitung (2011) (as well as the original LM tests) do not take into account the estimation of β, and hence may suffer from the problems of size distortion due mainly to the lack of centering and rescaling. Note that the numerators of the tests above are:

$$\begin{aligned}
\tilde{\varepsilon}'_n W_{1n} Y_n &= \varepsilon'_n M_n W_{1n} \varepsilon_n + \varepsilon'_n M_n \eta_n \\
\tilde{\varepsilon}'_n W_{2n} \tilde{\varepsilon}_n &= \varepsilon'_n M_n W_{2n} M_n \varepsilon_n.
\end{aligned}$$

It follows that $\mathrm{E}(\tilde{\varepsilon}_n' W_{1n} Y_n) = \sum_{i=1}^n \sigma_i^2 a_{1n,ii} \neq 0$, and $\mathrm{E}(\tilde{\varepsilon}_n' W_{2n} \tilde{\varepsilon}_n) = \sum_{i=1}^n \sigma_i^2 a_{2n,ii} \neq 0$, where $\{a_{1n,ii}\}$ are the diagonal elements of $A_{1n} = M_n W_n$, and $\{a_{2n,ii}\}$ are the diagonal elements of $A_{2n} = M_n W_n M_n$. Replacing W_{1n} by A_{1n} and W_{2n} by A_{2n} in (13)-(15), and applying Theorems 1 and 2, one immediately obtains a set of OPG variants of the LM tests which take into account the estimation of β. Applying Corollaries 1 and 2, one obtains another set of OPG variants of the LM tests which take into account the estimation of β and also center the tests properly. However, the feasible versions $Q_n^0(\tilde{\varepsilon})$ of $Q_n^0(\varepsilon)$ defined in (5), applied to SAR, SED and SARAR models, may not have zero mean, and hence further improvements can be made (see the proof of our next theorem for details).

For $r = 1, 2$, define $\mathcal{H}_{rn} = \mathrm{diag}(A_{rn})\mathrm{diag}(M_n)^{-2}$ and $A_{rn}^* = A_{rn} - M_n \mathcal{H}_{rn} M_n$, and decompose $A_{rn}^* = A_{rn}^{*u} + A_{rn}^{*l} + A_{rn}^{*d}$ as in (3). Let $\tilde{\xi}_{1n}^* = (A_{1n}^{*u\prime} + A_{1n}^{*l})\tilde{\varepsilon}_n + A_{1n}^{*d}\tilde{\varepsilon}_n + M_n \tilde{\eta}_n$ and $\tilde{\xi}_{2n}^* = (A_{2n}^{*u\prime} + A_{2n}^{*l})\tilde{\varepsilon}_n + A_{2n}^{*d}\tilde{\varepsilon}_n$. We have the following theorem.

Theorem 3. *Assume Assumption 1 holds for ε_n in Model (9). Assume further that (i) the diagonal elements of W_{rn} are zero for $r = 1, 2$, (ii) all row and column sums of W_{rn} are uniformly bounded for all n and $r = 1, 2$, and (iii) the elements of the $n \times k$ matrix X_n are uniformly bounded for all n, and $\lim_{n\to\infty} \frac{1}{n} X_n' X_n$ exists and is nonsingular. Then we have the following OPG variants of the LM tests with finite sample corrections:*

$$\mathrm{SLM}_{\mathtt{SAR}}^{\mathtt{OPG}} = \frac{\tilde{\varepsilon}_n' W_{1n} Y_n - \tilde{\varepsilon}_n' \mathcal{H}_{1n} \tilde{\varepsilon}_n'}{(\tilde{\varepsilon}_n^{2\,\prime}\, \tilde{\xi}_{1n}^{*2})^{\frac{1}{2}}}, \tag{16}$$

$$\mathrm{SLM}_{\mathtt{SED}}^{\mathtt{OPG}} = \frac{\tilde{\varepsilon}_n' (W_{2n} - \mathcal{H}_{2n}) \tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\,\prime}\, \tilde{\xi}_{2n}^{*2})^{\frac{1}{2}}}, \text{ and} \tag{17}$$

$$\mathrm{SLM}_{\mathtt{SARAR}}^{\mathtt{OPG}} = S_n' \begin{pmatrix} \tilde{\varepsilon}_n^{2\,\prime} \tilde{\xi}_{1n}^{*2} & \tilde{\varepsilon}_n^{2\,\prime}(\tilde{\xi}_{1n}^* \otimes \tilde{\xi}_{2n}^*) \\ \tilde{\varepsilon}_n^{2\,\prime}(\tilde{\xi}_{1n}^* \otimes \tilde{\xi}_{2n}^*) & \tilde{\varepsilon}_n^{2\,\prime} \tilde{\xi}_{2n}^{*2} \end{pmatrix}^{-1} S_n, \tag{18}$$

where $S_n = \{\tilde{\varepsilon}_n' W_{1n} Y_n - \tilde{\varepsilon}_n' \mathcal{H}_{1n} \tilde{\varepsilon}_n',\ \ \tilde{\varepsilon}_n' (W_{2n} - \mathcal{H}_{2n}) \tilde{\varepsilon}_n\}'$. Under H_0, $\mathrm{SLM}_{\mathtt{SAR}}^{\mathtt{OPG}} \xrightarrow{D} N(0,1)$, $\mathrm{SLM}_{\mathtt{SED}}^{\mathtt{OPG}} \xrightarrow{D} N(0,1)$, and $\mathrm{SLM}_{\mathtt{SARAR}}^{\mathtt{OPG}} \xrightarrow{D} \chi_2^2$.

3.2 Linear regression with spatial error components

The linear regression model with spatial error components (SEC) by Kelejian and Robinson (1995) takes the following form:

$$Y_n = X_n \beta + u_n \quad \text{ with } u_n = W_n \nu_n + \varepsilon_n, \tag{19}$$

where Y_n, X_n and W_n are defined as in the SARAR model. ν_n is an $n \times 1$ vector of errors that together with W_n incorporates the spatial dependence, and ε is an $n \times 1$ vector of location specific disturbance terms. The error components ν_n and ε_n are assumed to be independent, with independent and identically distributed (iid) elements of mean zero and variances σ_ν^2 and σ_ε^2, respectively. In this model, the null hypothesis of no spatial effect can be either $H_0 : \sigma_\nu^2 = 0$, or $\theta = \sigma_\nu^2/\sigma_\varepsilon^2 = 0$. The alternative hypothesis can only be one-sided, as σ_ν^2 is non-negative, i.e., $H_a : \sigma_\nu^2 > 0$, or $\theta > 0$. Anselin (2001) derived an LM test based on the assumptions that the errors are normally distributed. This LM test is of the form

$$\text{LM}_{\texttt{SEC}} = \frac{\tilde{\varepsilon}_n'(W_nW_n' - \frac{1}{n}T_{1n}I_n)\tilde{\varepsilon}_n}{\tilde{\sigma}_\varepsilon^2(2T_{2n} - \frac{2}{n}T_{1n}^2)^{\frac{1}{2}}}, \tag{20}$$

where $\tilde{\sigma}_\varepsilon^2 = \frac{1}{n}\tilde{\varepsilon}_n'\tilde{\varepsilon}_n$, $\tilde{\varepsilon}_n$ is the vector of OLS residuals, $T_{1n} = \text{tr}(W_nW_n')$ and $T_{2n} = \text{tr}(W_nW_n'W_nW_n')$. Under H_0, the positive part of LM_{SEC} converges to that of $N(0,1)$. This test is not robust against non-normality, and a robust version was proposed by Yang (2010):

$$\text{SLM}_{\texttt{SEC}} = \frac{\tilde{\varepsilon}_n'(W_nW_n' - \frac{1}{n}S_{1n}I_n)\tilde{\varepsilon}_n}{\tilde{\sigma}_\varepsilon^2(\tilde{\kappa}_\varepsilon S_{2n} + S_{3n})^{\frac{1}{2}}}, \tag{21}$$

where $S_{1n} = \frac{n}{n-k}\text{tr}(W_nW_n'M_n)$, $S_{2n} = \sum_i c_{n,ii}^2$ with $\{c_{n,ii}\}$ being the diagonal elements of $C_n = M_n(W_nW_n' - \frac{1}{n}S_{1n}I_n)M_n$, $S_{3n} = 2\text{tr}(C_n^2)$, and $\tilde{\kappa}_\varepsilon$ is the excess sample kurtosis of $\tilde{u}_n$. Yang (2010) showed that under H_0, (i) the positive part of $\text{SLM}_{\texttt{SEC}}$ converges to that of $N(0,1)$, and (ii) $\text{SLM}_{\texttt{SEC}}$ is asymptotically equivalent to $\text{LM}_{\texttt{SEC}}$ when $\kappa_\varepsilon = 0$.

Neither tests defined in (20) and (21) are robust against heteroskedasticity. The idea of Born and Breitung (2011) cannot be applied as in general the diagonal elements of $W_nW_n' - \frac{1}{n}T_{1n}I_n$ are not zero. However, the general method given in Theorem 1 and Corollary 1 still apply. Similar to the developments in Section 3.1 for linear regressions with spatial error dependence, we introduce two OPG-variants of the LM test given in (20), one without and one with finite sample corrections.

Let $A_n^\circ = W_nW_n' - \frac{1}{n}T_{1n}I_n$, $A_n = M_nA_n^\circ M_n$, $\mathcal{H}_n = \text{diag}(A_n)\text{diag}(M_n)^{-2}$, and $A_n^* = A_n - M_n\mathcal{H}_nM_n$. Decompose A_n° and A_n^* as in (3): $A_n^\circ = A_n^{\circ u} + A_n^{\circ l} + A_n^{\circ d}$ and $A_n^* = A_n^{*u} + A_n^{*l} + A_n^{*d}$. Define $\tilde{\xi}_n^\circ = (A_n^{\circ u\prime} + A_n^{\circ l})\tilde{\varepsilon}_n + A_n^{\circ d}\tilde{\varepsilon}_n$ and $\tilde{\xi}_n^* = (A_n^{*u\prime} + A_n^{*l})\tilde{\varepsilon}_n + A_n^{*d}\tilde{\varepsilon}_n$.

Theorem 4. *Assume Assumption 1 holds for ε_n in Model (19). Assume further that (i) the diagonal elements of W_n are zero, (ii) the sequence $\{W_n\}$ are uniformly bounded in both row and column sums, and (iii) the elements of the $n \times k$ matrix X_n are uniformly bounded for all n, and $\lim_{n\to\infty} \frac{1}{n}X_n'X_n$ exists and is nonsingular. Then we have the OPG-variant of the LM test without finite sample corrections (standardizations) as:*

$$\text{LM}_{\texttt{SEC}}^{\texttt{OPG}} = \frac{\tilde{\varepsilon}_n'A_n^\circ\tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\prime}\,\tilde{\xi}_n^{\circ 2})^{\frac{1}{2}}}, \tag{22}$$

and the OPG-variant of the LM test with finite sample corrections (standardizations) as:

$$\text{SLM}_{\texttt{SEC}}^{\texttt{OPG}} = \frac{\tilde{\varepsilon}_n'(A_n^\circ - \mathcal{H}_n)\tilde{\varepsilon}_n}{(\tilde{\varepsilon}_n^{2\prime}\,\tilde{\xi}_n^{*2})^{\frac{1}{2}}}. \tag{23}$$

Under H_0, the positive part of $\text{LM}_{\texttt{SEC}}^{\texttt{OPG}}$ converges to that of $N(0,1)$ if $\sqrt{n}\text{Cov}(\varpi_n, \varsigma_n^2) = o(1)$ where $\varpi_n = \text{diag}(W_nW_n')$; and $\text{SLM}_{\texttt{SEC}}^{\texttt{OPG}}$ converges to that of $N(0,1)$.

Note that $\text{LM}_{\texttt{SEC}}^{\texttt{OPG}}$ does not take into account the estimation of β, and does not have mean and variance corrections. For a row normalized spatial contiguity weight matrix W_n, we have $\varpi_{n,i} = n_i^{-1}$ where n_i is the number of neighbors that spatial unit i has. Thus, as long as the correlation between $\{n_i^{-1}\}$ and $\{\sigma_i^2\}$ is weak so that $\text{Cov}(\varpi_n, \varsigma_n^2) = o(n^{-1/2})$, the asymptotic null distribution of $\text{LM}_{\texttt{SEC}}^{\texttt{OPG}}$ will be centered at 0. This weak correlation occurs when the variations among $\{n_i^{-1}\}$ is small, or $\{\sigma_i^2\}$ depends on the regressors' values $\{x_{n,i}\}$ which are generated independently of $\{n_i^{-1}\}$, etc. A similar version taking into account the estimation of β can be obtained by replacing A_n° by A_n.

