11-2011

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SMU ECONOMICS & STATISTICS
WORKING PAPER SERIES

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November 2011

Paper No. 16-2011

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November 18, 2011

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Abstract

This paper develops a double asymptotic limit theory for the persistent parameter (θ) in an explosive continuous time model with a large number of time span (N) and a small number of sampling interval (h). The limit theory allows for the joint limits where $N \to \infty$ and $h \to 0$ simultaneously, the sequential limits where $N \to \infty$ is followed by $h \to 0$, and the sequential limits where $h \to 0$ is followed by $N \to \infty$. All three asymptotic distributions are the same. The initial condition, either fixed or random, appears in the limiting distribution. The simultaneous double asymptotic theory is derived by using results recently obtained in Phillips and Magdalinos (2007) for the mildly explosive discrete time model and so a invariance principle applies. However, our asymptotic distribution is different from what was reported in Perron (1991, *Econometrica*) where the sequential limits, $h \to 0$ followed by $N \to \infty$, were considered. It is shown that the limit theory in Perron is not correct and the correct sequential asymptotic distribution is identical to the simultaneous double asymptotic distribution.

1 Introduction

There has been a long-standing interest in statistics for developing the asymptotic theory for explosive processes. Two of the earliest studies are White (1958) and Anderson (1959) where the asymptotic distribution of the autoregressive (AR) coefficient was derived when the root is larger than unity. Phillips and Magdalinos (2007, PM hereafter) has provided an asymptotic theory and an invariance principle for mildly explosive processes where the root is moderately deviated from unity. In economics, there has recently been a growing interest on using explosive processes to model asset price bubbles. Phillips et al (2011) has developed a recursive method to detect bubbles in the discrete time AR model. Phillips and Yu (2011) applied the method to analyze the bubble episodes in various markets in the U.S. and documented the bubble migration mechanism during the subprime crisis.

All the above cited studies focus on discrete time models. Explosive behavior can also be described using continuous time models. Let T, h, N be the sample size, the sampling interval, and the time span of the data, respectively. Obviously $T = N/h$. While the asymptotic theory in discrete time models always corresponds to the scheme of $T \to \infty$, how to develop the asymptotic theory in continuous time is less a clear cut because $T \to \infty$ is achievable from different ways. In the literature, three alternative sampling schemes have been discussed (see, for example, Jeong and Park (2011) and Zhou and Yu (2011)), namely:

$$N \to \infty, \quad h \text{ is fixed}; \qquad (A1)$$

$$N \to \infty, \quad h \to 0; \qquad (A2)$$

$$h \to 0, \quad N \text{ is fixed}. \qquad (A3)$$

The purpose of the present paper is to develop the double asymptotic theory under scheme (A2) for an explosive continuous time model. In particular, three alternative double asymptotics are considered. In the first case, $N \to \infty$ and $h \to 0$ simultaneously. In the second case, a sequential asymptotic treatment is considered, i.e., $N \to \infty$ is followed by $h \to 0$. In the third case, another sequential asymptotic treatment is considered wherein, $h \to 0$ is followed by $N \to \infty$. We show that the asymptotic distributions under these three treatments are the same.

Different from PM, in our double asymptotic distribution, the initial condition, either fixed or random, appears in the limiting distribution. Interestingly, our limiting distribution is different from what was developed in Perron (1991) under the sequential limits, $h \to 0$ followed by $N \to \infty$. It is shown that the limit theory in Perron is not correct and his distribution can be very different from ours for a general initial condition.

The paper is organized as follows. Section 2 develops the double asymptotic distribution of the persistent parameter in an explosive continuous time model with $N \to \infty$ and $h \to 0$ simultaneously. Section 3 develops the sequential asymptotic distribution where $N \to \infty$ is followed by $h \to 0$. Section 4 develops the sequential asymptotic distribution where $h \to 0$ is followed by $N \to \infty$ and shows what goes wrong in the derivations of Perron. Section 5 concludes. Appendix collects the proof of the theoretical results.

2 Simultaneous Double Asymptotics

Consider the following continuous time Ornstein-Uhlenbeck (OU) process:

$$dy(t) = \theta y(t)dt + \sigma dW(t),\ y(0) = y_0, \tag{1}$$

where $W(t)$ is a standard Brownian motion. When $\theta < 0$, the process is strictly stationary and $-\theta$ captures the speed of mean reversion. When $\theta \geq 0$, the process becomes nonstationary. If $\theta > 0$, the process is explosive. $y(t)$ is assumed to be observed at discrete points in time, say $t = 0, h, 2h, \ldots, Th$.

The exact discrete time model corresponding to (1) is

$$y_{th} = a_h(\theta)y_{(t-1)h} + \sigma\sqrt{\frac{e^{2\theta h}-1}{2\theta}}\varepsilon_t,\ y_{0h} = y_0, \tag{2}$$

where $a_h(\theta) = e^{\theta h}$, $\varepsilon_t \overset{iid}{\sim} N(0,1)$. In the paper, we focus our analysis on the explosive case $\theta > 0$ which means that $a_h(\theta) > 1$. The initial value, y_0, may be a random variable, whose distribution is fixed and independent of the sampling interval h, or a constant. We simply write y_{th} as y_t when it causes no confusion. The least squares (LS) estimators of $a_h(\theta)$ and θ are, respectively,

$$\widehat{a}_h(\theta) = \frac{\sum_{t=1}^{T} y_{t-1}y_t}{\sum_{t=1}^{T} y_{t-1}^2},$$

and

$$\widehat{\theta} = \frac{1}{h}\ln\left(\widehat{a}_h(\theta)\right). \tag{3}$$

Letting $\lambda(h) = \sigma\sqrt{\frac{e^{2\theta h}-1}{2\theta}}$ which has the order $O(\sqrt{h})$, $x_t = y_t/\lambda(h)$, $x_0 = y_0/\lambda(h)$, and dividing both sides of Model (2) by $\lambda(h)$, we get the following explosive AR(1) model

$$x_t = a_h(\theta)x_{t-1} + \varepsilon_t = e^{\theta h}x_{t-1} + \varepsilon_t,\ x_0 = y_0/\lambda(h). \tag{4}$$

This model compares to Model (1) in PM,

$$x_t = \left(1 + \frac{\theta}{k_T}\right)x_{t-1} + \varepsilon_t,\ x_0 = o_p\left(\sqrt{k_T}\right),\ k_T \to \infty,\ \frac{k_T}{T} \to 0. \tag{5}$$

