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Testing for Multiple Bubbles

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Abstract

Identifying explosive bubbles that are characterized by periodically collapsing behavior over time has been a major concern in the literature and is of great importance for practitioners. The complexity of the nonlinear structure in multiple bubble phenomena diminishes the discriminatory power of existing tests, as evidenced in early simulations conducted by Evans (1991). Multiple collapsing bubble episodes within the same sample period make bubble diagnosis particularly difficult and complicate attempts at econometric dating. The present paper systematically investigates these issues and develops new procedures for practical implementation and surveillance strategies by central banks. We show how the testing procedure and dating algorithm of Phillips, Wu and Yu (2011, PWY) is affected by multiple bubbles and may fail to be consistent. To assist performance in such contexts, the present paper proposes a generalized version of the sup ADF test of PWY that addresses the difficulty. The asymptotic distribution of the generalized test is provided and the test is shown to significantly improve discriminatory power in simulations. The paper advances a new date-stamping strategy for the origination and termination of multiple bubbles that is based on this generalized test and consistency of the date-stamping algorithm is established. The new strategy leads to distinct power gains over the date-stamping strategy of PWY when multiple bubbles occur. Empirical applications are conducted with both tests along with their respective date-stamping technology to S&P 500 stock market data from January 1871 to December 2010. The new approach identifies many key historical episodes of exuberance and collapse over this period, whereas the strategy of PWY locates only two such episodes in the same sample range.

Keywords: Date-stamping strategy; Generalized sup ADF test; Multiple bubbles, Rational bubble; Periodically collapsing bubbles; Sup ADF test;
JEL classification: C15, C22

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Economists have taught us that it is unwise and unnecessary to combat asset price bubbles and excessive credit creation. Even if we were unwise enough to wish to prick an asset price bubble, we are told it is impossible to see the bubble while it is in its inflationary phase. (George Cooper, 2008)

If history repeats itself, and the unexpected always happens, how incapable must Man be of learning from experience. (George Bernard Shaw, 1903)

1 Introduction

As financial historians have argued recently (Ahamed, 2009; Ferguson, 2008), financial crises are often preceded by an asset market bubble or rampant credit growth. The global financial crisis of 2007-2009 is no exception. In its aftermath, central bank economists and policy makers are now affirming the recent Basil III accord to work to stabilize the financial system by way of guidelines on capital requirements and related measures to control "excessive credit creation". In this process of control, an important practical issue of market surveillance involves the assessment of what is "excessive". But as Cooper (2008) puts it in the header cited above from his recent bestseller, many economists have declared the task to be impossible and that it is imprudent to seek to combat asset price bubbles. How then can central banks and regulators work to offset a speculative bubble when they are unable to assess whether one exists and are considered unwise to take action if they believe one does exist?

One contribution that econometric techniques can offer in this complex exercise of market surveillance and policy action is the detection of exuberance in financial markets by explicit quantitative measures. These measures are not simply ex post detection techniques but anticipative dating algorithm that can assist regulators in their market monitoring behavior by means of early warning diagnostic tests. If history has a habit of repeating itself and human learning mechanisms do fail, as Shaw (1903) and others (notably, Ferguson, 2008[1]) assert, then quantitative warnings may serve as useful alert mechanisms to both market participants and regulators.

[1] "Nothing illustrates more clearly how hard human beings find it to learn from history than the repetitive history of stock market bubbles." Ferguson (2008).

Several attempts to develop econometric tests have been made in the literature going back some decades (see Gurkaynak, 2008, for a recent review). None of these tests have had much of an impact on empirical surveillance or policy. Most recently, Phillips, Wu and Yu (2010, PWY hereafter) propose a method which can detect exuberance in an asset price series during its inflationary phase. The approach is anticipative as an early warning alert system, so that it meets the needs of central bank surveillance teams and regulators, thereby addressing one of the key concerns articulated by Cooper (2008). The method is especially effective when there is a single bubble episode in the sample data, as in the 1990s Nasdaq episode analyzed in the PWY paper.

Just as historical experience confirms the existence of many financial crises (Ahamed reports 60 different financial crises since the 17th century[2]), when the sample period is long enough there will often be evidence of multiple asset price bubbles in the data. The econometric identification of multiple bubbles with periodically collapsing behavior over time is substantially more difficult than identifying a single bubble. The difficulty in practice arises from the complex nonlinear structure involved in multiple bubble phenomena which typically diminishes the discriminatory power of existing test mechanisms such as those given in PWY. These power reductions complicate attempts at econometric dating and enhance the need for new approaches that do not suffer from this problem.

The present paper responds to this need by providing a new framework for testing and dating bubble phenomena when there are potentially multiple bubbles in the data. The mechanisms developed here extend those of PWY by allowing for variable window widths in the recursive regressions on which the test procedures are based. The new mechanisms are shown in simulations to substantially increase discriminatory power in the tests and dating strategies. The paper contributes further by providing a limit theory for the new tests, by proving the consistency of the dating mechanisms, and by showing the inconsistency of certain versions of the PWY dating strategy when multiple bubbles occur. The final contribution of the paper is to

[2] "Financial booms and busts were, and continue to be, a feature of the economic landscape. These bubbles and crises seem to be deep-rooted in human nature and inherent to the capitalist system. By one count there have been 60 different crises since the 17th century." Ahamed (2009).

apply the techniques to a long historical series of US stock market data where multiple financial crises and episodes of exuberance and collapse have occured.