3.3 Spatial panel data models with fixed effects

The SARAR(1,1) model defined in (9) can be extended to the fixed effects panel data model with SARAR(1,1) dependence, and denoted by `panel SARAR(1,1)`in this paper:

$$Y_{nt} = \lambda_1 W_{1n} Y_{nt} + X_{nt}\beta + \mu_n + u_{nt}, \quad u_{nt} = \lambda_2 W_{2n} u_{nt} + \varepsilon_{nt}, \quad t = 1, \ldots, T, \tag{24}$$

where the individual specific effect μ_n may be correlated with the regressors. Similar to the linear SARAR(1,1) model, letting $\lambda_2 = 0$ gives a fixed effects `panel SAR` model, and letting $\lambda_1 = 0$ leads to a fixed effects `panel SED` model.

Lee and Yu (2010) studied the asymptotic properties of QML estimation of the panel SARAR(1,1) model with fixed effects. They used orthogonal transformations to wipe out the fixed effects so that the incidental parameter problem would not occur in case T is fixed. The resulting model takes the form:

$$Y_{nt}^* = \lambda_1 W_{1n} Y_{nt}^* + X_{nt}^*\beta + u_{nt}^*, \quad u_{nt}^* = \lambda_2 W_{2n} u_{nt}^* + \varepsilon_{nt}^*, \quad t = 1, \ldots, T-1, \tag{25}$$

where $(Y_{n1}^*, Y_{n2}^*, \cdots, Y_{n,T-1}^*) = (Y_{n1}, Y_{n2}, \cdots, Y_{nT})F_{T,T-1}$, $F_{T,T-1}$ is a $T \times (T-1)$ matrix whose columns are the eigenvectors of $I_T - \frac{1}{T}\iota_T \iota_T'$ corresponding to the eigenvalues of one, and ι_T is a vector of ones of dimension T. u_{nt}^*, ε_{nt}^*, and the columns of X_{nt}^* are similarly defined. Letting $\lambda_2 = 0$ or $\lambda_1 = 0$ in (25) gives the transformed panel SAR or the transformed panel SED model, respectively.

Debarsy and Ertur (2010) followed up with LM tests for spatial dependence for model (24) or (25). Similar to the case of a linear SARAR model, we are interested in the following three tests: $H_0^a : \lambda_1 = 0$ in the panel SAR model, $H_0^b : \lambda_2 = 0$ in the panel SED model, and $H_0^c : \lambda_1 = 0, \lambda_2 = 0$ in the panel SARAR model; and we develop LM tests that are robust against both heteroskedasticity and non-normality. First, the three standard LM tests derived by Debarsy and Ertur (2010) under normality and homoskedasticity are summarized below.

The LM test for $H_0^a : \lambda_1 = 0$ in the fixed effects panel SAR model takes the form:

$$\text{LM}_{\texttt{SAR}}^{\texttt{FE}} = \frac{\mathbb{N}}{\sqrt{\mathbb{S}_1 + \tilde{\mathbb{D}}}} \frac{\tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \mathbb{W}_1 Y_{\mathbb{N}}^*}{\tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \tilde{\varepsilon}_{\mathbb{N}}^*}, \tag{26}$$

where $\mathbb{N}=n(T-1)$, $\tilde{\varepsilon}^*_{\mathbb{N}}$ denotes the OLS residuals from regressing $Y^*_{\mathbb{N}}$ on $X^*_{\mathbb{N}}$, with $Y^*_{\mathbb{N}}$ being the stacked $\{Y^*_{nt}\}$ and $X^*_{\mathbb{N}}$ the stacked $\{X^*_{nt}\}$. $\mathbb{S}_1=\mathrm{tr}[(\mathbb{W}_1+\mathbb{W}_1')\mathbb{W}_1]$, $\mathbb{W}_1=I_{T-1}\otimes W_{1n}$, $\tilde{\mathbb{D}}=\tilde{\sigma}_{\mathbb{N}}^{-2}\tilde{\eta}_{\mathbb{N}}'\mathbb{M}\tilde{\eta}_{\mathbb{N}}$, $\tilde{\eta}_{\mathbb{N}}=\mathbb{W}_1X_{\mathbb{N}}\tilde{\beta}_{\mathbb{N}}$, $\mathbb{M}=I_{\mathbb{N}}-X^*_{\mathbb{N}}(X^{*\prime}_{\mathbb{N}}X^*_{\mathbb{N}})^{-1}X^{*\prime}_{\mathbb{N}}$, and $\tilde{\beta}_{\mathbb{N}}$ and $\tilde{\sigma}^2_{\mathbb{N}}$ are the OLS estimators of β and σ^2, respectively. The LM test for $H_0^b:\lambda_2=0$ in the fixed effects panel SED model takes the form:

$$\mathtt{LM}^{\mathtt{FE}}_{\mathtt{SED}}=\frac{\mathbb{N}}{\sqrt{\mathbb{S}_2}}\frac{\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}}{\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\tilde{\varepsilon}^*_{\mathbb{N}}}, \tag{27}$$

where $\mathbb{S}_2=\mathrm{tr}[(\mathbb{W}_2+\mathbb{W}_2')\mathbb{W}_2]$ and $\mathbb{W}_2=I_{T-1}\otimes W_{2n}$.[2] The joint LM test for $H_0^c:\lambda_1=0,\lambda_2=0$ in the fixed effects panel SARAR model has the following form:

$$\mathtt{LM}^{\mathtt{FE}}_{\mathtt{SARAR}}=\frac{1}{\tilde{\sigma}^4_{\mathbb{N}}}\begin{pmatrix}\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_1Y^*_{\mathbb{N}}\\ \tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}\end{pmatrix}'\begin{pmatrix}\mathbb{S}_1+\tilde{\mathbb{D}} & \mathbb{S}_3\\ \mathbb{S}_3 & \mathbb{S}_2\end{pmatrix}^{-1}\begin{pmatrix}\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_1Y^*_{\mathbb{N}}\\ \tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}\end{pmatrix}, \tag{28}$$

where $\mathbb{S}_3=\mathrm{tr}[(\mathbb{W}_2+\mathbb{W}_2')\mathbb{W}_1]$.

It can be shown that all of these tests including the standardized version of $\mathtt{LM}^{\mathtt{FE}}_{\mathtt{SED}}$ given in Baltagi and Yang (2013) are asymptotically robust against *non-normality*. However, none of these tests are robust against unknown *heteroskedasticity*. Note that $\tilde{\varepsilon}^*_{\mathbb{N}}=\mathbb{M}\varepsilon^*_{\mathbb{N}}$ where $\varepsilon^*_{\mathbb{N}}$ is the stacked $\{\varepsilon^*_{nt}\}$ and has uncorrelated elements. The tests given in (26)-(28) have identical structures as those given in (10)-(12). Thus, the method of Born and Breitung can be applied to give OPG-variants of the three LM tests given in (26)-(28):

$$\mathrm{LM}^{\mathtt{FMOPG}}_{\mathtt{SAR}}=\frac{\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_1Y^*_{\mathbb{N}}}{(\tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}\,\tilde{\xi}^{*2}_{1\mathbb{N}})^{\frac{1}{2}}}, \tag{29}$$

$$\mathrm{LM}^{\mathtt{FMOPG}}_{\mathtt{SED}}=\frac{\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}}{(\tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}\,\tilde{\xi}^{*2}_{2\mathbb{N}})^{\frac{1}{2}}},\ \text{and} \tag{30}$$

$$\mathrm{LM}^{\mathtt{FMOPG}}_{\mathtt{SARAR}}=\begin{pmatrix}\tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_1Y^*_{\mathbb{N}}\\ \tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}\end{pmatrix}'\begin{pmatrix}\tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}\tilde{\xi}^{*2}_{1\mathbb{N}} & \tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}(\tilde{\xi}^*_{1\mathbb{N}}\otimes\tilde{\xi}^*_{2\mathbb{N}})\\ \tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}(\tilde{\xi}^*_{1\mathbb{N}}\otimes\tilde{\xi}^*_{2\mathbb{N}}) & \tilde{\varepsilon}^{*2\,\prime}_{\mathbb{N}}\tilde{\xi}^2_{*2\mathbb{N}}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{\varepsilon}'_{\mathbb{N}}\mathbb{W}_1Y^*_{\mathbb{N}}\\ \tilde{\varepsilon}^{*\prime}_{\mathbb{N}}\mathbb{W}_2\tilde{\varepsilon}^*_{\mathbb{N}}\end{pmatrix} \tag{31}$$

where $\tilde{\xi}_{1\mathbb{N}}=(\mathbb{W}_1^l+\mathbb{W}_1^{u\prime})\tilde{\varepsilon}^*_{\mathbb{N}}+\mathbb{M}\tilde{\eta}_{\mathbb{N}}$ and $\tilde{\xi}_{2\mathbb{N}}=(\mathbb{W}_2^l+\mathbb{W}_2^{u\prime})\tilde{\varepsilon}^*_{\mathbb{N}}$.

The structure of the three LM tests (26)-(28) show that applications of the methods proposed in this paper (Theorems 1 and 2) would lead to OPG-variants of the LM tests that could improve their finite sample performance. Now, define $\mathbb{A}_1=\mathbb{M}\mathbb{W}_1$ and $\mathbb{A}_2=\mathbb{M}\mathbb{W}_2\mathbb{M}$. For $r=1,2$, let $\mathbb{H}_r=\mathrm{diag}(\mathbb{A}_r)\mathrm{diag}(\mathbb{M})^{-2}$ and $\mathbb{A}_r^\circ=\mathbb{A}_r-\mathbb{M}\mathbb{H}_r\mathbb{M}$, which is decomposed as $\mathbb{A}_r^\circ=\mathbb{A}_r^{ou}+\mathbb{A}_r^{ol}+\mathbb{A}_r^{od}$ as in (3). Let $\tilde{\xi}^\circ_{1\mathbb{N}}=(\mathbb{A}^{ou\prime}_{1\mathbb{N}}+\mathbb{A}^{ol}_{1\mathbb{N}})\tilde{\varepsilon}^*_{\mathbb{N}}+\mathbb{A}^{od}_{1\mathbb{N}}\tilde{\varepsilon}^*_{\mathbb{N}}+M\tilde{\eta}_{\mathbb{N}}$, and $\tilde{\xi}^\circ_{2\mathbb{N}}=(\mathbb{A}^{ou\prime}_{2\mathbb{N}}+\mathbb{A}^{ol}_{2\mathbb{N}})\tilde{\varepsilon}^*_{\mathbb{N}}+\mathbb{A}^{od}_{2\mathbb{N}}\tilde{\varepsilon}^*_{\mathbb{N}}$. We have the following theorem.

Theorem 5. *Assume Assumption 1 holds for ε_{nt} in Model (24), $t=1,\ldots,T$. Assume further that (i) the diagonal elements of W_{rn} are zero for $r=1,2$, (ii) the sequences $\{W_{rn}\}$*

[2] To test spatial error dependence in linear or panel regressions, Baltagi and Yang (2013) introduced a standardized version of $\mathtt{LM}^{\mathtt{FE}}_{\mathtt{SED}}$ which performed better in finite samples.

are uniformly bounded in both row and column sums, and (iii) the elements of the $\mathbb{N} \times k$ *matrix* $X_{\mathbb{N}}$ *are uniformly bounded for all* $\mathbb{N}$*, and* $\lim_{\mathbb{N}\to\infty} \frac{1}{\mathbb{N}} X'_{\mathbb{N}} X_{\mathbb{N}}$ *exists and is nonsingular. Then we have the following OPG-variants of the LM tests with finite sample corrections:*

$$\mathrm{SLM}_{\texttt{SAR}}^{\texttt{FMOPG}} = \frac{\tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \mathbb{W}_1 Y_{\mathbb{N}}^* - \tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \mathbb{H}_1 \tilde{\varepsilon}_{\mathbb{N}}^*}{(\tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime}\, \tilde{\xi}_{1\mathbb{N}}^{\circ 2})^{\frac{1}{2}}}, \tag{32}$$

$$\mathrm{SLM}_{\texttt{SED}}^{\texttt{FMOPG}} = \frac{\tilde{\varepsilon}_{\mathbb{N}}^{*\prime} (\mathbb{W}_2 - \mathbb{H}_2) \tilde{\varepsilon}_{\mathbb{N}}^*}{(\tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime}\, \tilde{\xi}_{2\mathbb{N}}^{\circ 2})^{\frac{1}{2}}}, \text{ and} \tag{33}$$

$$\mathrm{SLM}_{\texttt{SARAR}}^{\texttt{FMOPG}} = \mathbb{S}'_{\mathbb{N}} \begin{pmatrix} \tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime} \tilde{\xi}_{1\mathbb{N}}^{2\circ} & \tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime} (\tilde{\xi}_{1\mathbb{N}}^{\circ} \otimes \tilde{\xi}_{2\mathbb{N}}^{\circ}) \\ \tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime} (\tilde{\xi}_{1\mathbb{N}}^{\circ} \otimes \tilde{\xi}_{2\mathbb{N}}^{\circ}) & \tilde{\varepsilon}_{\mathbb{N}}^{*2\,\prime} \tilde{\xi}_{2\mathbb{N}}^{\circ 2} \end{pmatrix}^{-1} \mathbb{S}_{\mathbb{N}}, \tag{34}$$

where $\mathbb{S}_{\mathbb{N}} = (\tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \mathbb{W}_1 Y_{\mathbb{N}}^* - \tilde{\varepsilon}_{\mathbb{N}}^{*\prime} \mathbb{H}_1 \tilde{\varepsilon}_{\mathbb{N}}^*,\ \tilde{\varepsilon}_{\mathbb{N}}^{*\prime} (\mathbb{W}_2 - \mathbb{H}_2) \tilde{\varepsilon}_{\mathbb{N}}^*)'$. Under H_0, $\mathrm{SLM}_{\texttt{SAR}}^{\texttt{FEOPG}} \xrightarrow{D} N(0,1)$, $\mathrm{SLM}_{\texttt{SED}}^{\texttt{FEOPG}} \xrightarrow{D} N(0,1)$, and $\mathrm{SLM}_{\texttt{SARAR}}^{\texttt{FEOPG}} \xrightarrow{D} \chi_2^2$.