Let $k_T = 1/h$ so that Model (4) may be written as

$$x_t = a_h(\theta)x_{t-1} + \varepsilon_t = e^{\theta/k_T}x_{t-1} + \varepsilon_t,\quad x_0 = y_0/\lambda(h) = O_p\left(\sqrt{k_T}\right), \tag{6}$$

and, hence, $\widehat{a}_h(\theta)$ and $\widehat{\theta}$ can also be obtained from x_t. The double asymptotics, $h \to 0$ and $N \to \infty$, implies that

$$k_T = \frac{1}{h} \to \infty \quad \text{and} \quad \frac{k_T}{T} = \frac{1}{N} \to 0. \tag{7}$$

Model (6) is very similar to Model (1) of PM with two subtle differences. First, the AR coefficient in (6) is e^{θ/k_T} whereas it is $1 + \theta/k_T$ in PM. This difference is small since $e^{\theta/k_T} = 1 + \theta/k_T + O(k_T^{-2})$ and, not surprisingly, it has no impact on the limiting distribution. Second, the initial condition in (6) is $x_0 \sim O_p\left(\sqrt{k_T}\right)$, whereas it is assumed to be $o_p\left(\sqrt{k_T}\right)$ in PM.

In Model (1) of PM, the root, $1 + \theta/k_T$, represents moderate deviations from unity in the sense that it is in a larger neighborhood of one than the conventional local to unity root, $1 + \theta/T$. Therefore, under the double asymptotics the root in Model (6) is also moderately deviated from unity. With a different initial condition, our analysis can be regarded as an extension to PM from $x_0 \sim o_p\left(\sqrt{k_T}\right)$ to $x_0 \sim O_p\left(\sqrt{k_T}\right)$. It turns out this change of the order of magnitude in the initial condition leads to a change in the limiting distribution of the LS estimators, $\widehat{a}_h(\theta)$ and $\widehat{\theta}$.

Let

$$X_T = \frac{1}{\sqrt{k_T}}\sum_{t=1}^{T}\left(a_h(\theta)\right)^{-(T-t)-1}\varepsilon_t, \tag{8}$$

$$Y_T = \frac{1}{\sqrt{k_T}} \sum_{t=1}^{T} (a_h(\theta))^{-t} \varepsilon_t. \tag{9}$$

With a slight change of the notation, using the proof of PM, we can obtain the following lemma.

Lemma 1 *Let $a_h(\theta) = e^{\theta/k_T}$, $k_T = 1/h$, $T = N/h$ $\varepsilon_t \overset{iid}{\sim} N(0,1)$. When $h \to 0$ and $N \to \infty$, we have*

(a)

$$(a_h(\theta))^{-T} = o\left(\frac{k_T}{T}\right) = o\left(\frac{1}{N}\right); \tag{10}$$

(b)

$$\frac{(a_h(\theta))^{-T}}{k_T} \sum_{t=1}^{T} \sum_{j=t}^{T} (a_h(\theta))^{t-j-1} \varepsilon_j \varepsilon_t \overset{L_1}{\longrightarrow} 0; \tag{11}$$

(c)

$$(X_T, Y_T) \implies (X, Y), \tag{12}$$

where X and Y are independent $N\left(0, \frac{1}{2\theta}\right)$ random variables.

Note that $(a_h(\theta))^{-T} = e^{-\theta h T} = e^{-\theta N}$ and that $x_0/\sqrt{k_T} \overset{L_1}{\longrightarrow} y_0/\sigma$. Theorem 1 reports the double asymptotic distribution of $\widehat{a}_h(\theta)$ with $h \to 0$ and $N \to \infty$ simultaneously.

Theorem 2 *Let $a_h(\theta) = e^{\theta h}$, $\widehat{a}_h(\theta)$ be the LS estimator obtained from x_t, $\widehat{\theta} = (1/h)\ln(\widehat{a}_h(\theta))$. Under the simultaneous double asymptotics, we have*

(a)

$$he^{-\theta N} \sum_{t=1}^{T} x_{t-1}\varepsilon_t \Rightarrow \frac{1}{2\theta}\xi\,[\eta + d]; \tag{13}$$

(b)

$$2\theta h^2 e^{-2\theta N} \sum_{t=1}^{T} x_{t-1}^2 \Rightarrow \frac{1}{2\theta}[\eta + d]^2; \tag{14}$$

(c)

$$\frac{e^{\theta N}}{2\theta h}(\widehat{a}_h(\theta) - a_h(\theta)) \Rightarrow \frac{\xi}{\eta + d}; \tag{15}$$

(d)

$$\frac{e^{\theta N}}{2\theta}\left(\widehat{\theta} - \theta\right) \Rightarrow \frac{\xi}{\eta + d}, \tag{16}$$

where $\xi = \sqrt{2\theta}X$, $\eta = \sqrt{2\theta}Y$ are independent $N(0,1)$ random variables with (X,Y) defined in Lemma 1 and $d = y_0\sqrt{2\theta}/\sigma$.

Remark 1 *To facilitate the comparison of our results with those of PM, we may rewrite the limit theory in (15) as*

$$\frac{(a_h(\theta))^T k_T}{2\theta}(\widehat{a}_h(\theta)-a_h(\theta)) \Rightarrow \frac{X}{Y+y_0/\sigma}. \tag{17}$$

When $y_0 = 0$, the limiting distribution is Cauchy and the same as in PM. Since the finite sample distribution always depends on the initial value in continuous time models, we expect that the double asymptotic distribution in (17) is a better approximation than the Cauchy distribution when y_0 is different from 0.