Fig. 1 graphs the S&P 500 price-dividend ratio[3] over 140 years from January 1871 to December 2010. This period covers many historical crises and financial catastrophes, most notably the 1907 banking panic, the stock market crash of 1929 and ensuing great depression, black Monday in October 1987, the dotcom bubble of the late 1990s and the recent subprime mortgage crisis. As evident in the figure, the price-dividend ratio is volatile with some repeated steep peaks and downturns over this long historical period. Of particular note is the rise in the ratio from December 1917 and the sharp rise prior to September 1929. The October 1929 crash was followed by a continuing downturn that bottomed out in June 1932. This rise and fall or boom and bust cycle was repeated on a smaller scale at other times. A significant inflationary episode occurred over March 1994 to August 2000. During this period, the price-dividend ratio was 5.28 times larger at its peak than at initiation. The ratio then dropped rapidly so that by February 2003 it was only 3.03 times its starting value. The ratio was relatively stable over March 2003 to December 2007 but dropped a further 47.8% over the next fifteen months during the subprime mortgage crisis.

The econometric identification of these repeated episodes of exuberance and collapse is substantially more difficult than identifying a single bubble. If econometric methods are to be useful in practical work conducted by central bank surveillance teams then they need to be capable of identifying key financial episodes over such periods. Of particular concern in financial surveillance is the usefulness of a warning alert system that points to inflationary upturns in the market. Such warning systems ideally need to have a low false detection rate to avoid unnecessary policy measures and a high positive detection rate that ensures early and effective policy implementation. The techniques developed in the present paper, together with an extended version of those in PWY, are tested empirically on the S&P 500 data shown in Fig. 1. The results are reported in Figs. 6 and 7 and are discussed in detail in Section 6. Our empirical findings confirm the effectiveness of the new testing strategy: the new approach succeeds in identifying

[3] The series is normalized to 100 at the first observation.

the main recognized episodes of exuberance and collapse over this long historical period, while the strategy of PWY locates only two such episodes over the same sample period.

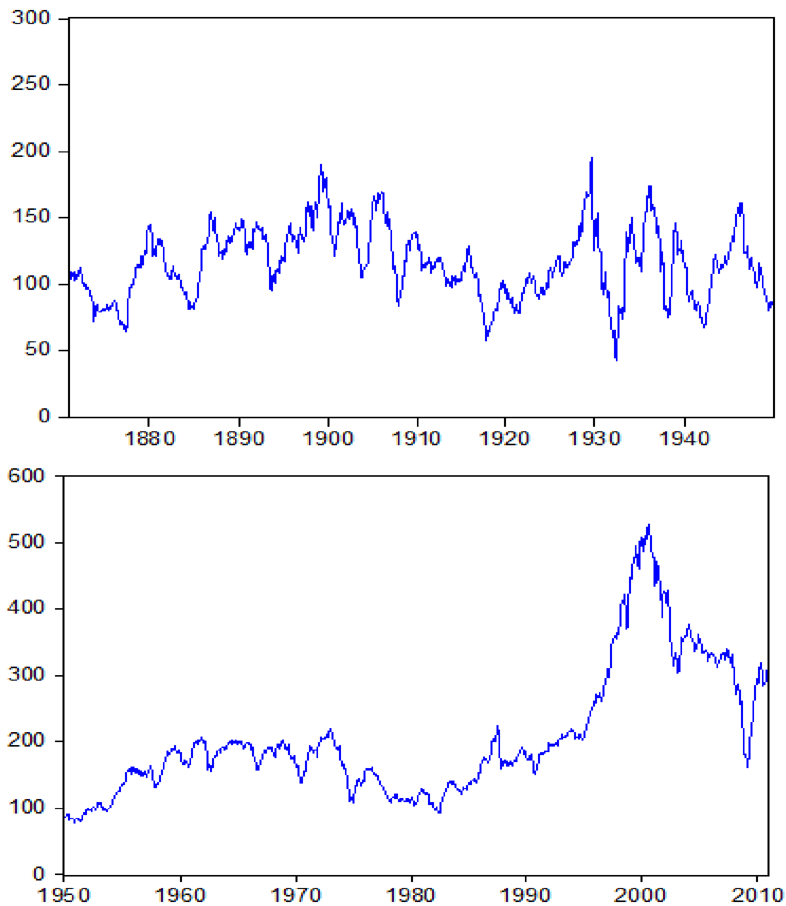

Figure 1: S&P 500 Price-Dividend ratio January 1871 to December 2010 (normalized to 100 at initiation).

The usual starting point in the analysis of financial bubbles is the standard asset pricing equation:

$$P_t = \sum_{i=0}^{\infty} \left(\frac{1}{1+r_f} \right)^i \mathbb{E}_t \left(D_{t+i} + U_{t+i} \right) + B_t, \tag{1}$$

where P_t is the after-dividend price of the asset, D_t is the payoff received from the asset (i.e.

dividend), r_f is the risk-free interest rate, U_t represents the unobservable fundamentals and B_t is the bubble component. The quantity $P_t^f = P_t - B_t$ is often called the market fundamental. Diba and Grossman (1988) argue that the bubble component has an explosive property characterized by the following submartingale property:

$$\mathbb{E}_t\left(B_{t+1}\right) = \left(1 + r_f\right) B_t. \tag{2}$$

In the absence of bubbles (i.e. $B_t = 0$), the degree of nonstationarity of the asset price is controlled by the character of the dividend series and unobservable fundamentals. For example, if D_t is an $I(1)$ process and U_t is either an $I(0)$ or an $I(1)$ process, then the asset price is at most an $I(1)$ process. On the other hand, given the submartingale behavior (2), asset prices will be explosive in the presence of bubbles. Therefore, when unobservable fundamentals are at most $I(1)$ and D_t is stationary after differencing, empirical evidence of explosive behavior in asset prices may be used to conclude the existence of bubbles.[4] Based on this argument, Diba and Grossman (1988) suggest conducting right-tailed unit root tests (against explosive alternatives) on the asset price and the observable fundamental (i.e. dividend) to detect the existence of bubbles. This method is then referred to as the conventional cointegration-based bubble test.