4 Monte Carlo Results

In this section, we describe Monte Carlo experiments and results for the finite sample performance of the LM tests discussed in Section 3. General methods for generating the spatial weight matrices, the model errors, the regressors values, and the heteroskedasticity to be used in the Monte Carlo experiments are described first, followed by the results for each of the three types of models considered earlier.

4.1 General settings

Spatial Weight Matrix. The spatial weight matrices used in the Monte Carlo experiments are generated according to `Rook Contiguity`, `Queen Contiguity` and `Group Interactions`. In the first two cases, the number of neighbors for each spatial unit stays the same (2-4 for Rook and 3-8 for Queen) and does not change when the sample size n increases. In the last case, the number of neighbors for each spatial unit increases with the sample size but at a slower rate, and changes from group to group.

The W_n matrix under *Rook* contiguity is generated as follows: (i) index the n spatial units by $\{1, 2, \cdots, n\}$. Randomly permute these indices and then allocate them into a lattice of $r \times m (\geq n)$ squares. (ii) Let $W_{n,ij} = 1$ if the index j is in a square which is on the immediate left, or right, or above, or below the square which contains the index i, otherwise $W_{n,ij} = 0$; and (iii) divide each element of W_n by its row sum. The W_n matrix under *Queen* contiguity is generated in a similar way, but with additional neighbors which share a common vertex with the unit of interest. To generate the W_n matrix according to the *group interaction* scheme: (i) Calculate the number of groups according to $g = \text{Round}(n^\delta)$, and the approximate average group size $m = n/g$; (ii) generate the group sizes $(n_1, n_2, \cdots, n_g)$

according to a discrete uniform distribution from $0.5m$ to $1.5m$; (iii) adjust the group sizes so that $\sum_{i=1}^{g} n_i = n$, and (iv) define $W_n = \text{diag}\{W_i/(n_i - 1), i = 1, \cdots, g\}$, a matrix formed by placing the sub-matrices W_i along the diagonal direction, where W_i is an $n_i \times n_i$ matrix with ones on the off-diagonal positions and zeros on the diagonal positions. Clearly, under Rook or Queen contiguity, each spatial unit has a bounded number of neighbors, whereas under group interaction it is divergent with rate $n^{1-\delta}$.

Error Distributions. Various distributions are considered in generating the model errors, including normal, normal mixture, lognormal, chi-square, normal-gamma mixture, etc. All distributions are standardized to have zero mean and unit variance. The standardized normal-mixture variates are generated according to

$$e_{n,i} = ((1 - v_i)Z_i + v_i \tau Z_i)/(1 - p + p * \tau^2)^{0.5},$$

where v_i is a Bernoulli random variable (with probability of success p) and Z_i is a standard normal, independent of v_i. The parameter p in this case also represents the proportion of mixing the two normal populations. In our experiments, we choose $p = 0.1$, meaning that 90% of the random variates are from standard normal and the remaining 10% are from another normal population with standard deviation τ. We choose $\tau = 4$ to simulate the situation where there are gross errors in the data.

Regressors. The DGPs used in the linear spatial regression models contain a constant and two regressors, and the DGPs used in the spatial panel data models with fixed effects contains three time-varying regressors. The simplest method for generating the values $\{x_i\}$ for a regressor X_n is to make random draws from a certain distribution, leading to a scheme `XVal-A`: $\{x_i\} \stackrel{iid}{\sim} N(0, 1)$. Alternatively, to allow for the possibility that there might be systematic differences in X_n values across the different sets of spatial units, e.g., spatial groups, spatial clusters, etc. In this case, the ith value in the jth 'group', or jth column of the lattice, $\{x_{ij}\}$ of X_n are generated according to scheme `XVal-B`: $\{x_{ij}\} = (2z_j + z_{ij})/\sqrt{5}$, where $\{z_j, z_{ij}, v_j, v_{ij}\} \stackrel{iid}{\sim} N(0, 1)$, across all i and j. Unlike the `XVal-A` scheme that gives iid X values, the `XVal-B` scheme gives non-iid X values, or different group means in terms of group interaction, see Lee (2004). Additional regressors are generated similarly and independently according to either `XVal-A` or `XVal-B` or a mix of the two. In case of a panel data model, a time trend $0.1t$ is added to each regressor.

Heteroskedasticity. The heteroskedasticity is generated by making it either proportional to the absolute values of a regressor, or to the group size when the group interaction spatial weight matrix is used. To be exact, the former is generated by setting $\sigma_i = |X_{n1,i}|$ or $2|X_{n1,i}|$, and the latter by setting $\sigma_i =$ twice the group size over the average group size.

In each Monte Carlo experiment, five different sample sizes are considered, i.e., $n = 50, 100, 200, 500$ and 1000. The number of Monte Carlo replications used is 10,000. The regressors are treated as fixed in the experiments. As size-adjusted powers are almost the same for comparable tests, only the empirical sizes of the tests are reported.

4.2 Linear regression with SARAR effects

For the SARAR(1,1) model, we use the following data generating process (DGP) in our Monte Carlo experiments:

$$Y_n = \lambda_1 W_{1n} Y_n + \beta_0 1_n + X_{1n}\beta_1 + X_{2n}\beta_2 + u_n, \quad u_n = \lambda_2 W_{2n} u_n + \varepsilon_n,$$

where $\varepsilon_{ni} = \sigma_i e_{ni}$ with $\{e_{ni}\}$ being iid(0, 1). The parameter values are set at $\beta = \{5, 1, 1\}'$.

Table 1 presents partial results for the empirical mean, sd and rejection frequencies for the three LM tests for spatial lag dependence, i.e., $\text{LM}_{\texttt{SAR}}$, $\text{LM}^{\texttt{OPG}}_{\texttt{SAR}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SAR}}$ for testing $H_0^a : \lambda_1 = 0$ in the SAR model. Table 2 presents partial results for the three LM tests for spatial error dependence, i.e., $\text{LM}_{\texttt{SDE}}$, $\text{LM}^{\texttt{OPG}}_{\texttt{SED}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SED}}$ for testing $H_0^b : \lambda_2 = 0$ in the SED model. Table 3 gives partial results for the three tests of SARAR(1,1) dependence, i.e., $\text{LM}_{\texttt{SARAR}}$, $\text{LM}^{\texttt{OPG}}_{\texttt{SARAR}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SARAR}}$ for testing $H_0^c : \lambda_1 = 0, \lambda_2 = 0$ in the SARAR model.

The following general observations arise from our results: (*i*) The null distributions of the three proposed tests ($\text{SLM}^{\texttt{OPG}}_{\texttt{SAR}}$, $\text{SLM}^{\texttt{OPG}}_{\texttt{SED}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SARAR}}$) are very close to their nominal ones; (*ii*) The three OPG-variants of the LM tests given in Born and Breitung (2011) can have severe finite sample distortions in size, mean and variance; and (*iii*) the three regular LM tests can have both finite and large sample distortions in their null distributions. It is interesting to note that even when the disturbances are homoskedastic, the three proposed tests still dominate the other two sets of tests, especially when the disturbances are non-normal (some Monte Carlo results are not reported to save space).

To illustrate the point that the existing tests perform poorer under heavier spatial dependence, we report two sets of results for the SED model, one under Queen contiguity with $r = 10$ (light spatial dependence, Table 2a), and one under group interaction with $g = n^{0.5}$ (heavy spatial dependence, Table 2b). The results indeed indicate that under the `Queen` design, the two OPG-based tests agree well, but under the `group` design, $\text{LM}^{\texttt{OPG}}_{\texttt{SED}}$ performs noticeably poorer than $\text{SLM}^{\texttt{OPG}}_{\texttt{SED}}$. The same is observed for the LM tests of SLD and LM tests of SARAR. However, as seen from the next subsection, the OPG-based LM tests without finite sample correction can perform poorly even under light spatial dependence.

4.3 Linear regression with spatial error components

For investigating the finite sample performance of the three tests: The regular LM test $\text{LM}_{\texttt{SED}}$, its OPG-variant without finite sample corrections $\text{LM}^{\texttt{OPG}}_{\texttt{SED}}$, and its OPG-variant with finite sample corrections $\text{SLM}^{\texttt{OPG}}_{\texttt{SED}}$, we use the following DGP in the Monte Carlo experiments:

$$Y_n = \beta_0 1_n + X_{n1}\beta_1 + X_{n2}\beta_2 + u_n \quad \text{with} \;\; u_n = W_n \nu_n + \varepsilon_n,$$

where again $\varepsilon_{ni} = \sigma_i e_{ni}$ with $\{e_{ni}\}$ being iid(0, 1), and $\beta = \{5, 1, 1\}'$.

Table 4 contains partial Monte Carlo results for the three LM tests. The results show that the proposed test $\text{SLM}^{\texttt{OPG}}_{\texttt{SEC}}$ dominates the regular LM test ($\text{LM}_{\texttt{SEC}}$) and another proposed test ($\text{LM}^{\texttt{OPG}}_{\texttt{SEC}}$) without finite sample corrections. While the results do show that $\text{LM}^{\texttt{OPG}}_{\texttt{SEC}}$ converges to $N(0, 1)$, its convergence rate can be very slow and as a result the finite sample performance of $\text{LM}^{\texttt{OPG}}_{\texttt{SEC}}$ can be poor, even when the spatial dependence (Queen contiguity) is quite light. The results (not reported for brevity) under a heavier spatial dependence (group interaction) show that $\text{LM}^{\texttt{OPG}}_{\texttt{SEC}}$ performs much poorer. In contrast, $\text{SLM}^{\texttt{OPG}}_{\texttt{SEC}}$ still performs reasonably well. This shows the importance of finite sample corrections. The results show that $\text{LM}_{\texttt{SEC}}$ is not robust against heteroskedasticity. The non-robust feature of $\text{LM}_{\texttt{SEC}}$ (against non-normality) is demonstrated in Yang (2010).

4.4 Fixed effects spatial panel data model with SARAR dependence

For the spatial panel data models with fixed effects, we use the following DGP:

$$\begin{aligned} Y_{nt} &= \lambda_1 W_{1n} Y_{nt} + X_{1n}\beta_1 + X_{2n}\beta_2 + X_{3n}\beta_3 + \mu_n + u_{nt}, \\ u_{nt} &= \lambda_2 W_{2n} u_{nt} + \varepsilon_{nt}, \ \ t = 1, \ldots, T, \end{aligned}$$

where the additional regressor X_{3n} is generated in a similar fashion as the earlier two except it is generated from a standardized lognormal distribution instead of the standard normal distribution. The fixed effects are generated by setting $\mu_n = \frac{1}{T}\sum_{t=1}^{T} X_{nt} + Z_n$ where $Z_n \sim N(0, I_n)$.

Tables 5-7 report partial Monte Carlo results, corresponding to the three null hypotheses, of the three sets of tests, namely, the regular LM tests ($\text{LM}^{\texttt{FE}}_{\texttt{SAR}}$, $\text{LM}^{\texttt{FE}}_{\texttt{SED}}$, $\text{LM}^{\texttt{FE}}_{\texttt{SARAR}}$), the OPG-variants without finite sample corrections ($\text{LM}^{\texttt{FEOPG}}_{\texttt{SAR}}$, $\text{LM}^{\texttt{FEOPG}}_{\texttt{SED}}$, $\text{LM}^{\texttt{FEOPG}}_{\texttt{SARAR}}$), and the OPG-variants with finite sample corrections ($\text{SLM}^{\texttt{FEOPG}}_{\texttt{SAR}}$, $\text{SLM}^{\texttt{FEOPG}}_{\texttt{SED}}$, $\text{SLM}^{\texttt{FEOPG}}_{\texttt{SARAR}}$). The results show the following: (*i*) The SLMs dominate the other two sets of tests in terms of null distributions and their robustness against non-normality and heteroskedasticity; (*ii*) The regular LMs are not robust against heteroskedasticity; and (*iii*) the OPG variants without finite sample corrections can perform poorly when the sample size is not large even under homoskedasticity. It is interesting to note that the SLMs dominate the other two sets of tests even under normality and homoskedasticity.