Remark 2 *Since our asymptotic treatments follow those of PM for moderate deviations, it is straightforward to show that the results in Lemma 1 and Theorem 2 continue to hold for non-Gaussian errors, such as the explosive Levy processes. This observation suggests that although the invariance principle does not cover the discrete time explosive model it covers the continuous time explosive model under the simultaneous double asymptotics.*

3 Sequential Asymptotics: $N \to \infty$ followed by $h \to 0$

When h is fixed, the discrete time model (6) is an explosive AR(1) model with Gaussian errors. Letting $N \to \infty$, Anderson (1959) showed that

$$\frac{(a_h(\theta))^T[\widehat{a}_h(\theta)-a_h(\theta)]}{(a_h(\theta))^2-1} \Longrightarrow \frac{Y_a}{Z_a+a_h(\theta)x_0} \overset{d}{=} \frac{N\left(0,1/\left[1-(a_h(\theta))^{-2}\right]\right)}{N\left(0,1/\left[1-(a_h(\theta))^{-2}\right]\right)+a_h(\theta)x_0},$$

where Y_a and Z_a are independent. The proof was done under the condition that x_0 is a constant, but it still holds when $x_0 \sim O_p(1)$. It is straightforward to show that

$$\frac{Y_a}{Z_a+a_h(\theta)x_0} \overset{d}{=} \frac{N(0,1)}{N(0,1)+x_0\sqrt{(a_h(\theta))^2-1}} \overset{d}{=} \frac{\xi}{\eta+d},$$

because $d = y_0\sqrt{2\theta}/\sigma$, and

$$x_0 = \frac{y_0}{\lambda(h)} = \frac{y_0}{\sigma}\sqrt{\frac{2\theta}{(a_h(\theta))^2-1}}.$$

Letting $h \to 0$, the sequential limiting distribution is

$$\lim_{h\to 0}\lim_{N\to\infty}\frac{\exp\{\theta N\}\left[\widehat{a}_h(\theta)-a_h(\theta)\right]}{2\theta h}=\lim_{h\to 0}\frac{(a_h(\theta))^2-1}{2\theta h}\frac{Y_a}{Z_a+a_h(\theta)x_0}\overset{d}{=}\frac{\xi}{\eta+d},$$

which is the same as the double asymptotic distribution derived in Section 2. We now collect these results together in the following theorem.

Theorem 3 *Let $a_h(\theta)=e^{\theta h}$, $\widehat{a}_h(\theta)$ be the LS estimator obtained from x_t, $\widehat{\theta}=(1/h)\ln(\widehat{a}_h(\theta))$, we have*

(a)

$$\lim_{h\to 0}\lim_{N\to\infty}\frac{\exp\{\theta N\}\left[\widehat{a}_h(\theta)-a_h(\theta)\right]}{2\theta h}\overset{d}{=}\frac{\xi}{\eta+d};$$

(b)

$$\lim_{h\to 0}\lim_{N\to\infty}\frac{\exp\{\theta N\}}{2\theta}\left(\widehat{\theta}-\theta\right)\overset{d}{=}\frac{\xi}{\eta+d},$$

where ξ, η are independent $N(0,1)$ random variables and $d=y_0\sqrt{2\theta}/\sigma$.

Remark 3 *Although the sequential asymptotic theory developed here is the same as the asymptotic distribution derived in Section 2, the advantage of using the setup of PM is clear, namely, the applicability of the invariance principle is easy to establish from PM but not from Anderson.*

4 Sequential Asymptotics: $h \to 0$ followed by $N \to \infty$

Perron (1991) derived a sequential limiting distribution for the LS estimator of the persistent parameter θ in the explosive OU process; see Corollary 1 (ii) on Page 217 and the corresponding proof on Page 234 in Perron (1991). The sequential asymptotics first requires $h \to 0$ and then $N \to \infty$. To our surprise, however, his sequential limiting distribution is different from the limiting distributions that we obtained in Section 2 and Section 3. It is important to find the reasons that cause this discrepancy. In this section we investigate the double asymptotic theory under the sequential limits where $h \to 0$ is followed by $N \to \infty$.

The continuous time OU process considered in Perron is given in (1) where the initial condition is assumed to be constant, $y_0 = b$. First, by letting time interval h goes to zero with fixed time span N, Perron developed the in-fill asymptotics for $\widehat{a}_h(\theta)$,

$$T\left(\widehat{a}_h(\theta) - a_h(\theta)\right) \Rightarrow \frac{A(\gamma, c)}{B(\gamma, c)}, \tag{18}$$

where

$$A(\gamma, c) = \gamma \int_0^1 \exp\{cr\} \, dW(r) + \int_0^1 J_c(r) \, dW(r), \tag{19}$$

$$B(\gamma, c) = \gamma^2 \frac{\exp\{2c\} - 1}{2c} + 2\gamma \int_0^1 \exp\{cr\} J_c(r) \, dr + \int_0^1 J_c(r)^2 \, dr, \tag{20}$$

and $J_c(r) = \int_0^1 \exp\{c(r-s)\} \, dW(s)$ is generated by the stochastic differential equation

$$dJ_c(r) = cJ_c(r) \, dr + dW(r),$$

with the initial condition $J_c(0) = 0$, $c = \theta N$, $\gamma = b/\left(\sigma\sqrt{N}\right)$.

To derive the sequential limiting distribution, he then let $N \to \infty$, namely, $c \to +\infty$, and showed that (see (vi) and (viii) of Lemma A.2 in his paper)

$$(2c)^{3/2} e^{-2c} \int_0^1 \exp\{cr\} J_c(r) \, dr \Rightarrow N(0,1),$$

and

$$(2c)^{1/2} e^{-c} \int_0^1 \exp\{cr\} \, dW(r) \Rightarrow N(0,1).$$

Then he argued, without a proof, that these two limiting distributions are identical (call it η). Based on this argument and the two results in Phillips (1987), Perron obtained the limiting distributions of $A(\gamma, c)$ and $B(\gamma, c)$, and the sequential limiting distribution for $\widehat{a}_h(\theta)$ and $\widehat{\theta}$,

$$\lim_{N\to\infty} \lim_{h\to 0} \frac{e^{\theta N}\left(\widehat{a}_h(\theta) - a_h(\theta)\right)}{2\theta h} = \lim_{c\to\infty} \frac{(2c) e^{-c} A(\gamma, c)}{(2c)^2 e^{-2c} B(\gamma, c)} = \frac{d\eta + \xi\eta}{[d+\eta]^2}, \tag{21}$$

$$\lim_{N\to\infty} \lim_{h\to 0} \frac{e^{\theta N}\left(\widehat{\theta} - \theta\right)}{2\theta} = \frac{d\eta + \xi\eta}{[d+\eta]^2}, \tag{22}$$

where ξ and η are independent $N(0,1)$ variates, $d = y_0\sqrt{2\theta}/\sigma = b\sqrt{2\theta}/\sigma$.