Evans (1991) demonstrated that this conventional cointegration-based test is not capable of detecting explosive bubbles when they manifest periodically collapsing behavior in the sample (Blanchard, 1979).[5] The Evans critique has led to a number of papers, which propose extended versions of the conventional cointegration-based test that have some power in detecting periodically collapsing bubbles.

[4] This argument also applies to the logarithmic asset price and the logarithmic dividend under certain conditions. This is due to the fact that in the absence of bubbles, equation (1) can be rewritten as

$$(1-\rho)\, p_t^f = \kappa + \rho e^{\bar{d}-\bar{p}} d_t + \rho e^{\bar{u}-\bar{p}} u_t + e^{\bar{d}-\bar{p}} \sum_{j=1}^{\infty} \rho^j \mathbb{E}_t\left[\triangle d_{t+j}\right] + e^{\bar{u}-\bar{p}} \sum_{j=1}^{\infty} \rho^j \mathbb{E}_t\left[\triangle u_{t+j}\right],$$

where $p_t^f = \log(P_t^f)$, $d_t = \log(D_t)$, $u_t = \log\left(U_t\right)$, $\rho = \left(1 + r_f\right)^{-1}$, κ is a constant, $\bar{p}, \bar{d}$ and $\bar{u}$ are the respective sample means of p_t^f, d_t and u_t. The degree of nonstationary of p_t^f is determined by that of d_t and u_t. Lee and Phillips (2011) provide a detailed analysis of the accuracy of this log linear approximation under various conditions.

[5] The failure of the cointegration based test is further studied in Charemza and Deadman (1995) within the setting of bubbles with stochastic explosive roots.

The approach of PWY (2011) is the sup ADF test (or forward recursive right-tailed ADF test). PWY suggest implementing the right-tailed ADF test repeatedly on a forward expanding sample sequence and performing inference based on the sup value of the corresponding ADF statistic sequence. They show that the sup ADF (SADF) test significantly improves power compared with the conventional cointegration-based test. This test gives rise to an associated dating strategy which identifies points of origination and termination of a bubble. When there is a single bubble in the data, it is known that this dating strategy is consistent, as first shown in the working paper by Phillips and Yu (2009). Extensive simulations conducted by Homm and Breitung (2010) indicate that the PWY procedure works well against other procedures such as CUSUM tests for structural breaks and is particularly effective as a real time bubble detection algorithm.

The present paper demonstrates that when the sample period includes multiple episodes of exuberance and collapse, the SADF test may suffer from reduced power and can be inconsistent, failing to reveal the existence of bubbles. This weakness is a particular drawback in analyzing long time series, like that in Fig. 1, or rapidly changing market data where more than one episode of exuberance is suspected. To overcome this weakness, we propose an alternative approach named the *generalized sup ADF* (GSADF) test. The GSADF test is also based on the idea of repeatedly implementing a right-tailed ADF test, but the new test extends the sample sequence to a broader and more flexible range. Instead of fixing the starting point of the sample (namely, on the first observation of the sample), the GSADF test extends the sample sequence by changing both the starting point and the ending point of the sample over a feasible range of flexible windows.

The sample sequences used in the SADF and GSADF tests are designed to: (a) capture any explosive behavior manifested within the overall sample; and (b) ensure that there are sufficient observations to achieve estimation efficiency. Since the GSADF test covers more subsamples of the data and has greater window flexibility, it is expected to outperform the SADF test in detecting explosive behavior in multiple episodes. This enhancement in performance by the GSADF test is demonstrated in simulations which compare the two tests in terms of their size and power in bubble detection. The paper also derives the asymptotic distribution of the GSADF statistic in comparison with that of the SADF statistic.

A further contribution of the paper is to develop a new dating strategy. The recursive ADF test is used in PWY to date stamp the origination and termination of a bubble. More specifically, the recursive procedure compares the ADF statistic sequence against critical values for the standard right-tailed ADF statistic and uses a first crossing time occurrence to date origination and collapse. For the generalized sup ADF test, we recommend a new date-stamping strategy, which compares the backward sup ADF (BSADF) statistic sequence with critical values for the sup ADF statistic, where the BSADF statistics are obtained from implementing the right-tailed ADF test on backward expanding sample sequences.

For a data generating process with only one bubble episode in the sample period, we show that both date-stamping strategies successfully estimate the origination and termination of a single bubble consistently. We then consider a situation in which there are two bubbles in the sample period and allow the duration of the first bubble to be longer or shorter than the second one. We demonstrate that the date-stamping strategy of PWY cannot consistently estimate the origination and termination of a (shorter) second bubble, whereas the strategy proposed in this paper can consistently estimate the origination and termination of each bubble. The same technology is applicable and similar results apply in multiple bubble scenarios.