5 Conclusion and Discussion

We have presented a general methodology to robustify the standard LM tests to allow for non-normality and unknown heteroskedasticity. General ideas and methods for correcting the robustified LM tests to obtain better finite sample performance are also presented.

These ideas and methods are demonstrated in details using the three popular spatial models. In addition, extensive Monte Carlo experiments are performed, where the spatially autocorrelated regressors as in Pace et al. (2011) are also considered. The results show that these tests work very well. While many popular spatial LM tests are of the form specified above, some are not. For example, the LM test for spatial lag dependence allowing for the presence of spatial error dependence and vice versa. In these cases, the matrices A_{rn} and the vectors $b_{rn}, r = 1, \ldots, k$ contain estimated parameter(s). Thus, it is necessary to further extend the above ideas to deal with these cases.

Appendix: Proofs of the Theorems

To prove the theorems, we need the following central limit theorem (CLT) for the linear-quadratic form $Q_n(\varepsilon_n) = \varepsilon_n' A_n \varepsilon_n + b_n' \varepsilon_n$ defined in (2).

Theorem A.1 (Kelejian and Prucha, 2001): *Suppose ε_n, A_n and b_n satisfy Assumptions 1-2. If $\frac{1}{n}\tau_n^2 \geq c$ for some $c > 0$ and large enough n, then*

$$\frac{Q_n(\varepsilon_n) - \mu_n}{\tau_n} \xrightarrow{D} N(0, 1), \tag{A-1}$$

where $\mu_n = E[Q_n(\varepsilon_n)] = \sum_{i=1}^n a_{n,ii}\sigma_i^2$, and $\tau_n^2 = Var[Q_n(\varepsilon_n)] = 2\sum_{i=1}^n \sum_{j=1}^n a_{n,ij}^2 \sigma_i^2 \sigma_j^2 + \sum_{i=1}^n b_{n,i}^2 \sigma_i^2 + \sum_{i=1}^n [a_{n,ii}^2 \sigma_i^4 \kappa_i + 2 b_{n,i} a_{n,ii} \sigma_i^3 \gamma_i]$, with γ_i and κ_i being, respectively, the skewness and excess kurtosis of $\varepsilon_{n,i}$.

Note that the above result requires that A_n be symmetric. When A_n is not symmetric, it can be replaced by $\frac{1}{2}(A_n + A_n')$. The above result allows the elements of ε_n to depend upon n. When $\{\varepsilon_{n,i}\}$ are normal, $\gamma_i = \kappa_i = 0$ and the last term in τ_n^2 vanishes. A multivariate extension of this result is the CLT for a $k \times 1$ vector of linear quadratic forms given in Kelejian and Prucha (2010, p. 63).

Proof of Theorem 1: It suffices to show that $\frac{1}{n}(\sum_{i=1}^n \varepsilon_{n,i}^2 \xi_{n,i}^2 - \tau_n^2) \xrightarrow{p} 0$. Recall $\xi_{n,i} = \zeta_{n,i} + a_{n,ii}\varepsilon_{n,i} + b_{n,i}$ and $\zeta_{n,i}$ is the ith element of $\zeta_n = (A_n^l + A_n^{u\prime})\varepsilon_n$. We have

$$\begin{aligned}
&\frac{1}{n}\left(\sum_{i=1}^n \varepsilon_{n,i}^2 \xi_{n,i}^2 - \tau_n^2\right) \\
= \;& \frac{1}{n}\sum_{i=1}^n a_{n,ii}^2 \left(\varepsilon_{n,i}^4 - \mathrm{E}(\varepsilon_{n,i}^4)\right) + \frac{2}{n}\sum_{i=1}^n a_{n,ii} b_{n,ii} \left(\varepsilon_{n,i}^3 - \mathrm{E}(\varepsilon_{n,i}^3)\right) + \frac{1}{n}\sum_{i=1}^n b_{n,ii}^2 \left(\varepsilon_{n,i}^2 - \sigma_i^2\right) \\
&+ \frac{1}{n}\sum_{i=1}^n \left(\varepsilon_{n,i}^2 \zeta_{n,i}^2 - \sigma_i^2 c_{n,i}\right) + \frac{2}{n}\sum_{i=1}^n a_{n,ii}\; \varepsilon_{n,i}^3 \zeta_{n,i} + \frac{2}{n}\sum_{i=1}^n b_{n,ii}\; \varepsilon_{n,i}^2 \zeta_{n,i} \equiv \sum_{k=1}^6 H_{kn},
\end{aligned}$$

where $c_{n,i} = 4\sum_{j=1}^{i-1} a_{n,ij}^2 \sigma_j^2$. The result of the theorem follows by showing that $H_{kn} \xrightarrow{p} 0$ for $k = 1, \ldots, 6$, which is done by using the weak law for large numbers (WLLN) for martingale difference arrays in Davidson (1994, p. 299). Let $\mathcal{F}_{n,i}$ be the increasing σ-field generated by $\{\varepsilon_{n,1}, \ldots \varepsilon_{n,i}\}$, and note that $\zeta_{n,i}$ is $\mathcal{F}_{n,i-1}$-measurable and $\varepsilon_{n,i}$ is independent of $\zeta_{n,i}$.

To show that $H_{1n} = \frac{1}{n}\sum_{i=1}^n a_{n,ii}^2(\varepsilon_{n,i}^4 - \mathrm{E}\varepsilon_{n,i}^4) \xrightarrow{p} 0$, note that under Assumption 1 the $\{\varepsilon_{n,i}^4 - \mathrm{E}\varepsilon_{n,i}^4\}$ are independent with mean zero and that $\mathrm{E}|\varepsilon_{n,i}^4 - \mathrm{E}\varepsilon_{n,i}^4|^{1+\delta} \leq K_\varepsilon < \infty$ for $\delta > 0$. Thus, the $\{\varepsilon_{n,i}^4 - \mathrm{E}\varepsilon_{n,i}^4\}$ are uniformly integrable. Furthermore, under Assumption 2, we have $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n a_{n,ii}^2 \leq K_a^2 < \infty$, and $\limsup_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^n a_{n,ii}^4 \leq \limsup_{n\to\infty} \frac{1}{n}K_a^4 = 0$. It follows from the WLLN for martingale difference arrays in Davidson (1994, p. 299) that $H_{1n} \xrightarrow{p} 0$. Similar arguments lead to $H_{2n} = \frac{2}{n}\sum_{i=1}^n a_{n,ii}b_{n,ii}(\varepsilon_{n,i}^3 - \mathrm{E}\varepsilon_{n,i}^3) \xrightarrow{p} 0$, and $H_{3n} = \frac{1}{n}\sum_{i=1}^n b_{n,ii}^2(\varepsilon_{n,i}^2 - \sigma_i^2) \xrightarrow{p} 0$.

To prove $H_{4n} = \frac{1}{n}\sum_{i=1}^n(\varepsilon_{n,i}^2\zeta_{n,i}^2 - \sigma_i^2 c_{n,i}) \xrightarrow{p} 0$, write $H_{4n} = H_{4n}^a + H_{4n}^b$, where $H_{4n}^a = \frac{1}{n}\sum_{i=1}^n(\varepsilon_{n,i}^2 - \sigma_i^2)\zeta_{n,i}^2$, and $H_{4n}^b = \frac{1}{n}\sum_{i=1}^n \sigma_i^2(\zeta_{n,i}^2 - c_{n,i})$. For H_{4n}^a, we note that $(\varepsilon_{n,i}^2 - \sigma_i^2)\zeta_{n,i}^2$ is $\mathcal{F}_{n,i}$-measurable and that $\mathrm{E}(\varepsilon_{n,i}^2 - \sigma_i^2)\zeta_{n,i}^2|\mathcal{F}_{n,i-1}) = 0$. It follows that $\{\varepsilon_{n,i}^2 - \sigma_i^2)\zeta_{n,i}^2, 1 \leq i \leq n\}$ forms a martingale difference array. Thus, under Assumption 1 the WLLN for martingale difference arrays applies which leads to $H_{4n}^a \xrightarrow{p} 0$.

For H_{4n}^b, it is easy to see that $\zeta_{n,i} = 2\sum_{j=1}^{i-1} a_{n,ij}\varepsilon_{n,j}$, $\mathrm{E}\zeta_{n,i}^2 = 4\sum_{j=1}^{i-1} a_{n,ij}^2\sigma_j^2 = c_{n,i}$, and

$$
\begin{aligned}
H_{4n}^b &= \frac{1}{n}\sum_{i=1}^n \sigma_i^2(\zeta_{n,i}^2 - c_{n,i}) \\
&= \frac{4}{n}\sum_{i=1}^n \sigma_i^2 \sum_{j=1}^{i-1} a_{n,ij}^2(\varepsilon_{n,j}^2 - \sigma_j^2) + \frac{8}{n}\sum_{i=1}^n \sigma_i^2 \sum_{j=1}^{i-1}\sum_{k=1}^{j-1} a_{n,ij}a_{n,ik}\varepsilon_{n,j}\varepsilon_{n,k} \\
&= \sum_{i=1}^{n-1} \phi_{n,i}(\varepsilon_{n,i}^2 - \sigma_i^2) + \frac{1}{n}\sum_{i=1}^{n-1} \varepsilon_{n,i}V_{n,i},
\end{aligned}
$$

where $\phi_{n,i} = \frac{4}{n}\sum_{j=i+1}^n \sigma_j^2 a_{n,ji}^2$, $V_{n,i} = \sum_{j=1}^{i-1} \varphi_{n,ij}\varepsilon_{n,j}$, and $\varphi_{n,ij} = 8\sum_{k=i+1}^n \sigma_k^2 a_{n,ki}a_{n,kj}$. Thus, H_{n4}^b is written as two sums of martingale difference arrays. It is easy to verify the conditions of the WLLN for martingale difference arrays. It follows that $H_{4n}^b \xrightarrow{p} 0$. Similarly, $H_{5n} = \frac{2}{n}\sum_{i=1}^n a_{n,ii}\ \varepsilon_{n,i}^3\zeta_{n,i} \xrightarrow{p} 0$, and $H_{6n} = \frac{2}{n}\sum_{i=1}^n b_{n,ii}\ \varepsilon_{n,i}^2\zeta_{n,i} \xrightarrow{p} 0$. ■

Proof of Corollary 1: Follow the same arguments as those for proving Theorem 1.

Proof of Theorem 2: Without loss of generality, we prove the theorem for the case of $k = 2$. With the result of Theorem 1 and the multivariate CLT for a vector of linear quadratic forms of Kelejian and Prucha (2010, p. 63), it suffices to show that $\frac{1}{n}[\sum_{i=1}^n \varepsilon_{n,i}^2\xi_{1n,i}\xi_{2n,i} - \mathrm{Cov}(Q_{1n}, Q_{2n})] \xrightarrow{p} 0$, where $\xi_{rn,i} = \zeta_{rn,i} + a_{rn,ii}\varepsilon_{n,i} + b_{rn,i}$, $\zeta_{rn,i}$ is the ith element of $\zeta_{rn} = (A_{rn}^l + A_{rn}^{u\prime})\varepsilon_n$, and $Q_{rn} = \varepsilon_{n,i}'A_{rn}\varepsilon_{n,i} + b_{rn}'\varepsilon_{n,i} = \varepsilon_{n,i}'\xi_{rn,i}$, $r = 1, 2$. It is easy to verify that $\varepsilon_{n,i}'\xi_{rn,i}$ and $\varepsilon_{n,j}'\xi_{sn,j}$ are uncorrelated, for $i \neq j$ and $r, s = 1, 2$. It follows that

$$
\begin{aligned}
\mathrm{Cov}(Q_{1n}, Q_{2n}) &= \sum_{i=1}^n \mathrm{Cov}(\varepsilon_{n,i}'\xi_{1n,i},\ \varepsilon_{n,i}'\xi_{2n,i}) \\
&= 4\sum_{i=1}^n\sum_{j=1}^{i-1} a_{1n,ij}a_{2n,ij}\sigma_i^2\sigma_j^2 + \sum_{i=1}^n a_{1n,ii}a_{2n,ii}(\mathrm{E}\varepsilon_{n,i}^4 - \sigma_i^4) \\
&\quad + \sum_{i=1}^n(a_{1n,ii}b_{2n,i} + a_{2n,ii}b_{1n,i})\mathrm{E}\varepsilon_{n,i}^3 + \sum_{i=1}^n b_{1n,i}b_{2n,i}\sigma_i^2.
\end{aligned}
$$

The above result allows us to write $\frac{1}{n}[\sum_{i=1}^{n} \varepsilon_{n,i}^2 \xi_{1n,i}\xi_{2n,i} - \mathrm{Cov}(Q_{1n}, Q_{2n})]$ as sums of martingale difference arrays, and the rest is similar to the proof of Theorem 1. ■

Proof of Corollary 2: Follow the same arguments as those for proving Theorem 2.