The limiting distribution in (21) (or (22)) is different from that in (15) (or (16)) unless $y_0 = b = 0$ where the two limiting distributions become the Cauchy distribution. In this section we will show that the limiting distributions of $(2c)^{3/2} e^{-2c} \int_0^1 \exp\{cr\} J_c(r)\, dr$ and $(2c)^{1/2} e^{-c} \int_0^1 \exp\{cr\}\, dW(r)$ are not identical and hence, his sequential limiting distribution is not correct. Instead, the two limiting distributions are independent. The correct sequential limiting distribution turns out to be identical to the simultaneous double asymptotic distribution developed in Section 2.

Let us start the investigation from the joint moment generating function (MGF) of $A(\gamma, c)$ and $B(\gamma, c)$ given by Perron. Firstly, we derive the limiting joint MGF in Theorem 4, from which we obtain the sequential limiting distribution. Secondly, in Theorem 5, we give the correct limiting distributions of $(2c)^{3/2} e^{-2c} \int_0^1 \exp\{cr\} J_c(r)\, dr$ and $(2c)^{1/2} e^{-c} \int_0^1 \exp\{cr\}\, dW(r)$ and show that they are actually independent.

Theorem 4 *Let $d = b\sqrt{2\theta}/\sigma$, $\gamma = b/\left(\sigma\sqrt{N}\right)$. When $N \to +\infty$, $c = \theta N \to +\infty$,we have*

(a) The limiting joint MGF of $(2c)\, e^{-c} A(\gamma, c)$ and $(2c)^2 e^{-2c} B(\gamma, c)$ is

$$\begin{aligned} \lim_{c \to +\infty} M(\widetilde{v}, \widetilde{u}) &= \lim_{c \to +\infty} E\left[\exp\left(\widetilde{v}\,(2c)\, e^{-c} A(\gamma, c) + \widetilde{u}\,(2c)^2 e^{-2c} B(\gamma, c)\right)\right] \\ &= \frac{1}{\{1 - 2\widetilde{u} - \widetilde{v}^2\}^{1/2}} \exp\left\{\frac{d^2\left[2\widetilde{u} + \widetilde{v}^2\right]}{2\left(1 - 2\widetilde{u} - \widetilde{v}^2\right)}\right\}. \end{aligned}$$

(b) Letting ξ and η be independent $N(0,1)$ random variables, then

$$\left((2c)\, e^{-c} A(\gamma, c), (2c)^2 e^{-2c} B(\gamma, c)\right) \Longrightarrow \left(\xi\,[d + \eta], [d + \eta]^2\right).$$

(c)

$$\lim_{N \to \infty} \lim_{h \to 0} \frac{e^{\theta N}\left(\widehat{a}_h(\theta) - a_h(\theta)\right)}{2\theta h} = \lim_{c \to \infty} \frac{(2c)\, e^{-c} A(\gamma, c)}{(2c)^2 e^{-2c} B(\gamma, c)} = \frac{\xi\,[d + \eta]}{[d + \eta]^2} = \frac{\xi}{d + \eta},$$

$$\lim_{N \to \infty} \lim_{h \to 0} \frac{e^{\theta N}\left(\widehat{\theta} - \theta\right)}{2\theta} = \frac{\xi}{d + \eta}.$$

Remark 4 *The new sequential limiting distribution wherein $h \to 0$ is followed by $N \to \infty$ is the same as the simultaneous double asymptotic distribution derived in Section 2 and the sequential limiting distribution wherein $N \to \infty$ is followed by $h \to 0$ derived in Section 3.*

Remark 5 *Anderson (1959) proved that, when the error term in the explosive AR(1) model is independent over time and the initial condition is a constant, the limit distribution for the LS estimator should be a ratio of two independent random variables. Our* *new sequential limiting distributions reported in Theorem 4 and Theorem 3, and the double asymptotic distribution are consistent with this result. However, the asymptotic distribution developed in Perron (1991) is at odds with Anderson's result.*

Remark 6 *It is easy to show that the joint MGF of $d\eta + \xi\eta$ and $[d+\eta]^2$ is*

$$\frac{1}{\left\{1-2\widetilde{u}-\widetilde{v}^2\right\}^{1/2}} \exp\left\{\frac{d^2\left[2\widetilde{u}-2\widetilde{u}\widetilde{v}^2+4\widetilde{u}\widetilde{v}+\widetilde{v}^2\right]}{2\left(1-2\widetilde{u}-\widetilde{v}^2\right)}\right\}, \tag{23}$$

which is different from the limiting joint MGF of $(2c)\,e^{-c}A(\gamma,c)$ and $(2c)^2\,e^{-2c}B(\gamma,c)$. This supports the conclusion that the limiting distribution developed in Corollary 1 in Perron (1991) is not correct.

Theorem 5 *Let $J_c(r) = \int_0^1 \exp\{c(r-s)\}\,dW(s)$, ξ and η be independent $N(0,1)$ variates, $N \to +\infty$, and $c = \theta N \to +\infty$. Then*

(a)

$$(2c)^2\,e^{-2c}\int_0^1 J_c(r)^2\,dr \Rightarrow \eta^2; \tag{24}$$

(b)

$$(2c)\,e^{-c}\int_0^1 J_c(r)\,dW(r) \Rightarrow \xi\eta; \tag{25}$$

(c)

$$\left\{(2c)^{3/2}\,e^{-2c}\int_0^1 \exp\{cr\}\,J_c(r)\,dr\right\}^2 \Rightarrow \eta^2; \tag{26}$$

(d)

$$(2c)^{1/2}\,e^{-c}\int_0^1 \exp\{cr\}\,dW(r) \Rightarrow \xi. \tag{27}$$

Comparing results in Theorem 5 to those in (ii), (iv), (vi) and (viii) of Lemma A.2 in Perron, we notice that we disagree on the limit of $(2c)^{1/2}\,e^{-c}\int_0^1 \exp\{cr\}\,dW(r)$. While Perron argued the limit is the same as that of $(2c)^{3/2}\,e^{-2c}\int_0^1 \exp\{cr\}\,J_c(r)\,dr$, they should be independent. This difference is the source of the discrepancy in the sequential limits. Interestingly, the sequential limiting distribution holds true even when y_0 is a random variable. This is reported in the following Corollary. Therefore, the simultaneous and sequential double asymptotic distributions are identical to each other no matter what the initial condition is, fixed or random.