The organization of the paper is as follows. The two sup ADF tests along with their limit distributions, are given in Section 2. Section 3 demonstrates the shortcomings of the SADF test in simulations. Size and power comparisons are conducted in Section 4. Section 5 proposes a date-stamping strategy based on the GSADF test and derives the consistency properties of this strategy and the PWY strategy under both single bubble and twin bubble alternatives. An alternative sequential implementation of the PWY procedure is developed which is shown to be capable of consistent date estimation in a twin bubble scenario. Both SADF and GSADF test procedures are applied to the S&P 500 price-dividend ratio data in Section 6. Both find evidence of bubbles but the new date-stamping strategy reveals many more crisis episodes over the 140 year time period and these correspond very closely with historical evidence. Section 7 concludes and summarizes the key steps involved in the implementation of these methods in practice. Two appendices contain supporting lemmas and derivations for the limit theory presented in the paper covering both single and multiple bubble scenarios. A technical supplement to the paper (Phillips, Shi and Yu, 2011)[6] provides a complete set of mathematical derivations of the limit theory presented here.

2 Sup ADF Tests

A common issue that arises in unit root testing is the specification of the model used for estimation purposes, not least because of its impact on the appropriate asymptotic theory and the critical values that are used in testing. Related issues arise in right-tailed unit root tests of the type used in bubble detection. The impact of hypothesis formulation and model specification on right-tailed unit root tests has been studied recently in Shi, Phillips and Yu (2010). Their analysis allowed for a null random walk process with an asymptotically negligible drift, namely

$$y_t = dT^{-\eta} + \theta y_{t-1} + \varepsilon_t, \ \varepsilon_t \overset{iid}{\sim} N\left(0, \sigma^2\right), \quad \theta = 1 \tag{3}$$

where d is a constant, T is the sample size and $\eta > 1/2$, and their recommended empirical regression model for bubble detection follows (3) and therefore includes an intercept but no fitted time trend in the regression. Suppose a regression sample starts from the r_1^{th} fraction of the total sample and ends at the r_2^{th} fraction of the sample, where $r_2 = r_1 + r_w$ and r_w is the (fractional) window size of the regression. The empirical regression model is

$$\Delta y_t = \alpha_{r_1,r_2} + \beta_{r_1,r_2} y_{t-1} + \sum_{i=1}^{k} \psi_{r_1,r_2}^{i} \Delta y_{t-i} + \varepsilon_t, \tag{4}$$

where k is the lag order and $\varepsilon_t \overset{iid}{\sim} N\left(0, \sigma_{r_1,r_2}^2\right)$. The number of observations in the regression is $T_w = \lfloor T r_w \rfloor$, where $\lfloor . \rfloor$ signifies the integer part of the argument. The ADF statistic (t-ratio) based on this regression is denoted by $ADF_{r_1}^{r_2}$.

The SADF test estimates the ADF model repeatedly on a forward expanding sample sequence and conducts a hypothesis test based on the sup value of the corresponding ADF statistic

[6] It is downloadable from `https://sites.google.com/site/shupingshi/TN_GSADF.pdf?attredirects=0&d=1`.

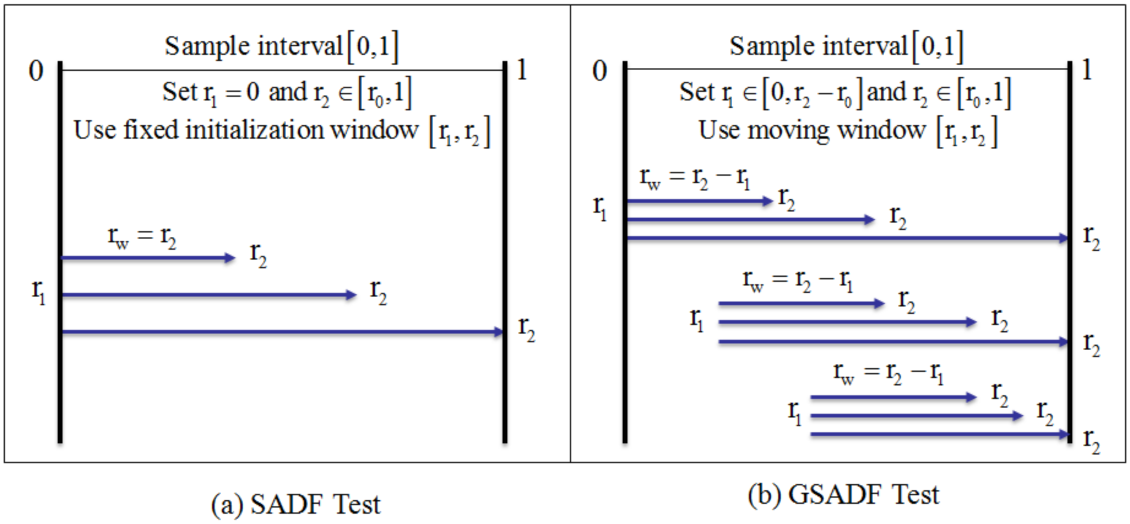

Figure 2: The sample sequences and window widths of the SADF test and the GSADF test

sequence. The window size r_w expands from r_0 to 1, where r_0 is the smallest sample window (selected to ensure estimation efficiency) and 1 is the largest sample window (the total sample size). The starting point r_1 of the sample sequence is fixed at 0, so the ending point of each sample r_2 is equal to r_w, changing from r_0 to 1. The ADF statistic for a sample that runs from 0 to r_2 is denoted by $ADF_0^{r_2}$. The SADF statistic is defined as $\sup_{r_2 \in [r_0,1]} ADF_0^{r_2}$, and is denoted by $SADF\,(r_0)$.