Proof of Theorem 3: The main part of the proof parallels that of the proof of Theorems 1 and 2. We focus on the finite sample corrections. Consider the quadratic form $\varepsilon_n' A_n \varepsilon_n$ and note that $\mu_n = \mathrm{E}(\varepsilon_n' A_n \varepsilon_n) = \sum_{i=1}^n a_{n,ii}\sigma_i^2$. A natural estimator for μ_n is $\hat{\mu}_n = \sum_{i=1}^n a_{n,ii}\tilde{\varepsilon}_{n,i}^2 = \tilde{\varepsilon}_n' A_n^d \tilde{\varepsilon}_n$, where $\tilde{\varepsilon}_n'$ is the vector of OLS residuals. Clearly, $\hat{\mu}_n$ is a biased estimator as $\mathrm{E}(\hat{\mu}_n) = \mathrm{E}(\tilde{\varepsilon}_n' A_n^d \tilde{\varepsilon}_n) = \mathrm{E}(\varepsilon_n' M_n A_n^d M_n \varepsilon_n) = \sum_{i=1}^n b_{n,ii}\sigma_i^2 \neq 0$. In this case, $b_{n,ii}$ are the diagonal elements of $M_n A_n^d M_n$, which are of the form

$$b_{n,ii} = m_{n,ii}^2 a_{n,ii} + \sum_{j=1(\neq i)}^{n} m_{n,ij}^2 a_{n,jj},$$

where $m_{n,ij}$ are the elements of the projection matrix M_n defined above (10). This immediately suggests a new estimator $\hat{\mu}_n^* = \sum_{i=1}^n a_{n,ii} m_{n,ii}^{-2} \tilde{\varepsilon}_{n,i}^2$ that is nearly unbiased. In fact, the quantities leading to the bias, $\sum_{j=1(\neq i)}^n (m_{n,ij}/m_{m,ii})^2 a_{n,jj}$, becomes negligible by the properties of the projection matrix M_n. Clearly, these arguments and methods can be applied to give finite sample corrections to all tests where the null model is either the classical linear regression model, or the panel data model with fixed effects.

Proof of Theorem 4: Similar to the proof of Theorem 3.

Proof of Theorem 5: Similar to the proof of Theorem 3.

References

[1] Anselin, L. (1988a). Lagrange multiplier test diagnostics for spatial dependence and heterogeneity. *Geographical Analysis* **20**, 1-17.

[2] Anselin, L. (1988b). *Spatial Econometrics: Methods and Models.* Kluwer Academic, Dordrecht.

[3] Anselin, L. (2001). Rao's score test in spatial econometrics. *Journal of Statistical Planning and Inference* **97**, 113-139.

[4] Anselin L. and Bera, A. K. (1998). Spatial dependence in linear regression models with an introduction to spatial econometrics. In: *Handbook of Applied Economic Statistics, Edited by Aman Ullah and David E. A. Giles.* New York: Marcel Dekker.

[5] Baltagi, B. H., Song, S. H. and Koh W. (2003). Testing panel data regression models with spatial error correlation. *Journal of Econometrics* **117**, 123-150.

[6] Baltagi, B. H. and Yang Z. L. (2013). Standardized LM tests of spatial error dependence in linear or panel regressions. *Econometrics Journal* **16**, 103-134.

[7] Born, B. and Breitung, J. (2011). Simple regression based tests for spatial dependence. *Econometrics Journal*, **14**, 330-342.

[8] Davidson, J. (1994). *Stochastic Limit Theory.* Oxford University Press, Oxford.

[9] Davidson, R. and MacKinnon, J. (1985). Heteroscedasticity-robust tests in regression directions. *Annales De L'INSEE*, **59/60**, 183-217.

[10] Debarsy, N. and Ertur, C. (2010). Testing for spatial autocorrelation in a fixed effects panel data model. *Regional Science and Urban Economics*, **40**, 453–70.

[11] Glaeser, E. L., Sacerdote, B., and Scheinkman, J. A. (1996). Crime and social interactions. *Quarterly Journal of Economics*, **111**, 507-548.

[12] Kelejian H. H. and Prucha, I. R. (2001). On the asymptotic distribution of the Moran I test statistic with applications. *Journal of Econometrics* **104**, 219-257.

[13] Kelejian H. H. and Prucha, I. R. (2010). Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances. *Journal of Econometrics* **157**, 53-67.

[14] Kelejian H. H. and Robinson, D. P. (1995). Spatial correlation: a suggested alternative to the autoregressive models. In: Anselin, L., Florax, R. J. G. M. (Eds.), *New Directions in Spatial Econometrics.* Springer-Verlag, Berlin.

[15] Lee, L. F. and Yu, J. (2010). Estimation of spatial autoregressive panel data models with fixed effects. *Journal of Econometrics* **154**, 165-185.

[16] LeSage, J. P. (1999). The theory and practice of spatial econometrics. www.spatialeconometrics.com.

[17] Lin, X. and Lee, L. F. (2010). GMM estimation of spatial autoregressive models with unknown heteroskedasticity. *Journal of Econometrics*, **157**, 34-52.

[18] Pace, R. K., LeSage, J. P. and Zhu, S. (2011). Spatial dependence in regressors and its effect on estimator performance. Working Paper.

[19] White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica*, 48, 817-838.

[20] Yang, Z. L. (2010). A robust LM test for spatial error components. *Regional Science and Urban Economics* **40**, 299-310.

Table 1. Mean, sd, and Rejection Frequencies: LM Tests for Spatial Lag Dependence

n	Heteroskedasticity = $\lvert X_1\rvert$ mean	sd	10%	5%	1%	Heteroskedasticity = $2\lvert X_1\rvert$ mean	sd	10%	5%	1%
	`Normal Errors`									
50	-0.3159	0.8135	.0545	.0202	.0006	-0.6587	0.9206	.1236	.0439	.0071
	-0.3927	0.9761	.1160	.0545	.0060	-0.7272	0.9269	.1618	.0744	.0070
	0.0286	1.0266	.1090	.0506	.0075	-0.0419	1.0634	.1224	.0600	.0111
100	-0.3739	1.0574	.1476	.0753	.0152	-0.9043	1.0325	.2697	.1412	.0152
	-0.3897	1.0168	.1344	.0686	.0121	-0.7769	0.8635	.1563	.0722	.0089
	-0.0382	1.0393	.1133	.0531	.0091	-0.0838	1.0448	.1167	.0573	.0108
200	-0.3524	1.0486	.1355	.0706	.0143	-0.5287	1.0522	.1660	.0823	.0164
	-0.4003	0.9854	.1221	.0613	.0104	-0.5682	0.9670	.1484	.0753	.0127
	-0.0708	1.0118	.1064	.0523	.0094	-0.0946	1.0184	.1083	.0526	.0073
500	-0.2483	0.9595	.0991	.0454	.0082	-0.3044	1.0002	.1113	.0548	.0103
	-0.2755	0.9823	.1091	.0526	.0107	-0.3391	0.9802	.1117	.0562	.0094
	-0.0224	1.0020	.1033	.0505	.0098	-0.0488	0.9970	.0956	.0472	.0082
1000	-0.1594	1.0135	.1067	.0539	.0115	-0.3239	1.2137	.1910	.1127	.0332
	-0.1792	0.9900	.1009	.0491	.0095	-0.3479	0.9893	.1186	.0595	.0116
	-0.0346	0.9966	.0981	.0472	.0093	-0.0892	1.0028	.1023	.0523	.0091
	`Normal Mixture`									
50	-0.2939	0.8118	.0542	.0188	.0012	-0.6282	0.8593	.1053	.0368	.0037
	-0.3471	0.9675	.1040	.0440	.0049	-0.6896	0.9015	.1437	.0609	.0051
	0.0472	1.0134	.0985	.0432	.0056	0.0146	1.0438	.1125	.0532	.0089
100	-0.3589	1.0369	.1377	.0722	.0128	-0.8301	1.0115	.2383	.1206	.0144
	-0.3572	0.9977	.1163	.0537	.0083	-0.7169	0.8810	.1413	.0649	.0080
	-0.0297	1.0249	.1036	.0463	.0055	-0.0397	1.0326	.1060	.0529	.0091
200	-0.3458	1.0239	.1270	.0601	.0122	-0.5202	1.0135	.1484	.0730	.0138
	-0.3719	0.9829	.1106	.0558	.0087	-0.5449	0.9577	.1357	.0606	.0086
	-0.0492	1.0196	.1031	.0481	.0083	-0.0660	1.0235	.1028	.0494	.0086
500	-0.2473	0.9662	.1032	.0477	.0069	-0.2926	1.0022	.1144	.0569	.0109
	-0.2719	0.9922	.1137	.0534	.0091	-0.3192	0.9892	.1155	.0546	.0097
	-0.0215	1.0123	.1085	.0507	.0071	-0.0323	1.0071	.1025	.0505	.0079
1000	-0.1441	1.0299	.1159	.0581	.0133	-0.3142	1.2148	.1824	.1140	.0355
	-0.1620	1.0037	.1070	.0528	.0097	-0.3298	0.9952	.1169	.0596	.0107
	-0.0192	1.0096	.1016	.0517	.0106	-0.0725	1.0102	.1067	.0529	.0085
	`Lognormal Errors`									
50	-0.2218	0.8217	.0494	.0162	.0010	-0.6056	0.9030	.1167	.0471	.0068
	-0.1984	0.9878	.0885	.0373	.0037	-0.6596	0.9333	.1417	.0618	.0070
	0.1722	1.0186	.1064	.0484	.0063	0.0157	1.0346	.1039	.0451	.0080
100	-0.3635	1.0022	.1273	.0640	.0126	-0.5443	1.3511	.2984	.1692	.0390
	-0.3529	0.9793	.1057	.0486	.0062	-0.4425	1.1426	.1805	.1010	.0228
	-0.0395	1.0262	.1034	.0450	.0067	0.1488	1.1496	.1535	.0911	.0291
200	-0.3661	0.9702	.1092	.0534	.0096	-0.4247	1.0034	.1245	.0598	.0096
	-0.4303	0.9807	.1212	.0595	.0097	-0.4489	0.9766	.1201	.0566	.0100
	-0.1111	1.0098	.1030	.0461	.0072	0.0290	1.0346	.1080	.0557	.0102
500	-0.2450	0.9531	.0958	.0455	.0089	-0.2905	0.9819	.1054	.0513	.0098
	-0.2661	0.9902	.1078	.0500	.0087	-0.3369	0.9906	.1107	.0551	.0098
	-0.0172	1.0143	.1036	.0483	.0086	-0.0527	1.0030	.0977	.0458	.0078
1000	-0.1362	1.0178	.1112	.0593	.0127	-0.2998	1.1520	.1615	.0888	.0251
	-0.1525	0.9978	.1038	.0497	.0078	-0.3380	0.9804	.1067	.0526	.0110
	-0.0131	1.0031	.1041	.0497	.0071	-0.0777	1.0002	.0962	.0440	.0082

Note: Three rows under each n: $\text{LM}_{\texttt{SAR}}$, $\text{LM}^{\texttt{OPG}}_{\texttt{SAR}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SAR}}$; `Group,` $g = n^{0.5}$; `XVal-B`.