d	Limit Theory	0.5%	1%	2.5%	5%	10%	90%	95%	97.5%	99%	99.5%
$N(0,1)$	Correct	-45.0	-22.5	-9.0	-4.5	-2.2	2.2	4.5	9.0	22.5	45.0
	Perron	-3583	-895	-143	-35.4	-8.6	1.0	2.6	8.8	51.9	205
$N(0,4)$	Correct	-28.5	-14.2	-5.7	-2.8	-1.4	1.4	2.8	5.7	14.2	28.5
	Perron	-2280	-570	-91.0	-22.6	-5.5	0.5	1.1	3.3	18.0	70.3
$N(0,1/4)$	Correct	-56.9	-28.5	-11.4	-5.6	-2.8	2.8	5.6	11.4	28.5	56.9
	Perron	-2470	-617	-99.0	-24.9	-6.4	1.7	4.7	16.0	94.3	374
$N(5,1)$	Correct	-0.75	-0.62	-0.47	-0.37	-0.28	0.28	0.37	0.47	0.62	0.75
	Perron	-3.82	-2.36	-1.30	-0.82	-0.49	0.16	0.20	0.23	0.26	0.29
$N(5,0)$	Correct	-0.60	-0.53	-0.43	-0.35	-0.27	0.27	0.35	0.43	0.53	0.60
	Perron	-2.25	-1.66	-1.07	-0.73	-0.46	0.16	0.19	0.22	0.24	0.26

Table 1: This table reports various percentiles of the two limiting distributions, (22) and (16), for five initial conditions. The last initial condition is simply a constant 5.

Corollary 6 *Let y_0 be any random variable whose distribution is fixed and independent of sampling interval h or be a constant. Then*

$$\lim_{N\to\infty}\lim_{h\to 0}\frac{e^{\theta N}\left(\widehat{a}_h(\theta)-a_h(\theta)\right)}{2\theta h}\overset{d}{=}\lim_{c\to\infty}\frac{(2c)\,e^{-c}A\left(\gamma,c\right)}{\left(2c\right)^2e^{-2c}B\left(\gamma,c\right)}\overset{d}{=}\frac{\xi\left[d+\eta\right]}{\left[d+\eta\right]^2}=\frac{\xi}{d+\eta},$$

$$\lim_{N\to\infty}\lim_{h\to 0}\frac{e^{\theta N}\left(\widehat{\theta}-\theta\right)}{2\theta}=\frac{\xi}{d+\eta}.$$

To understand the difference between the two limiting distributions, (22) and (16), in Table 1, we report the 0.5%, 1%, 2.5%, 5%, 10%, 90%, 95%, 97.5%, 99%, 99.5% percentiles of the two distributions for five different initial conditions. Several conclusions can be made. First, in all cases, the difference between two distributions are very substantial. For example when $d\sim N(0,1)$, the 99% confidence interval for Perron's distribution is more than 40 times wider than that for the distribution derived in the present paper. Second, the difference in the left tail is more substantial. Third, Perron's distribution is skewed to the left.

5 Conclusion

This paper develops the double asymptotic limit theory for the persistent parameter in an explosive continuous time model with a large number of time span (N) and a small number of sampling interval (h). The simultaneous limits and the alternative sequential limits have been considered. The three limiting distributions are identical and the expression works for both the fixed and the random initial condition. However, they are different from the sequential limit theory derived in Perron (1991). We have identified the source of the discrepancy. While Perron argued the limit of $(2c)^{1/2} e^{-c} \int_0^1 \exp\{cr\} dW(r)$ is the same as that of $(2c)^{3/2} e^{-2c} \int_0^1 \exp\{cr\} J_c(r) dr$, the two limits should actually be independently distributed.

Notations

$:=$	definition
$\overset{iid}{\sim}$	independent and identically distributed
$o_p(1)$	tends to zero in probability
$O_p(1)$	bounded in the limit in probability
$\overset{p}{\to}$	converge in probability
$\overset{L_p}{\to}$	converge in L_p norm
$\Rightarrow$	weak convergence
$\overset{d}{=}$	distributional equivalence

Appendix

Proof of Lemma 1 is similar to that in PM and is omitted.

Proof of Theorem 2:

(a) Denote $a_h(\theta) = a_h$ when there is no confusion. From Model (6) we get

$$x_t = a_h x_{t-1} + \varepsilon_t = \sum_{j=1}^{t} a_h^{t-j} \varepsilon_j + a_h^t x_0,$$

and

$$\begin{aligned}
\sum_{t=1}^{T} x_{t-1}\varepsilon_t &= \sum_{t=1}^{T}\left[\sum_{j=1}^{t-1} a_h^{t-j-1}\varepsilon_j + a_h^{t-1}x_0\right]\varepsilon_t \\
&= \sum_{t=1}^{T}\left[\sum_{j=1}^{t-1} a_h^{t-j-1}\varepsilon_j\right]\varepsilon_t + x_0\sum_{t=1}^{T} a_h^{t-1}\varepsilon_t \\
&= \sum_{t=1}^{T}\left[\sum_{j=1}^{T} a_h^{t-j-1}\varepsilon_j\right]\varepsilon_t - \sum_{t=1}^{T}\left[\sum_{j=t}^{T} a_h^{t-j-1}\varepsilon_j\right]\varepsilon_t + x_0\sum_{t=1}^{T} a_h^{t-1}\varepsilon_t \\
&= \left[\sum_{t=1}^{T} a_h^{t-1}\varepsilon_t\right]\left[\sum_{j=1}^{T} a_h^{-j}\varepsilon_j\right] - \sum_{t=1}^{T}\left[\sum_{j=t}^{T} a_h^{t-j-1}\varepsilon_j\right]\varepsilon_t + x_0\sum_{t=1}^{T} a_h^{t-1}\varepsilon_t.
\end{aligned}$$