The GSADF test continues the idea of repeatedly estimating the ADF test regression (4) on a sample sequence. However, the sample sequence is broader than that of the SADF test. Besides varying the end point of the regression r_2 from r_0 to 1, the GSADF test allows the starting points r_1 to change within a feasible range, which is from 0 to $r_2 - r_0$. Figure 2 illustrates the sample sequences of the SADF test and the GSADF test. We define the GSADF statistic to be the largest ADF statistic over the feasible ranges of r_1 and r_2, and we denote this statistic by $GSADF\,(r_0)$. That is,

$$GSADF\,(r_0) = \sup_{\substack{r_2 \in [r_0,1] \\ r_1 \in [0, r_2 - r_0]}} \left\{ ADF_{r_1}^{r_2} \right\}.$$

Proposition 1 *When the regression model includes an intercept and the null hypothesis is a random walk with an asymptotically negligible drift (i.e. $dT^{-\eta}$ with $\eta > 1/2$ and constant d), the limit distribution of the GSADF test statistic is:*

$$\sup_{\substack{r_2 \in [r_0,1] \\ r_1 \in [0, r_2 - r_0]}} \left\{ \frac{\frac{1}{2} r_w \left[W(r_2)^2 - W(r_1)^2 - r_w \right] - \int_{r_1}^{r_2} W(r)\, dr \left[W(r_2) - W(r_1) \right]}{r_w^{1/2} \left\{ r_w \int_{r_1}^{r_2} W(r)^2 dr - \left[\int_{r_1}^{r_2} W(r)\, dr \right]^2 \right\}^{1/2}} \right\}, \quad (5)$$

where $r_w = r_2 - r_1$ and W is a standard Wiener process.

The proof of Proposition 1 is similar to that of PWY and is therefore given in a separate technical supplement which is downloadable from `https://sites.google.com/site/shupingshi/TN_GSADF.pdf?attredirects=0&d=1`. The technical note of Shi, Phillips and Yu (2010) provides further details. Note that the limit distribution of the GSADF statistic is identical to that of the case when the regression model includes an intercept and the null hypothesis is a random walk without drift. The usual limit distribution of the ADF statistic is a special case of equation (5) with $r_1 = 0$ and $r_2 = r_w = 1$ while the limit distribution of the SADF statistic is a further special case of equation (5) with $r_1 = 0$ and $r_2 = r_w \in [r_0, 1]$ (see Shi, Phillips and Yu, 2010).

Similar to the SADF statistic, the asymptotic GSADF distribution depends on the smallest window size r_0. In practice, r_0 needs to be chosen according to the total number of observations T. If T is small, r_0 needs to be large enough to achieve estimation efficiency. If T is large, r_0 can be set to be a smaller number so that the test does not miss any opportunity to detect an early explosive episode. In our empirical application we use $r_0 = 36/1680$.

Critical values of the SADF and GSADF statistics are displayed in Table 1. The asymptotic critical values are obtained by numerical simulations, where the Wiener process is approximated by partial sums of $2,000$ independent $N(0,1)$ variates and the number of replications is $2,000$. The finite sample critical values are obtained from $5,000$ Monte Carlo replications. The lag order k is set to zero. The parameters, d and η, in the null hypothesis are set to unity.[7]

[7] From Shi, Phillips and Yu (2010), we know that when $d = 1$ and $\eta > 1/2$, the finite sample distribution of the SADF statistic is almost invariant to the value of η.

Table 1: Critical values of the SADF and GSADF tests against an explosive alternative

(a) The asymptotic critical values

	$r_0 = 0.4$		$r_0 = 0.2$		$r_0 = 0.1$	
	SADF	GSADF	SADF	GSADF	SADF	GSADF
90%	0.86	1.25	1.04	1.66	1.18	1.89
95%	1.18	1.56	1.38	1.92	1.49	2.14
99%	1.79	2.18	1.91	2.44	2.01	2.57

(b) The finite sample critical values

	$T = 100$ and $r_0 = 0.4$		$T = 200$ and $r_0 = 0.4$		$T = 400$ and $r_0 = 0.4$	
	SADF	GSADF	SADF	GSADF	SADF	GSADF
90%	0.72	1.16	0.75	1.21	0.78	1.27
95%	1.05	1.48	1.08	1.52	1.10	1.55
99%	1.66	2.08	1.75	2.18	1.75	2.12

(c) The finite sample critical values

	$T = 100$ and $r_0 = 0.4$		$T = 200$ and $r_0 = 0.2$		$T = 400$ and $r_0 = 0.1$	
	SADF	GSADF	SADF	GSADF	SADF	GSADF
90%	0.72	1.16	0.97	1.64	1.19	1.97
95%	1.05	1.48	1.30	1.88	1.50	2.21
99%	1.66	2.08	1.86	2.46	1.98	2.71

Note: the asymptotic critical values are obtained by numerical simulations with 2,000 iterations. The Wiener process is approximated by partial sums of $N(0,1)$ with $2,000$ steps. The finite sample critical values are obtained from the $5,000$ Monte Carlo simulations. The parameters, d and η, are set to unity.