Table 2a. Mean, sd, and Rejection Frequency: LM Tests for Spatial Error Dependence

	Heteroskedasticity = $\lvert X_1 \rvert$					Heteroskedasticity = 1				
n	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	-0.2843	0.8682	.0694	.0270	.0031	-0.2653	0.9354	.0870	.0368	.0059
	-0.3196	0.9637	.1016	.0404	.0035	-0.3348	0.9950	.1189	.0536	.0079
	0.0012	1.0343	.1090	.0478	.0053	-0.0395	1.0313	.1116	.0512	.0065
100	-0.1167	0.9274	.0759	.0351	.0051	-0.1921	0.9672	.0934	.0454	.0071
	-0.1623	1.0025	.1016	.0433	.0038	-0.2421	1.0024	.1116	.0542	.0092
	-0.0233	1.0157	.1025	.0422	.0038	-0.0460	1.0183	.1076	.0534	.0080
200	-0.1535	0.9674	.0923	.0449	.0073	-0.1171	0.9812	.0979	.0462	.0081
	-0.1885	0.9930	.1043	.0462	.0063	-0.1499	0.9972	.1017	.0515	.0097
	-0.0207	1.0043	.1000	.0461	.0063	-0.0129	1.0061	.1035	.0498	.0094
500	-0.0619	0.9731	.0891	.0439	.0073	-0.0935	0.9781	.0945	.0461	.0097
	-0.0849	1.0110	.1045	.0511	.0088	-0.1138	0.9877	.0980	.0490	.0104
	-0.0099	1.0168	.1045	.0520	.0094	-0.0313	0.9910	.0962	.0503	.0101
1000	-0.0588	1.0017	.0990	.0496	.0102	-0.0645	0.9969	.1002	.0498	.0091
	-0.0740	0.9991	.0976	.0522	.0097	-0.0804	1.0012	.1017	.0507	.0096
	-0.0121	1.0021	.0984	.0522	.0094	-0.0204	1.0027	.1006	.0502	.0093
	`Normal Mixture`									
50	-0.2962	0.8410	.0620	.0232	.0020	-0.2676	0.8698	.0660	.0275	.0040
	-0.3355	0.9887	.1105	.0443	.0029	-0.3365	0.9792	.1090	.0450	.0048
	0.0033	1.0286	.1037	.0423	.0030	-0.0258	1.0133	.0997	.0428	.0050
100	-0.1421	0.8892	.0700	.0306	.0052	-0.1861	0.9339	.0824	.0379	.0049
	-0.1832	1.0051	.0984	.0385	.0043	-0.2325	1.0019	.1027	.0483	.0063
	-0.0353	1.0121	.0966	.0369	.0031	-0.0277	1.0209	.1033	.0468	.0051
200	-0.1492	0.9259	.0772	.0372	.0077	-0.1200	0.9579	.0865	.0410	.0085
	-0.1841	0.9844	.0964	.0396	.0052	-0.1497	0.9920	.1006	.0403	.0056
	-0.0051	0.9957	.0938	.0389	.0050	-0.0084	1.0010	.0976	.0405	.0056
500	-0.0780	0.9387	.0804	.0399	.0079	-0.0791	0.9888	.0972	.0484	.0109
	-0.0986	1.0026	.0995	.0460	.0087	-0.1037	0.9999	.1002	.0492	.0092
	-0.0185	1.0064	.0984	.0455	.0085	-0.0196	1.0032	.1013	.0503	.0084
1000	-0.0777	0.9956	.0963	.0459	.0109	-0.0579	1.0033	.1005	.0519	.0090
	-0.0908	1.0019	.0986	.0440	.0074	-0.0743	1.0111	.1054	.0522	.0082
	-0.0265	1.0048	.0992	.0426	.0067	-0.0138	1.0128	.1035	.0514	.0084
	`Lognormal Errors`									
50	-0.2773	0.8259	.0525	.0199	.0033	-0.2576	0.8496	.0560	.0229	.0049
	-0.3805	0.9843	.1114	.0436	.0036	-0.4010	0.9699	.1071	.0439	.0063
	-0.0405	0.9972	.0875	.0335	.0027	-0.0859	0.9956	.0903	.0341	.0037
100	-0.1466	0.8576	.0550	.0262	.0058	-0.1811	0.8949	.0609	.0278	.0073
	-0.2866	0.9991	.1045	.0445	.0049	-0.3341	1.0001	.1136	.0520	.0067
	-0.1299	1.0031	.0898	.0339	.0037	-0.1183	1.0065	.0996	.0422	.0038
200	-0.1571	0.8934	.0668	.0299	.0066	-0.1331	0.9259	.0718	.0359	.0091
	-0.3002	0.9979	.1057	.0519	.0083	-0.2883	1.0032	.1128	.0563	.0101
	-0.1137	0.9980	.0926	.0423	.0056	-0.1394	1.0022	.1013	.0460	.0070
500	-0.0570	0.9321	.0722	.0378	.0103	-0.0969	0.9656	.0843	.0421	.0094
	-0.2245	1.0222	.1122	.0566	.0109	-0.2424	1.0260	.1161	.0602	.0128
	-0.1382	1.0210	.1044	.0514	.0085	-0.1532	1.0233	.1084	.0558	.0112
1000	-0.0662	0.9645	.0803	.0400	.0116	-0.0582	0.9853	.0888	.0441	.0104
	-0.2196	1.0193	.1104	.0571	.0108	-0.1856	1.0320	.1163	.0593	.0126
	-0.1497	1.0158	.1037	.0529	.0093	-0.1219	1.0289	.1111	.0557	.0117

Note: Three rows under each n: $\mathrm{LM}_{\texttt{SED}}$, $\mathrm{LM}_{\texttt{SED}}^{\texttt{OPG}}$ and $\mathrm{SLM}_{\texttt{SED}}^{\texttt{OPG}}$; `Queen,` $r = 10$; `XVal-B`.

Table 2b. Mean, sd, and Rejection Frequency: LM Tests for Spatial Error Dependence

n	Heteroskedasticity = $\lvert X_1 \rvert$					Heteroskedasticity = 1				
	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	-0.6348	0.8522	.0690	.0175	.0054	-0.6884	0.8265	.0919	.0140	.0026
	-0.8025	0.9268	.1840	.0851	.0105	-0.9054	0.9857	.2536	.1409	.0239
	-0.1347	1.0888	.1301	.0576	.0057	-0.1456	1.0930	.1423	.0725	.0111
100	-0.6374	0.7301	.0296	.0053	.0014	-0.5490	0.8634	.0880	.0226	.0029
	-0.8386	0.9019	.1951	.0989	.0156	-0.7230	0.9995	.1973	.1088	.0254
	-0.1436	1.0931	.1383	.0703	.0120	-0.1380	1.0565	.1234	.0618	.0118
200	-0.6993	1.0170	.1723	.0584	.0099	-0.4741	0.8978	.0909	.0286	.0029
	-0.7688	0.9368	.1798	.0939	.0170	-0.6187	1.0045	.1676	.0938	.0223
	-0.2137	1.0201	.1127	.0559	.0098	-0.1468	1.0378	.1131	.0579	.0115
500	-0.4728	1.0855	.1595	.0758	.0154	-0.3338	0.9500	.0949	.0409	.0064
	-0.5436	0.9846	.1466	.0772	.0156	-0.4425	1.0143	.1401	.0771	.0183
	-0.1317	1.0184	.1116	.0578	.0096	-0.0748	1.0308	.1120	.0607	.0116
1000	-0.5406	1.2492	.2324	.1321	.0342	-0.3079	0.9693	.1024	.0456	.0065
	-0.5689	0.9907	.1544	.0813	.0170	-0.3993	1.0208	.1369	.0739	.0171
	-0.1694	1.0300	.1148	.0616	.0106	-0.0855	1.0295	.1111	.0587	.0130
	`Normal Mixture`									
50	-0.6520	0.7139	.0480	.0154	.0028	-0.6899	0.6950	.0600	.0161	.0013
	-0.8268	0.8365	.1496	.0637	.0065	-0.8741	0.8453	.1679	.0800	.0099
	-0.0626	1.0191	.0952	.0431	.0061	-0.0395	1.0107	.0950	.0427	.0056
100	-0.6177	0.6613	.0246	.0068	.0013	-0.5512	0.7724	.0585	.0231	.0056
	-0.8159	0.8469	.1501	.0738	.0105	-0.7349	0.9191	.1555	.0730	.0115
	-0.0464	1.0542	.1107	.0572	.0107	-0.0639	1.0266	.0985	.0454	.0067
200	-0.6497	0.8456	.1034	.0377	.0086	-0.4462	0.8411	.0733	.0289	.0037
	-0.7297	0.8823	.1429	.0673	.0102	-0.5859	0.9673	.1369	.0654	.0118
	-0.1053	0.9861	.0874	.0377	.0041	-0.0505	1.0178	.0997	.0421	.0057
500	-0.4680	0.9789	.1236	.0579	.0120	-0.3474	0.9095	.0851	.0356	.0053
	-0.5279	0.9570	.1191	.0536	.0083	-0.4536	0.9920	.1318	.0610	.0106
	-0.0708	1.0172	.0983	.0419	.0053	-0.0603	1.0182	.1031	.0455	.0066
1000	-0.5213	1.1371	.1903	.1052	.0254	-0.3109	0.9407	.0930	.0380	.0049
	-0.5281	0.9723	.1290	.0603	.0092	-0.3988	1.0007	.1253	.0648	.0113
	-0.0900	1.0412	.1096	.0500	.0082	-0.0726	1.0176	.1043	.0494	.0069
	`Lognormal Errors`									
50	-0.6082	0.8130	.0602	.0195	.0046	-0.6884	0.7423	.0616	.0118	.0022
	-0.7622	0.8997	.1561	.0761	.0115	-0.9131	0.9026	.2075	.1087	.0208
	-0.0516	1.0641	.1181	.0525	.0063	-0.1093	1.0494	.1166	.0597	.0090
100	-0.5992	0.7250	.0248	.0081	.0028	-0.5404	0.8073	.0627	.0180	.0028
	-0.8143	0.9095	.1769	.0927	.0187	-0.7660	0.9591	.1833	.0992	.0221
	-0.1062	1.0648	.1232	.0623	.0108	-0.1374	1.0448	.1146	.0572	.0100
200	-0.6139	0.9651	.1361	.0512	.0104	-0.4523	0.8500	.0687	.0208	.0048
	-0.7155	0.9591	.1688	.0942	.0221	-0.6477	0.9735	.1639	.0866	.0204
	-0.1231	1.0477	.1177	.0605	.0118	-0.1420	1.0109	.1025	.0497	.0090
500	-0.4555	1.0267	.1354	.0613	.0124	-0.3329	0.9074	.0772	.0310	.0056
	-0.5570	0.9971	.1466	.0794	.0206	-0.4911	0.9906	.1391	.0734	.0167
	-0.1281	1.0472	.1178	.0559	.0124	-0.1070	1.0063	.1005	.0451	.0094
1000	-0.5002	1.1937	.2066	.1102	.0264	-0.3023	0.9240	.0807	.0334	.0054
	-0.5415	1.0080	.1494	.0822	.0179	-0.4375	0.9957	.1304	.0713	.0151
	-0.1229	1.0618	.1180	.0614	.0161	-0.1106	0.9994	.0982	.0492	.0092

Note: Three rows under each n: $\text{LM}_{\texttt{SED}}$, $\text{LM}^{\texttt{OPG}}_{\texttt{SED}}$ and $\text{SLM}^{\texttt{OPG}}_{\texttt{SED}}$; `Group`, $g = n^{0.5}$; `XVal-B`.

Table 3. Mean, sd, and Rejection Frequency: Joint LM Tests for SARAR Dependence

	Heteroskedasticity = $\lvert X_1 \rvert$					Heteroskedasticity = $2\lvert X_1 \rvert$				
n	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	2.2886	1.4830	.0613	.0180	.0018	2.4416	2.1771	.1081	.0472	.0138
	2.6891	1.8687	.1494	.0592	.0051	2.4771	1.8823	.1364	.0517	.0052
	2.2328	1.9698	.1201	.0539	.0080	2.2385	1.8381	.1149	.0455	.0037
100	2.3192	2.1102	.1014	.0503	.0116	2.2836	1.8988	.0903	.0378	.0091
	2.5021	2.0250	.1450	.0686	.0081	2.5870	2.0651	.1557	.0724	.0091
	2.1200	1.8979	.1038	.0478	.0063	2.1832	1.9700	.1117	.0498	.0081
200	2.5567	2.0947	.1286	.0505	.0103	2.8150	2.9793	.1686	.0766	.0185
	2.4096	2.0472	.1366	.0683	.0099	2.5063	2.1511	.1528	.0763	.0123
	2.2554	2.1731	.1246	.0653	.0140	2.1532	2.0063	.1121	.0540	.0087
500	2.7570	2.8166	.1743	.0934	.0244	2.6424	2.7786	.1593	.0820	.0211
	2.3415	2.1658	.1389	.0697	.0130	2.2700	2.1095	.1305	.0636	.0113
	2.1228	2.0155	.1130	.0557	.0088	2.1090	2.0027	.1098	.0539	.0089
1000	2.3977	2.3921	.1318	.0676	.0181	2.5871	2.5542	.1605	.0831	.0209
	2.2587	2.1948	.1284	.0687	.0137	2.2352	2.1468	.1264	.0657	.0126
	2.0670	2.0385	.1059	.0556	.0107	2.0765	2.0116	.1094	.0520	.0102
	`Normal Mixture`									
50	2.1706	1.4789	.0546	.0164	.0018	2.1743	1.8616	.0895	.0394	.0074
	2.5894	1.7446	.1265	.0465	.0032	2.3768	1.7502	.1132	.0437	.0023
	2.1809	1.8714	.1055	.0479	.0073	2.1529	1.7201	.0939	.0347	.0020
100	2.1496	1.9206	.0894	.0430	.0089	2.1344	1.7725	.0838	.0347	.0056
	2.3976	1.8953	.1244	.0555	.0058	2.4772	1.9166	.1339	.0583	.0059
	2.1028	1.7966	.0969	.0392	.0049	2.1594	1.8613	.1032	.0430	.0061
200	2.3802	1.9236	.1119	.0439	.0080	2.5839	2.3989	.1525	.0737	.0153
	2.3394	1.9379	.1228	.0562	.0071	2.4137	1.9868	.1349	.0622	.0082
	2.2335	2.1201	.1207	.0601	.0117	2.1232	1.8890	.1025	.0454	.0064
500	2.6161	2.6828	.1591	.0845	.0209	2.5565	2.4927	.1556	.0791	.0189
	2.2481	1.9890	.1179	.0550	.0080	2.2760	2.0051	.1237	.0566	.0097
	2.0777	1.9016	.0989	.0467	.0073	2.1270	1.9423	.1064	.0498	.0075
1000	2.3695	2.3712	.1342	.0676	.0158	2.5007	2.5717	.1535	.0831	.0200
	2.2267	2.0895	.1216	.0611	.0101	2.1974	2.0589	.1197	.0590	.0098
	2.0651	1.9807	.1061	.0500	.0085	2.0582	1.9657	.1044	.0483	.0083
	`Lognormal Errors`									
50	2.1689	1.5147	.0527	.0169	.0017	2.0630	1.8949	.0755	.0330	.0085
	2.5160	1.7353	.1234	.0444	.0027	2.3862	1.8016	.1196	.0478	.0028
	2.0377	1.7615	.0890	.0343	.0049	2.1161	1.7444	.0926	.0392	.0037
100	2.1353	2.2535	.0821	.0434	.0143	2.1039	1.9902	.0768	.0338	.0086
	2.4211	1.9868	.1300	.0642	.0088	2.5107	2.0459	.1426	.0703	.0103
	2.1930	1.9544	.1164	.0524	.0083	2.2294	2.0110	.1099	.0545	.0110
200	2.5451	2.5838	.1261	.0561	.0168	2.4693	2.3420	.1321	.0578	.0129
	2.5126	2.1653	.1462	.0792	.0143	2.4749	2.1352	.1432	.0683	.0142
	2.3739	2.3203	.1383	.0754	.0200	2.2189	2.0168	.1157	.0518	.0102
500	2.5566	2.7368	.1442	.0771	.0241	2.3771	2.5619	.1310	.0631	.0172
	2.3736	2.1520	.1340	.0685	.0134	2.3298	2.1134	.1322	.0649	.0122
	2.1850	2.0532	.1157	.0558	.0105	2.1413	1.9842	.1070	.0533	.0087
1000	2.2785	2.6453	.1146	.0591	.0167	2.4533	2.8658	.1379	.0706	.0211
	2.2545	2.1161	.1254	.0647	.0133	2.3116	2.1562	.1295	.0688	.0133
	2.0782	1.9999	.1052	.0534	.0098	2.1387	2.0244	.1103	.0541	.0091