From Lemma 1 (b), we have $\frac{a_h^{-T}}{k_T}\sum_{t=1}^{T}\left[\sum_{j=t}^{T} a_h^{t-j-1}\varepsilon_j\right]\varepsilon_t \xrightarrow{L_1} 0$, and hence,

$$\begin{aligned}
&\frac{a_h^{-T}}{k_T}\sum_{t=1}^{T} x_{t-1}\varepsilon_t \\
=\; &\left[\frac{1}{\sqrt{k_T}}\sum_{t=1}^{T} a_h^{-(T-t)-1}\varepsilon_t\right]\left[\frac{1}{\sqrt{k_T}}\sum_{j=1}^{T} a_h^{-j}\varepsilon_j\right] + \frac{x_0}{\sqrt{k_T}}\frac{1}{\sqrt{k_T}}\sum_{t=1}^{T} a_h^{-(T-t)-1}\varepsilon_t + o_p(1) \\
=\; &X_T Y_T + \frac{x_0}{\sqrt{k_T}}X_T + o_p(1) \\
\Rightarrow\; &XY + \frac{y_0}{\sigma}X \\
=\; &\frac{1}{2\theta}\left[\sqrt{2\theta}X\right]\left[\sqrt{2\theta}Y\right] + \frac{y_0}{\sigma\sqrt{2\theta}}\left[\sqrt{2\theta}X\right] = \frac{1}{2\theta}\xi\eta + \frac{y_0}{\sigma\sqrt{2\theta}}\xi \\
=\; &\frac{1}{2\theta}\xi\left[\eta + \frac{y_0\sqrt{2\theta}}{\sigma}\right] = \frac{1}{2\theta}\xi\left[\eta + d\right],
\end{aligned}$$

where by Lemma 1 (c) ξ and η are independent $N(0,1)$ variates. The convergence rate is $a_h^{-T}/k_T = h\left(e^{\theta h}\right)^{-T} = he^{\theta N}$.

(b) Since $x_t = a_h x_{t-1} + \varepsilon_t$, we get

$$x_t^2 = a_h^2 x_{t-1}^2 + 2a_h x_{t-1}\varepsilon_t + \varepsilon_t^2,$$

and

$$x_t^2 - x_{t-1}^2 = \left(a_h^2 - 1\right)x_{t-1}^2 + 2a_h x_{t-1}\varepsilon_t + \varepsilon_t^2.$$

Hence,

$$\left(a_h^2 - 1\right)\sum_{t=1}^{T} x_{t-1}^2 = \left(x_T^2 - x_0^2\right) - 2a_h\sum_{t=1}^{T} x_{t-1}\varepsilon_t - \sum_{t=1}^{T}\varepsilon_t^2,$$

and

$$k_T\left(a_h^2-1\right)\frac{a_h^{-2T}}{k_T^2}\sum_{t=1}^{T}x_{t-1}^2=\frac{a_h^{-2T}}{k_T}x_T^2-a_h^{-2T}\frac{x_0^2}{k_T}-2a_h^{-T+1}\frac{a_h^{-T}}{k_T}\sum_{t=1}^{T}x_{t-1}\varepsilon_t-\frac{Ta_h^{-2T}}{k_T}\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2.$$

$\frac{x_0^2}{k_T}\to\left(\frac{y_0}{\sigma}\right)^2$ implies that $a_h^{-2T}\frac{x_0^2}{k_T}\to 0$. The proof of Part (a) suggests that $\frac{a_h^{-T}}{k_T}\sum_{t=1}^{T}x_{t-1}\varepsilon_t$ is $O_p(1)$ because $a_h^{-T+1}\to 0$, and hence, the third term of the right side goes to 0. Since $\frac{Ta_h^{-2T}}{k_T}\to 0$ by Lemma 1 (a) and $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2\xrightarrow{p}1$ by the law of large numbers, the last term of the right side goes to 0. Hence,

$$\begin{aligned}
k_T\left(a_h^2-1\right)\frac{a_h^{-2T}}{k_T^2}\sum_{t=1}^{T}x_{t-1}^2 &= \frac{a_h^{-2T}}{k_T}x_T^2+o_p(1)\\
&= \frac{a_h^{-2T}}{k_T}\left(\sum_{j=1}^{T}a_h^{T-j}\varepsilon_j+a_h^Tx_0\right)^2+o_p(1)\\
&= \left(\frac{1}{\sqrt{k_T}}\sum_{j=1}^{T}a_h^{-j}\varepsilon_j+\frac{x_0}{\sqrt{k_T}}\right)^2+o_p(1)\\
&= \left(Y_T+\frac{x_0}{\sqrt{k_T}}\right)^2+o_p(1)\\
&\Rightarrow \left(Y+\frac{y_0}{\sigma}\right)^2\\
&= \frac{1}{2\theta}\left(\sqrt{2\theta}Y+\frac{y_0\sqrt{2\theta}}{\sigma}\right)^2=\frac{1}{2\theta}\left(\eta+d\right)^2.
\end{aligned}$$

Note that $k_T\left(a_h^2-1\right)=\left[e^{2\theta h}-1\right]/h\to 2\theta$ as $h\to 0$ and we get

$$2\theta\frac{a_h^{-2T}}{k_T^2}\sum_{t=1}^{T}x_{t-1}^2\Rightarrow\frac{1}{2\theta}\left(\eta+d\right)^2. \tag{28}$$

(c) This is an immediate consequence of (a) and (b).

(d) Since $\theta=(1/h)\ln\left(a_h\left(\theta\right)\right)$ and $\widehat{\theta}=(1/h)\ln\left(\widehat{a}_h(\theta)\right)$, by the mean value theorem,

$$h\left(\widehat{\theta}-\theta\right)=\ln\left(\widehat{a}_h(\theta)\right)-\ln\left(a_h(\theta)\right)=\frac{1}{\widetilde{a}_h(\theta)}\left(\widehat{a}_h(\theta)-a_h(\theta)\right) \tag{29}$$

for some $\widetilde{a}_h(\theta)$ whose value is between $\widehat{a}_h(\theta)$ and $a_h(\theta)$. The *Delta* method is not directly applicable since $a_h(\theta)$ is a not constant but a real sequence that goes to 1 as

$h \to 0$. However, if we can show $\widetilde{a}_h(\theta) \xrightarrow{p} 1$, we can obtain the limiting distribution for $\widehat{\theta}$. For any $\varepsilon > 0$,

$$\begin{aligned}
\Pr\{|\widetilde{a}_h(\theta) - 1| > \varepsilon\} &= \Pr\{|\widetilde{a}_h(\theta) - a_h(\theta) + a_h(\theta) - 1| > \varepsilon\} \\
&\leq \Pr\{|\widetilde{a}_h(\theta) - a_h(\theta)| + |a_h(\theta) - 1| > \varepsilon\} \\
&\leq \Pr\{|\widehat{a}_h(\theta) - a_h(\theta)| + |a_h(\theta) - 1| > \varepsilon\} \\
&\leq \frac{E\left([|\widehat{a}_h(\theta) - a_h(\theta)| + |a_h(\theta) - 1|]^2\right)}{\varepsilon^2} \\
&= \frac{E\left(|\widehat{a}_h(\theta) - a_h(\theta)|^2\right) + |a_h(\theta) - 1|^2 + 2\,|a_h(\theta) - 1|\,E\left(|\widehat{a}_h(\theta) - a_h(\theta)|\right)}{\varepsilon^2} \\
&\to 0,
\end{aligned}$$

where the first inequality is the triangular inequality, the second comes from the fact that $\widetilde{a}_h(\theta)$ is between $\widehat{a}_h(\theta)$ and $a_h(\theta)$, the third is Chebyshev's inequality. Hence, $\widetilde{a}_h(\theta) \xrightarrow{p} 1$ and

$$\frac{e^{\theta N}}{2\theta}\left(\widehat{\theta} - \theta\right) = \frac{e^{\theta N}}{2\theta h}\frac{1}{\widetilde{a}_h(\theta)}\left(\widehat{a}_h(\theta) - a_h(\theta)\right) \Rightarrow \frac{\xi}{\eta + d}. \tag{30}$$