We observe the following phenomena. First, as the minimum window size r_0 decreases, critical values of the test statistic (including the SADF statistic and the GSADF statistic) increase. For instance, when r_0 decreases from 0.4 to 0.1, the 95% asymptotic critical value of the GSADF statistic rises from 1.56 to 2.14 and the 95% finite sample critical value of the test statistic with sample size 400 increases from 1.48 to 2.21. Second, for a given r_0, the finite sample critical values of the test statistic are almost invariant. Third, critical values for the GSADF statistic are larger than those of the SADF statistic. As a case in point, when $T = 400$ and $r_0 = 0.1$, the 95% critical value of the GSADF statistic is 2.21 while that of the SADF statistic is 1.50. Figure 3 shows the asymptotic distribution of the ADF, $SADF\,(0.1)$ and $GSADF\,(0.1)$ statistics. The distributions move sequentially to the right and have greater concentration in the order ADF, $SADF\,(0.1)$ and $GSADF\,(0.1)$.

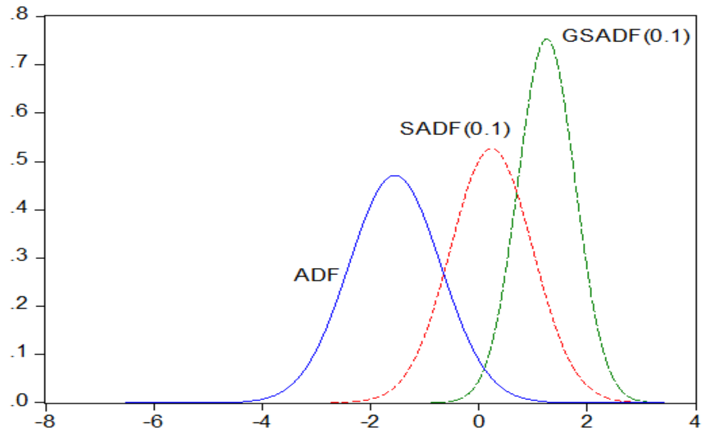

Figure 3: Asymptotic distributions of the ADF and supADF statistics ($r_0 = 0.1$)

3 Simulation Study

This section investigates the performance of the SADF and GSADF tests when the test sample contains multiple collapsing episodes.

3.1 Generating the test sample

We first simulate an asset price series based on the Lucas asset pricing model and the Evans (1991) bubble model. The simulated asset prices consist of a market fundamental component P_t^f, which combines a random walk dividend process and equation (1) with $U_t = 0$ and $B_t = 0$

for all t to obtain[8]

$$D_t = \mu + D_{t-1} + \varepsilon_{Dt}, \quad \varepsilon_{Dt} \sim N\left(0, \sigma_D^2\right) \tag{6}$$

$$P_t^f = \frac{\mu\rho}{(1-\rho)^2} + \frac{\rho}{1-\rho} D_t, \tag{7}$$

and the Evans bubble component

$$B_{t+1} = \rho^{-1} B_t \varepsilon_{B,t+1}, \qquad \text{if } \; B_t < b \tag{8}$$

$$B_{t+1} = \left[\zeta + (\pi\rho)^{-1} \theta_{t+1} \left(B_t - \rho\zeta\right)\right] \varepsilon_{B,t+1}, \qquad \text{if } \; B_t \geq b \; . \tag{9}$$

This series has the submartingale property $\mathbb{E}_t \left(B_{t+1}\right) = \left(1 + r_f\right) B_t$. Parameter μ is the drift of the dividend process, σ_D^2 is the variance of the dividend, $\rho^{-1} = 1 + r_f > 1$ and $\varepsilon_{B,t} = \exp\left(y_t - \tau^2/2\right)$ with $y_t \sim NID\left(0, \tau^2\right)$. The quantity ζ is the re-initializing value after the bubble collapse. The series θ_t follows a Bernoulli process which takes the value 1 with probability π and 0 with probability $1 - \pi$. Equations (8) - (9) state that a bubble grows explosively at rate ρ^{-1} when its size is less than b while if the size is greater than b, the bubble grows at a faster rate $(\pi\rho)^{-1}$ but with a $1 - \pi$ probability of collapsing. The asset price is the sum of the market fundamental and the bubble component, namely $P_t = P_t^f + \kappa B_t$, where $\kappa > 0$ controls the relative magnitudes of these two components.

The parameter settings used by Evans (1991) are displayed in the top line of Table 2 and labeled *yearly*. The parameter values for μ and σ_D^2 were originally obtained by West (1988), by matching the sample mean and sample variance of first differenced real S&P 500 stock price index dividends from 1871 to 1980. The value for the discount factor ρ is equivalent to a 5% yearly interest rate.

Due to the availability of higher frequency data, we apply the SADF test and the GSADF test to monthly data. The parameters μ and σ_D^2 are set to correspond to the sample mean

[8] An alternative data generating process, which assumes that the logarithmic dividend is a random walk with drift, is as follows:

$$\ln D_t = \mu + \ln D_{t-1} + \varepsilon_t, \varepsilon_t \sim N\left(0, \sigma_d^2\right)$$

$$P_t^f = \frac{\rho \exp\left(\mu + \frac{1}{2}\sigma_d^2\right)}{1 - \rho \exp\left(\mu + \frac{1}{2}\sigma_d^2\right)} D_t.$$

Table 2: Parameter settings

	μ	σ_D^2	D_0	ρ	b	B_0	π	ζ	τ	κ
Yearly	0.0373	0.1574	1.3	0.952	1	0.50	0.85	0.50	0.05	20
Monthly	0.0024	0.0010	1.0	0.985	1	0.50	0.85	0.50	0.05	50

and sample variance of the first differenced monthly real S&P 500 stock price index dividend described in the application section below, so that the settings are in accordance with our empirical application. The discount value ρ equals 0.985 (we allow ρ to vary from 0.975 to 0.999 in the size and power comparisons section). The new setting is labeled *monthly* in Table 2.