Each n: $\mathrm{LM}_{\mathtt{SARAR}}$, $\mathrm{LM}^{\mathtt{OPG}}_{\mathtt{SARAR}}$ and $\mathrm{SLM}^{\mathtt{OPG}}_{\mathtt{SARAR}}$; W_{1n}=`Queen`, $r = 5$; W_{2n}=`Group`, $g = n^{0.5}$; `XVal-B`.

Table 4. Mean, sd, and Rejection Frequency: LM Tests for Spatial Error Components

n	Heteroskedasticity = $\lvert X_1\rvert$					Heteroskedasticity = 1				
	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	-0.2856	0.8409	.0430	.0226	.0132	-0.3637	0.8740	.0470	.0266	.0159
	-0.4012	0.9313	.0392	.0150	.0056	-0.5228	0.9820	.0374	.0164	.0068
	-0.0604	1.0052	.1005	.0458	.0193	-0.1018	1.0368	.0965	.0485	.0245
100	-0.1061	0.9223	.0743	.0399	.0233	-0.2825	0.9230	.0574	.0319	.0192
	-0.1920	0.9787	.0650	.0283	.0129	-0.3918	0.9834	.0487	.0213	.0088
	-0.0764	1.0251	.0955	.0465	.0217	-0.0758	1.0146	.0947	.0456	.0217
200	-0.2907	0.9635	.0605	.0323	.0187	-0.1953	0.9585	.0710	.0356	.0203
	-0.3458	0.9756	.0507	.0240	.0099	-0.2743	0.9924	.0565	.0262	.0113
	-0.0631	1.0128	.0968	.0475	.0232	-0.0773	1.0058	.0883	.0427	.0194
500	-0.1102	0.9846	.0852	.0461	.0263	-0.1260	0.9814	.0809	.0418	.0223
	-0.1634	0.9758	.0674	.0323	.0146	-0.1749	0.9928	.0681	.0312	.0156
	-0.0391	1.0016	.0957	.0479	.0232	-0.0543	1.0030	.0910	.0431	.0211
1000	0.0020	1.0335	.1131	.0616	.0352	-0.0701	0.9918	.0906	.0461	.0243
	-0.0306	0.9895	.0941	.0462	.0219	-0.1042	0.9929	.0806	.0380	.0181
	-0.0164	1.0113	.1019	.0509	.0259	-0.0217	1.0030	.0961	.0490	.0241
	`Normal Mixture`									
50	-0.2822	0.8868	.0572	.0336	.0195	-0.3656	0.8907	.0490	.0265	.0142
	-0.3996	0.9414	.0456	.0166	.0055	-0.4996	0.9656	.0384	.0143	.0060
	-0.0333	1.0039	.1081	.0493	.0185	-0.0848	1.0255	.1004	.0453	.0195
100	-0.0987	0.9657	.0811	.0451	.0274	-0.2829	0.9730	.0672	.0386	.0217
	-0.1727	0.9681	.0664	.0274	.0106	-0.3840	0.9848	.0498	.0197	.0073
	-0.0420	1.0033	.1022	.0451	.0192	-0.0619	1.0100	.0965	.0459	.0215
200	-0.2802	1.0704	.0756	.0465	.0303	-0.1931	1.0274	.0822	.0482	.0288
	-0.3347	0.9689	.0532	.0213	.0087	-0.2648	1.0038	.0654	.0282	.0125
	-0.0330	1.0050	.0996	.0474	.0212	-0.0616	1.0152	.0954	.0475	.0223
500	-0.1289	1.1099	.1081	.0605	.0349	-0.1275	1.0424	.0900	.0492	.0290
	-0.1735	0.9937	.0706	.0298	.0122	-0.1682	0.9926	.0718	.0322	.0150
	-0.0313	1.0045	.1001	.0466	.0230	-0.0411	0.9973	.0981	.0467	.0216
1000	-0.0340	1.1345	.1222	.0717	.0435	-0.0835	1.0500	.0994	.0537	.0312
	-0.0641	0.9791	.0848	.0380	.0170	-0.1148	0.9938	.0815	.0371	.0175
	-0.0407	0.9941	.0938	.0448	.0221	-0.0296	0.9983	.0973	.0465	.0237
	`Chi-Square, df = 4`									
50	-0.2899	0.8641	.0501	.0277	.0169	-0.3590	0.8760	.0497	.0260	.0147
	-0.4150	0.9359	.0432	.0174	.0067	-0.5233	0.9739	.0362	.0136	.0051
	-0.0606	1.0080	.1007	.0479	.0218	-0.0996	1.0204	.0934	.0456	.0202
100	-0.1020	0.9352	.0812	.0442	.0247	-0.2947	0.9419	.0614	.0342	.0194
	-0.1969	0.9739	.0673	.0278	.0116	-0.4164	0.9894	.0448	.0187	.0064
	-0.0900	1.0184	.0929	.0441	.0192	-0.0976	1.0139	.0904	.0422	.0205
200	-0.2945	0.9877	.0638	.0376	.0201	-0.1929	0.9817	.0762	.0406	.0225
	-0.3585	0.9671	.0475	.0196	.0086	-0.2822	1.0004	.0577	.0233	.0095
	-0.0804	1.0096	.0902	.0415	.0195	-0.0823	1.0118	.0903	.0416	.0183
500	-0.1184	1.0265	.0899	.0491	.0293	-0.1189	1.0041	.0872	.0443	.0261
	-0.1831	0.9806	.0635	.0268	.0116	-0.1772	0.9951	.0703	.0309	.0160
	-0.0604	1.0021	.0901	.0426	.0195	-0.0556	1.0013	.0878	.0439	.0202
1000	-0.0298	1.0769	.1145	.0642	.0373	-0.0884	1.0161	.0937	.0486	.0273
	-0.0757	0.9963	.0873	.0386	.0180	-0.1322	0.9999	.0766	.0364	.0158
	-0.0609	1.0096	.0946	.0442	.0221	-0.0495	1.0030	.0933	.0442	.0208

Note: Three rows under each n: $\mathrm{LM}_{\texttt{SEC}}$, $\mathrm{LM}^{\texttt{OPG}}_{\texttt{SEC}}$ and $\mathrm{SLM}^{\texttt{OPG}}_{\texttt{SEC}}$; W_n=`Queen`, $r = 5$, `XVal-A`.

Table 5. Monte Carlo results: LM Tests for Fixed Effects Panel SAR Model, $T = 3$

	Heteroskedasticity $\propto$ group size					Heteroskedasticity = 1				
n	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	Normal Errors									
50	-0.3253	0.8610	.0607	.0209	.0032	-0.1970	0.9908	.1017	.0498	.0098
	-0.4452	0.9846	.1298	.0646	.0100	-0.2340	1.0235	.1186	.0591	.0112
	-0.0699	1.0249	.1096	.0534	.0075	-0.0453	1.0327	.1134	.0557	.0099
100	-0.2568	0.9231	.0817	.0372	.0056	-0.1633	0.9840	.0989	.0485	.0075
	-0.3465	0.9965	.1202	.0629	.0127	-0.1995	0.9999	.1102	.0547	.0107
	-0.0558	1.0059	.1038	.0536	.0098	-0.0311	1.0091	.1045	.0518	.0096
200	-0.2194	0.9466	.0851	.0364	.0063	-0.1599	0.9943	.1021	.0507	.0097
	-0.2834	1.0015	.1109	.0580	.0121	-0.1765	1.0046	.1052	.0547	.0113
	-0.0416	1.0123	.1027	.0504	.0105	-0.0180	1.0100	.1039	.0512	.0101
500	-0.1587	0.9665	.0904	.0434	.0081	-0.0901	0.9856	.0963	.0461	.0089
	-0.2023	1.0026	.1060	.0543	.0120	-0.0979	0.9891	.0987	.0467	.0091
	-0.0442	1.0086	.1025	.0515	.0107	0.0015	0.9913	.0966	.0458	.0089
1000	-0.1141	0.9576	.0869	.0426	.0088	-0.0705	0.9959	.0998	.0505	.0085
	-0.1472	1.0008	.1047	.0548	.0126	-0.0782	1.0006	.1030	.0512	.0092
	-0.0290	1.0043	.1023	.0525	.0124	-0.0120	1.0015	.1018	.0515	.0083
	Normal Mixture									
50	-0.3299	0.8416	.0577	.0190	.0024	-0.1623	0.9962	.1036	.0509	.0095
	-0.4366	0.9748	.1225	.0570	.0091	-0.1902	1.0287	.1179	.0588	.0089
	-0.0614	1.0211	.1062	.0497	.0062	-0.0066	1.0389	.1158	.0547	.0078
100	-0.2562	0.9227	.0785	.0350	.0067	-0.1706	0.9784	.0942	.0441	.0080
	-0.3378	1.0000	.1202	.0597	.0111	-0.2062	0.9999	.1062	.0524	.0083
	-0.0449	1.0136	.1040	.0507	.0092	-0.0380	1.0086	.1015	.0486	.0078
200	-0.2249	0.9235	.0770	.0349	.0063	-0.1542	0.9793	.0976	.0492	.0094
	-0.2839	0.9804	.1058	.0521	.0108	-0.1694	0.9928	.1047	.0518	.0102
	-0.0428	0.9920	.0950	.0484	.0103	-0.0112	0.9980	.0973	.0475	.0090
500	-0.1411	0.9710	.0948	.0452	.0079	-0.1102	1.0016	.1014	.0527	.0101
	-0.1835	1.0080	.1100	.0561	.0111	-0.1186	1.0039	.1037	.0521	.0106
	-0.0250	1.0146	.1047	.0546	.0097	-0.0192	1.0061	.1023	.0517	.0103
1000	-0.1230	0.9531	.0873	.0419	.0066	-0.0688	1.0029	.1009	.0529	.0095
	-0.1550	0.9993	.1011	.0542	.0108	-0.0764	1.0049	.1016	.0517	.0097
	-0.0366	1.0028	.0994	.0505	.0089	-0.0102	1.0061	.1019	.0515	.0104
	Lognormal errors									
50	-0.3234	0.8164	.0554	.0186	.0030	-0.1856	0.9603	.0929	.0442	.0066
	-0.4302	0.9518	.1121	.0516	.0053	-0.2086	1.0116	.1107	.0503	.0069
	-0.0469	1.0001	.0988	.0447	.0057	-0.0293	1.0256	.1064	.0490	.0072
100	-0.2630	0.8978	.0716	.0324	.0055	-0.1404	0.9737	.0925	.0442	.0077
	-0.3345	0.9764	.1069	.0519	.0081	-0.1694	0.9988	.1022	.0488	.0079
	-0.0424	0.9938	.0966	.0432	.0068	-0.0039	1.0052	.0978	.0462	.0077
200	-0.2446	0.9243	.0814	.0375	.0063	-0.1699	0.9667	.0930	.0432	.0075
	-0.3003	0.9917	.1081	.0561	.0100	-0.1768	0.9834	.0964	.0466	.0068
	-0.0606	1.0058	.1000	.0480	.0088	-0.0216	0.9914	.0952	.0445	.0073
500	-0.1225	0.9450	.0836	.0393	.0069	-0.0776	0.9941	.0972	.0475	.0092
	-0.1650	0.9853	.0982	.0457	.0083	-0.0721	0.9968	.0993	.0474	.0082
	-0.0066	0.9921	.0968	.0465	.0075	0.0268	1.0020	.1003	.0464	.0079
1000	-0.0902	0.9596	.0868	.0398	.0080	-0.0622	0.9901	.0955	.0487	.0091
	-0.1186	1.0044	.1015	.0520	.0103	-0.0650	0.9938	.0974	.0468	.0079
	-0.0003	1.0079	.1011	.0496	.0105	0.0008	0.9955	.0986	.0482	.0080

Note: Three rows under each n: $\mathrm{LM}_{\mathtt{SAR}}^{\mathtt{FE}}$, $\mathrm{LM}_{\mathtt{SAR}}^{\mathtt{FEOPG}}$ and $\mathrm{SLM}_{\mathtt{SAR}}^{\mathtt{FEOPG}}$; W_{1n} = `Group`, $g = n^{0.5}$; `XVal-B`.