Proof of Theorem 4:

(a) Letting $\gamma = b/\left(\sigma\sqrt{N}\right)$, Perron (1991) derived the joint MGF of $A(\gamma, c)$ and $B(\gamma, c)$ as

$$\begin{aligned}
& M(v, u) \\
= \; & E\left[\exp\left(vA(\gamma, c) + uB(\gamma, c)\right)\right] \\
= \; & \Psi_c(v, u) \exp\left\{-\left(\frac{\gamma^2}{2}\right)(v + c - \lambda)\left[1 - \exp(v + c + \lambda)\Psi_c^2(v, u)\right]\right\} \\
= \; & \underbrace{\Psi_c(v, u)}_{I}\underbrace{\exp\left\{-\left(\frac{\gamma^2}{2}\right)(v + c - \lambda)\right\}}_{II}\underbrace{\exp\left\{\left(\frac{\gamma^2}{2}\right)(v + c - \lambda)\exp(v + c + \lambda)\Psi_c^2(v, u)\right\}}_{III},
\end{aligned}$$

where

$$\lambda = \left(c^2 + 2cv - 2u\right)^{1/2},$$

$$\Psi_c(v, u) = \left[\frac{2\lambda \exp\{-(v + c)\}}{(\lambda + (v + c))\exp\{-\lambda\} + (\lambda - (v + c))\exp\{\lambda\}}\right]^{1/2}.$$

Let $v = \widetilde{v}(2c)e^{-c}$ and $u = \widetilde{u}(2c)^2 e^{-2c}$. The joint MGF of $(2c)\,e^{-c}A(\gamma, c)$ and $(2c)^2\,e^{-2c}B(\gamma, c)$ is

$$M(\widetilde{v}, \widetilde{u}) = E\left[\exp\left(\widetilde{v}(2c)e^{-c}A(\gamma, c) + \widetilde{u}(2c)^2 e^{-2c}B(\gamma, c)\right)\right].$$

We get

$$\begin{aligned}\lambda &= \left\{c^2+(2c)^2e^{-c}\widetilde{v}-2(2c)^2e^{-2c}\widetilde{u}\right\}^{1/2}\\ &= \left\{\left[c+(2c)e^{-c}\widetilde{v}-2(2c)e^{-2c}\widetilde{u}-(2c)\,e^{-2c}\widetilde{v}^2\right]^2+O\left(e^{-3c}\right)\right\}^{1/2}\\ &= c+(2c)e^{-c}\widetilde{v}-2(2c)e^{-2c}\widetilde{u}-(2c)\,e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right),\end{aligned}$$

$$\lambda+(v+c)=2c+2(2c)e^{-c}\widetilde{v}-2(2c)e^{-2c}\widetilde{u}-(2c)e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right),$$

$$\lambda-(v+c)=-2\,(2c)\,e^{-2c}\widetilde{u}-(2c)\,e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right),$$

$$e^{-\lambda}=e^{-c}-(2c)e^{-2c}\widetilde{v}+O\left(e^{-3c}\right),$$

and

$$(\lambda-(v+c))\,e^{\lambda}=-\,(2c)\,e^{-c}\left[2\widetilde{u}+\widetilde{v}^2\right]+O\left(e^{-2c}\right).$$

The denominator of $\Psi_c^2\,(v,u)$ is

$$(\lambda+(v+c))\,e^{-\lambda}+(\lambda-(v+c))\,e^{\lambda}=(2c)\,e^{-c}\left[1-2\widetilde{u}-\widetilde{v}^2\right]+O\left(e^{-2c}\right).$$

The numerator of $\Psi_c^2\,(v,u)$ is

$$2\lambda\exp\left\{-\,(v+c)\right\}=2\lambda\exp\left\{-(2c)e^{-c}\widetilde{v}-c\right\}=(2c)\,e^{-c}+O\left(e^{-2c}\right).$$

Hence,

$$I=\Psi_c\,(v,u)=\left\{\frac{(2c)\,e^{-c}+O\left(e^{-2c}\right)}{(2c)\,e^{-c}\left[1-2\widetilde{u}-\widetilde{v}^2\right]+O\left(e^{-2c}\right)}\right\}^{1/2}\to\left\{\frac{1}{1-2\widetilde{u}-\widetilde{v}^2}\right\}^{1/2}.$$

It is easy to show that $II\to 1$ because

$$-\left(\frac{\gamma^2}{2}\right)(v+c-\lambda)=\left(\frac{b^2\theta}{2\sigma^2c}\right)\left[-2\,(2c)\,e^{-2c}\widetilde{u}-(2c)\,e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right)\right]\to 0.$$

Since

$$\exp\left\{\lambda+v+c\right\}=e^{2c}\exp\left\{2(2c)e^{-c}\widetilde{v}-2(2c)e^{-2c}\widetilde{u}-(2c)e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right)\right\},$$

letting $d=b\sqrt{2\theta}/\sigma$, we get $(2c)\,\gamma^2=d^2$ and

$$\begin{aligned}&\left(\frac{\gamma^2}{2}\right)(v+c-\lambda)\exp\left\{\lambda+v+c\right\}\\ =\;&\frac{d^2}{2}\left[2\widetilde{u}+\widetilde{v}^2+O\left(e^{-c}\right)\right]\exp\left\{2(2c)e^{-c}\widetilde{v}-2(2c)e^{-2c}\widetilde{u}-(2c)e^{-2c}\widetilde{v}^2+O\left(e^{-3c}\right)\right\}\\ \to\;&\frac{d^2}{2}\left[2\widetilde{u}+\widetilde{v}^2\right].\end{aligned}$$

Therefore,

$$III \to \exp\left\{\frac{d^2\left[2\widetilde{u}+\widetilde{v}^2\right]}{2\left[1-2\widetilde{u}-\widetilde{v}^2\right]}\right\}.$$

The limiting behavior of I, II and III gives rise to the limiting joint MGF of $(2c)\, e^{-c}A(\gamma, c)$ and $(2c)^2\, e^{-2c}B(\gamma, c)$.