Figure 4 depicts one realization of the data generating process with the monthly parameter settings. As we observe in this graph, there are several obvious collapsing episodes of different magnitudes within this particular sample trajectory.

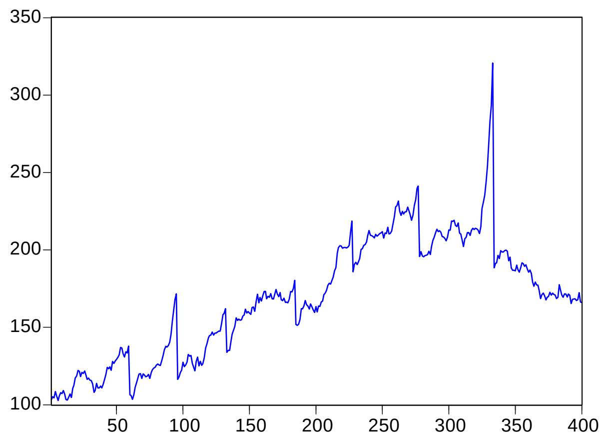

Figure 4: Simulated time series with sample size 400.

3.2 Performing the sup ADF test and the generalized sup ADF test

We first implement the SADF test on the whole sample range. We repeat the test on a subsample which contains fewer collapsing episodes to illustrate the instability of the SADF test. Furthermore, we conduct the test on the same simulated data series (over the whole sample range) to show the advantage of the GSADF test.

The lag order k is set to zero for all tests in this paper.[9] The smallest window size considered in the SADF test for the whole sample contains 40 observations ($r_0 = 0.1$). The SADF statistic for the simulated data series is 0.71, which is smaller than the 10% finite sample critical value 1.19 (see Table 1). Therefore, we conclude that there are no bubbles in this sample. Now suppose that the SADF test starts from the 201^{st} observation, and the smallest regression window also contains 40 observations ($r_0 = 0.2$). The SADF statistic obtained from this sample is 1.39 and it is greater than 1.30 (Table 1). In this case, we reject the null hypothesis of no bubble at the 5% significance level.

Evidently the SADF test fails to find bubbles when the whole sample is utilized, whereas by re-selecting the starting point of the sample to exclude some of the collapse episodes, it succeeds in finding evidence of bubbles. Each of the above experiments can be viewed as special cases of the GSADF test in which the sample starting points are fixed. In the first experiment, the sample starting point of the GSADF test r_1 is set to 0. The sample starting point r_1 of the second experiment is fixed at 0.502. The conflicting results obtained from these two experiments demonstrates the importance of using variable starting points, as is done in the GSADF test.

We then apply the GSADF test to the simulated asset prices. The GSADF statistic of the simulated data is 8.59, which is substantially greater than the 1% finite sample critical value 2.71 (Table 1). Thus, the GSADF test finds strong evidence of bubbles. Compared to the SADF test, the GSADF identifies bubbles without re-selecting the sample starting point, giving an obvious improvement that is particularly useful in empirical applications.[10]

[9] In PWY, the lag order is determined by significance testing, as in Campbell and Perron (1991). However, we demonstrate in the size and power comparison section that this lag selection criteria results in significant size distortion and reduces the power of both the SADF and GSADF tests.

[10] We observe similar phenomena from the alternative data generating process where the logarithmic dividend is a random walk with drift. Parameters in the alternative data generating process (monthly) are set as follows: $B_0 = 0.5, b = 1, \pi = 0.85, \zeta = 0.5, \rho = 0.985, \tau = 0.05, \mu = 0.001, \ln D_0 = 1$, $\sigma^2_{\ln D} = 0.0001$, and $P_t = P_t^f + 500B_t$.

4 Size and Power Comparisons

This section compares the sizes and powers of the SADF and GSADF tests. The data generating process for the size comparison is the null hypothesis in equation (3) with $d = \eta = 1$. We calculate size based on the asymptotic critical values displayed in Table 1. The nominal size is 5%. The number of replications is $5,000$. We observe from Table 3 that the size distortion of the GSADF test is smaller than that of the SADF test. For example, when $T = 400$ and $r_0 = 0.1$, the size distortion of the GSADF test is 0.9% whereas that of the SADF test is 1.6%.[11]

Table 3: Sizes of the SADF and GSADF tests with asymptotic critical values. The data generating process is equation (3) with $d = \eta = 1$. The nominal size is 5%.

	$T = 100$	$T = 200$		$T = 400$	
	$r_0 = 0.4$	$r_0 = 0.4$	$r_0 = 0.2$	$r_0 = 0.4$	$r_0 = 0.1$
SADF	0.043	0.040	0.038	0.041	0.034
GSADF	0.048	0.041	0.044	0.045	0.059

Note: size calculations are based on 5000 replications..

Powers in Table 4 and 5 are calculated with the 95% quantiles of the finite sample distributions (Table 1), and the number of iterations for the calculation is $5,000$. The smallest window size for both the SADF test and the GSADF test has 40 observations. The data generating process of the power comparison is the periodically collapsing explosive process, equation (6) - (9). For comparison with the literature, we first set the parameters in the DGP as in Evans (1991) with sample sizes of 100 and 200. From the left panel of Table 4 (labeled *yearly*), the power of the GSADF test is 7% and 15.2% higher than those of the SADF test when the sample size is 100 and 200.[12]

Table 4 also displays powers of the SADF and GSADF tests under the DGP with monthly parameter settings and with sample sizes 100, 200 and 400. From the right panel of the table,

[11] Suppose the lag order is determined by significance testing as in Campbell and Perron (1991) with a maximum lag order of 12. When $T = 400$ and $r_0 = 0.1$, the sizes of the SADF test and the GSADF test are 0.130 and 0.790 (the nominal size is 5%), indicating size distortion in both tests and a particularly large size distortion for the GSADF test.