Table 6. Monte Carlo results: LM Tests for Fixed Effects Panel SED Model, $T = 3$

	Heteroskedasticity $\propto$ group size					Heteroskedasticity $= 1$				
n	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	-0.3231	0.8613	.0524	.0173	.0043	-0.4076	0.9258	.0926	.0345	.0041
	-0.4803	1.0012	.1406	.0717	.0136	-0.5256	1.0103	.1499	.0816	.0170
	-0.1354	1.0579	.1224	.0632	.0126	-0.0887	1.0539	.1225	.0597	.0110
100	-0.2876	0.9175	.0773	.0295	.0048	-0.3306	0.9483	.0937	.0380	.0046
	-0.4094	1.0044	.1318	.0705	.0132	-0.4327	1.0129	.1378	.0732	.0146
	-0.1034	1.0239	.1120	.0580	.0100	-0.0874	1.0379	.1154	.0572	.0102
200	-0.2709	0.9169	.0739	.0285	.0052	-0.2827	0.9548	.0927	.0390	.0066
	-0.3835	0.9935	.1229	.0629	.0137	-0.3716	1.0051	.1273	.0676	.0147
	-0.0987	1.0152	.1073	.0548	.0100	-0.0668	1.0194	.1073	.0542	.0106
500	-0.2300	0.9333	.0790	.0334	.0063	-0.2451	0.9818	.1022	.0471	.0089
	-0.3352	1.0073	.1213	.0606	.0142	-0.3171	1.0163	.1229	.0654	.0156
	-0.0984	1.0155	.1067	.0542	.0105	-0.0773	1.0243	.1101	.0559	.0126
1000	-0.2367	0.9328	.0823	.0349	.0062	-0.1864	0.9716	.0978	.0447	.0077
	-0.3250	1.0003	.1168	.0608	.0145	-0.2437	0.9941	.1092	.0585	.0114
	-0.0891	1.0078	.1024	.0524	.0119	-0.0627	0.9999	.1015	.0517	.0096
	`Normal Mixture`									
50	-0.3185	0.8280	.0442	.0146	.0034	-0.4324	0.8930	.0913	.0321	.0024
	-0.4623	0.9745	.1248	.0608	.0086	-0.5414	0.9841	.1448	.0752	.0143
	-0.1098	1.0331	.1134	.0558	.0076	-0.0952	1.0373	.1141	.0547	.0088
100	-0.2767	0.9077	.0705	.0261	.0053	-0.3434	0.9299	.0878	.0369	.0060
	-0.3910	0.9956	.1272	.0624	.0113	-0.4399	0.9979	.1316	.0685	.0137
	-0.0790	1.0188	.1051	.0523	.0082	-0.0888	1.0267	.1081	.0548	.0105
200	-0.2822	0.9041	.0740	.0265	.0047	-0.2984	0.9253	.0813	.0345	.0054
	-0.3898	0.9922	.1211	.0613	.0129	-0.3816	0.9788	.1189	.0596	.0103
	-0.1015	1.0180	.1092	.0540	.0099	-0.0733	0.9943	.0975	.0457	.0090
500	-0.2451	0.9134	.0743	.0275	.0049	-0.2293	0.9686	.0942	.0431	.0064
	-0.3471	0.9970	.1170	.0597	.0133	-0.2980	1.0024	.1161	.0580	.0112
	-0.1091	1.0068	.1033	.0503	.0097	-0.0574	1.0110	.1052	.0492	.0094
1000	-0.2318	0.9306	.0814	.0319	.0057	-0.1797	0.9751	.0953	.0434	.0083
	-0.3199	1.0017	.1189	.0615	.0140	-0.2360	0.9952	.1100	.0551	.0100
	-0.0838	1.0094	.1050	.0528	.0109	-0.0546	1.0010	.1029	.0493	.0096
	`Lognormal Errors`									
50	-0.3231	0.8057	.0382	.0131	.0039	-0.3989	0.8701	.0706	.0242	.0032
	-0.4800	0.9669	.1253	.0583	.0073	-0.5309	0.9812	.1410	.0666	.0105
	-0.1099	1.0207	.1058	.0474	.0071	-0.0607	1.0242	.1035	.0470	.0066
100	-0.2792	0.8806	.0607	.0245	.0055	-0.3250	0.9069	.0763	.0333	.0068
	-0.4103	0.9920	.1252	.0614	.0091	-0.4399	0.9887	.1281	.0590	.0115
	-0.0788	1.0141	.1031	.0490	.0070	-0.0709	1.0129	.1017	.0462	.0070
200	-0.2910	0.8975	.0653	.0259	.0063	-0.2939	0.9230	.0801	.0339	.0068
	-0.4155	0.9985	.1305	.0641	.0113	-0.3968	0.9944	.1215	.0618	.0126
	-0.1160	1.0176	.1072	.0501	.0082	-0.0785	1.0078	.1024	.0470	.0083
500	-0.2188	0.9046	.0684	.0286	.0052	-0.2354	0.9472	.0879	.0370	.0060
	-0.3245	0.9938	.1153	.0571	.0119	-0.3145	0.9945	.1128	.0565	.0109
	-0.0810	1.0048	.1030	.0512	.0089	-0.0627	1.0004	.0990	.0462	.0081
1000	-0.2181	0.9361	.0806	.0316	.0055	-0.2000	0.9766	.0960	.0457	.0076
	-0.3127	1.0109	.1184	.0585	.0129	-0.2668	1.0107	.1173	.0595	.0119
	-0.0739	1.0188	.1072	.0524	.0100	-0.0642	1.0158	.1071	.0534	.0098

Note: Three rows under each n: $\text{LM}_{\texttt{SED}}^{\texttt{FE}}$, $\text{LM}_{\texttt{SED}}^{\texttt{FEOPG}}$ and $\text{SLM}_{\texttt{SED}}^{\texttt{FEOPG}}$; W_{2n} = `Group`, $g = n^{0.5}$; `XVal-B`.

Table 7. Monte Carlo results: LM Tests for Fixed Effects Panel SARAR Model, $T = 3$

	Heteroskedasticity $\propto$ group size					Heteroskedasticity = 1				
n	mean	sd	10%	5%	1%	mean	sd	10%	5%	1%
	`Normal Errors`									
50	1.8475	1.7698	.0702	.0306	.0061	1.9882	1.8950	.0880	.0431	.0084
	2.2877	2.0462	.1310	.0620	.0100	2.2342	2.0594	.1236	.0624	.0104
	2.1617	1.9856	.1175	.0539	.0077	2.1093	1.9760	.1092	.0523	.0084
100	1.8967	1.8868	.0731	.0348	.0086	1.9887	1.8975	.0850	.0397	.0082
	2.2495	2.1328	.1310	.0661	.0124	2.2646	2.1610	.1286	.0654	.0136
	2.0986	2.0037	.1101	.0528	.0095	2.1072	2.0321	.1107	.0560	.0106
200	1.8844	1.8150	.0794	.0345	.0062	1.9774	1.9044	.0896	.0435	.0084
	2.2110	2.1588	.1236	.0628	.0130	2.1567	2.0882	.1170	.0620	.0117
	2.0704	2.0488	.1099	.0534	.0111	2.0467	1.9972	.1059	.0526	.0097
500	1.9370	1.9192	.0848	.0390	.0087	2.0093	2.0198	.0982	.0463	.0094
	2.1424	2.1107	.1222	.0613	.0114	2.1147	2.1101	.1144	.0576	.0126
	2.0377	2.0138	.1027	.0512	.0105	2.0492	2.0549	.1046	.0538	.0118
1000	1.9527	1.9511	.0907	.0444	.0090	1.9837	1.9384	.0952	.0434	.0086
	2.0930	2.0803	.1141	.0591	.0112	2.0706	2.0503	.1041	.0529	.0115
	2.0383	2.0335	.1065	.0532	.0107	2.0098	1.9949	.0999	.0491	.0108
	`Normal Mixture`									
50	1.7835	1.7222	.0626	.0268	.0059	1.9417	1.9156	.0851	.0398	.0094
	2.2488	1.9511	.1190	.0563	.0068	2.2105	1.9364	.1172	.0554	.0066
	2.1386	1.9122	.1071	.0511	.0067	2.0945	1.8742	.1034	.0475	.0052
100	1.8511	1.7889	.0697	.0341	.0069	1.9745	1.8478	.0859	.0374	.0071
	2.2567	2.0837	.1243	.0618	.0112	2.2528	2.0556	.1230	.0592	.0109
	2.0949	1.9784	.1095	.0486	.0089	2.0979	1.9381	.1061	.0492	.0074
200	1.8491	1.8272	.0767	.0348	.0070	1.9458	1.8929	.0867	.0386	.0082
	2.1792	2.1047	.1181	.0621	.0128	2.1271	2.0206	.1137	.0542	.0085
	2.0437	1.9938	.1048	.0530	.0086	2.0275	1.9425	.1012	.0458	.0081
500	1.8883	1.8336	.0791	.0362	.0073	1.9872	1.9464	.0945	.0453	.0083
	2.1018	2.0185	.1092	.0561	.0101	2.0992	2.0569	.1114	.0565	.0104
	2.0081	1.9430	.0998	.0492	.0076	2.0345	2.0052	.1029	.0532	.0090
1000	1.9304	1.9345	.0864	.0417	.0091	2.0028	2.0047	.0985	.0512	.0101
	2.0690	2.0586	.1039	.0540	.0125	2.0891	2.1085	.1114	.0575	.0122
	2.0211	2.0064	.1008	.0491	.0105	2.0373	2.0604	.1070	.0549	.0103
	`Lognormal Errors`									
50	1.6484	1.6401	.0499	.0246	.0054	1.8401	1.9910	.0724	.0346	.0089
	2.2424	1.9181	.1149	.0534	.0060	2.2157	1.8932	.1122	.0486	.0065
	2.0917	1.8562	.0996	.0447	.0053	2.0671	1.8043	.0956	.0398	.0052
100	1.7922	1.8153	.0688	.0321	.0074	1.8906	1.8987	.0797	.0385	.0081
	2.2755	2.0395	.1235	.0591	.0105	2.2403	2.0305	.1188	.0579	.0099
	2.0908	1.9104	.1002	.0467	.0076	2.0575	1.8992	.0988	.0484	.0076
200	1.7899	1.7512	.0690	.0307	.0061	1.9355	1.9223	.0874	.0407	.0092
	2.1999	2.0088	.1174	.0571	.0094	2.1670	1.9633	.1133	.0531	.0075
	2.0485	1.9124	.1017	.0489	.0069	2.0503	1.8708	.1017	.0446	.0048
500	1.8536	1.9127	.0785	.0357	.0092	1.9202	1.8952	.0838	.0384	.0082
	2.1259	2.0422	.1127	.0553	.0108	2.0790	1.9645	.1002	.0508	.0084
	2.0156	1.9389	.0998	.0473	.0086	2.0117	1.9100	.0939	.0462	.0080
1000	1.9047	1.9584	.0856	.0436	.0089	1.9925	2.0059	.0999	.0480	.0093
	2.0683	1.9870	.1072	.0489	.0096	2.1012	2.0611	.1115	.0559	.0118
	2.0159	1.9403	.1010	.0465	.0079	2.0424	2.0051	.1036	.0512	.0096

Note: $\text{LM}^{\texttt{FE}}_{\texttt{SARAR}}$, $\text{LM}^{\texttt{FEOPG}}_{\texttt{SARAR}}$ and $\text{SLM}^{\texttt{FEOPG}}_{\texttt{SARAR}}$; W_{1n}=`Queen`, $r = 5$; W_{2n}=`Group,` $g = n^{0.5}$; `XVal-B`.