(b) Since ξ and η are independent $N(0,1)$ random variables and $d = b\sqrt{2\theta}/\sigma$ is a constant, we have

$$\begin{aligned} M(\widetilde{v}, \widetilde{u}) &= E\left\{\exp\left(\xi\left[d+\eta\right]\widetilde{v} + \left[d+\eta\right]^2\widetilde{u}\right)\right\} \\ &= E\left\{E\left[\exp\left(\xi\left[d+\eta\right]\widetilde{v} + \left[d+\eta\right]^2\widetilde{u}\right)\middle| \digamma_\xi\right]\right\} \\ &= E\left\{\exp\left(\left[d+\eta\right]^2\widetilde{u}\right)\exp\left(\frac{\left[d+\eta\right]^2\widetilde{v}^2}{2}\right)\right\} \\ &= \frac{1}{\left\{1-2\widetilde{u}-\widetilde{v}^2\right\}^{1/2}}\exp\left\{\frac{d^2\left[2\widetilde{u}+\widetilde{v}^2\right]}{2\left[1-2\widetilde{u}-\widetilde{v}^2\right]}\right\}. \end{aligned}$$

This is the joint MGF of $\xi\left[d+\eta\right]$ and $\left[d+\eta\right]^2$ and is equivalent to the result in (a).

(c) This is an immediate consequence of (b).

Proof of Theorem 5:

(a) and (b) are the classical results from Phillips (1987) and also identical to (ii) and (iv) of Lemma A.2 in Perron (1991).

(c) Note that $2c\int_0^1 \exp\{cr\}\, J_c(r)\, dr = e^c J_c(1) - \int_0^1 \exp\{cr\}\, dW(r)$ and hence,

$$\begin{aligned} &\left\{(2c)^{3/2}\, e^{-2c}\int_0^1 \exp\{cr\}\, J_c(r)\, dr\right\}^2 \\ =\ & (2c)\, e^{-4c}\left\{e^c J_c(1) - \int_0^1 \exp\{cr\}\, dW(r)\right\}^2 \\ =\ & (2c)\, e^{-2c}J_c(1)^2 + e^{-2c}\left[(2c)^{1/2}\, e^{-c}\int_0^1 \exp\{cr\}\, dW(r)\right]^2 \\ & -2e^{-c}\left[(2c)^{1/2}\, e^{-c}J_c(1)\right]\left[(2c)^{1/2}\, e^{-c}\int_0^1 \exp\{cr\}\, dW(r)\right]. \end{aligned}$$

By stochastic differentiation of $\left\{\int_0^r \exp\{-cs\}\, dW(s)\right\}^2$, we deduce the following useful relationship, as pointed out in Phillips (1987),

$$\{J_c(1)\}^2 = 1 + 2c\int_0^1 J_c(r)^2\, dr + 2\int_0^1 J_c(r)\, dW(r).$$

From (a) and (b), we get

$$\begin{aligned}
&\left\{(2c)^{1/2}\, e^{-c} J_c(1)\right\}^2 \\
=\;& (2c)\, e^{-2c} + (2c)^2\, e^{-2c} \int_0^1 J_c\,(r)^2\, dr + 2\,(2c)\, e^{-2c} \int_0^1 J_c\,(r)\, dW(r) \Rightarrow \eta^2.
\end{aligned}$$

As $\int_0^1 \exp\{cr\}\, dW(r) \sim N\left(0, \frac{\exp\{2c\}-1}{2c}\right)$, $(2c)^{1/2}\, e^{-c} \int_0^1 \exp\{cr\}\, dW(r)$ is $O_p(1)$, we have

$$\left\{(2c)^{3/2}\, e^{-2c} \int_0^1 \exp\{cr\}\, J_c\,(r)\, dr\right\}^2 = \left\{(2c)^{1/2}\, e^{-c} J_c(1)\right\}^2 + o_p(1) \Rightarrow \eta^2.$$

(d) Based on the results in (a), (b), (c), and $2c\gamma^2 = 2\theta b^2/\sigma^2 = d^2$, we get

$$\begin{aligned}
&\frac{(2c)\, e^{-c} A\,(\gamma, c)}{(2c)^2\, e^{-2c} B\,(\gamma, c)} \\
=\;& \frac{(2c)^{1/2}\,\gamma \left[(2c)^{1/2}\, e^{-c} \int_0^1 \exp\{cr\}\, dW(r)\right] + (2c)\, e^{-c} \int_0^1 J_c\,(r)\, dW(r)}{\gamma^2\,(2c)\,[1-e^{-2c}] + 2\gamma\,(2c)^{1/2}\left[(2c)^{3/2}\, e^{-2c} \int_0^1 \exp\{cr\}\, J_c\,(r)\, dr\right] + (2c)^2\, e^{-2c} \int_0^1 J_c\,(r)^2\, dr} \\
=\;& \frac{d\left[(2c)^{1/2}\, e^{-c} \int_0^1 \exp\{cr\}\, dW(r)\right] + [\xi\eta + o_p\,(1)]}{[d^2 + o\,(1)] + 2d\,[\eta^2 + o_p\,(1)]^{1/2} + [\eta^2 + o_p\,(1)]} \\
=\;& \frac{d\left[(2c)^{1/2}\, e^{-c} \int_0^1 \exp\{cr\}\, dW(r)\right] + [\xi\eta + o_p\,(1)]}{[d+\eta]^2 + o_p\,(1)}.
\end{aligned}$$

From Theorem 4 we have

$$\left((2c)\, e^{-c} A\,(\gamma, c)\,, (2c)^2\, e^{-2c} B\,(\gamma, c)\right) \Longrightarrow \left(\xi\,[d+\eta]\,, [d+\eta]^2\right).$$

Therefore, $(2c)^{1/2}\, e^{-c} \int_0^1 \exp\{cr\}\, dW(r) \Rightarrow \xi$.

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