[12] Suppose the lag order is determined by significance testing as in Campbell and Perron (1991) with a maximum lag order of 12. When $T = 200$ and $r_0 = 0.2$, the powers of the SADF test and the GSADF test are 0.565 and 0.661, which are smaller than those in Table 4.

Table 4: Powers of the SADF and GSADF tests. The data generating process is equation (6)-(9).

	Yearly		*Monthly*	
	SADF	GSADF	SADF	GSADF
$T = 100$ and $r_0 = 0.4$	0.408	0.478	0.509	0.556
$T = 200$ and $r_0 = 0.2$	0.634	0.786	0.699	0.833
$T = 400$ and $r_0 = 0.1$	-	-	0.832	0.977

Note: power calculations are based on 5000 replications.

when the sample size $T = 400$, the GSADF test raises test power from 83.2% to 97.7%, giving a 14.5% improvement. The power improvement of the GSADF test is 4.7% and 13.4% when the sample size is 100 and 200. Due to the fact that, for any given bubble collapsing probability π in the Evans model, the sample period is more likely to include multiple collapsing episodes when the sample size T is larger, the advantage of the GSADF test is more evident under these circumstances.

In Table 5, we compare powers of the SADF and GSADF tests with the discount factor ρ varying from 0.975 to 0.990, under the DGP with the monthly parameter settings. First, due to the fact that the rate of bubble expansion is inversely related to the discount factor, powers of both SADF test and GSADF tests are expected to decrease as ρ increases. The power of the SADF (GSADF) test declines from 84.5% to 76.9% (from 99.3% to 91.0%) as the discount factor rises from 0.975 to 0.990 (see Table 5). Second, we observe from Table 5 that the GSADF test has greater discriminatory power for detecting bubbles than the SADF test. The power improvement is 14.8%, 14.8%, 14.5% and 14.1% for $\rho = \{0.975, 0.980, 0.985, 0.990\}$.

Table 5: Powers of the SADF and GSADF tests. The data generating process is equation (6)-(9) with the *monthly* parameter settings and sample size 400 ($r_0 = 0.1$).

ρ	0.975	0.980	0.985	0.990
SADF	0.845	0.840	0.832	0.769
GSADF	0.993	0.988	0.977	0.910

Note: power calculations are based on 5000 replications.

5 Date-stamping Strategies for Bubble Episodes

Suppose that one is interested in knowing whether any particular observation, such as the point $\lfloor Tr_2 \rfloor$, belongs to a bubble phase in the trajectory. PWY suggest conducting a right-tailed ADF test recursively using information up to this observation (i.e. $I_{\lfloor Tr_2 \rfloor} = \{y_1, y_2, \cdots, y_{\lfloor Tr_2 \rfloor}\}$). Since it is possible that $I_{\lfloor Tr_2 \rfloor}$ includes one or more collapsing episodes of bubbles, like the conventional cointegration-based test for bubbles, the ADF test may result in finding *pseudo stationary* behavior. We therefore recommend performing a *backward sup ADF test* on $I_{\lfloor Tr_2 \rfloor}$ to improve identification accuracy.

The backward SADF test performs a sup ADF test on a backward expanding sample sequence, where the ending points of the samples are fixed at r_2 and the starting point varies from 0 to $r_2 - r_0$. Suppose we label the ADF statistic for each regression using its starting point r_1 and ending point r_2 to obtain $BADF_{r_1}^{r_2}$. The corresponding ADF statistic sequence is $\{BADF_{r_1}^{r_2}\}_{r_1 \in [0, r_2 - r_0]}$. The backward SADF statistic is defined as the sup value of the ADF statistic sequence, denoted by

$$BSADF_{r_2}(r_0) : BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} \left\{ BADF_{r_1}^{r_2} \right\}.$$

The backward ADF test is a special case of the backward sup ADF test with $r_1 = 0$. We denote the backward ADF statistic by $BADF_{r_2}$. Figure 5 illustrates the difference between the backward ADF test and the backward SADF test. PWY proposes comparing $BADF_{r_2}$ with the (right-tail) critical values of the standard ADF statistic to identify the explosiveness of observation $\lfloor Tr_2 \rfloor$. The feasible range of r_2 runs from r_0 to 1. The origination date of a bubble $\lfloor Tr_e \rfloor$ is calculated as the first chronological observation whose backward ADF statistic exceeds the critical value. We denote the calculated origination date by $\lfloor T\hat{r}_e \rfloor$. The estimated termination date of a bubble $\lfloor T\hat{r}_f \rfloor$ is the first chronological observation after $\lfloor T\hat{r}_e \rfloor + \log(T)$ whose backward ADF statistic goes below the critical value. PWY impose the condition that the duration of a bubble is longer than $\log(T)$. Namely,

$$\hat{r}_e = \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BADF_{r_2} > cv_{r_2}^{\beta_T} \right\} \text{ and } \hat{r}_f = \inf_{r_2 \in [\hat{r}_e + \log(T)/T, 1]} \left\{ r_2 : BADF_{r_2} < cv_{r_2}^{\beta_T} \right\}, \quad (10)$$

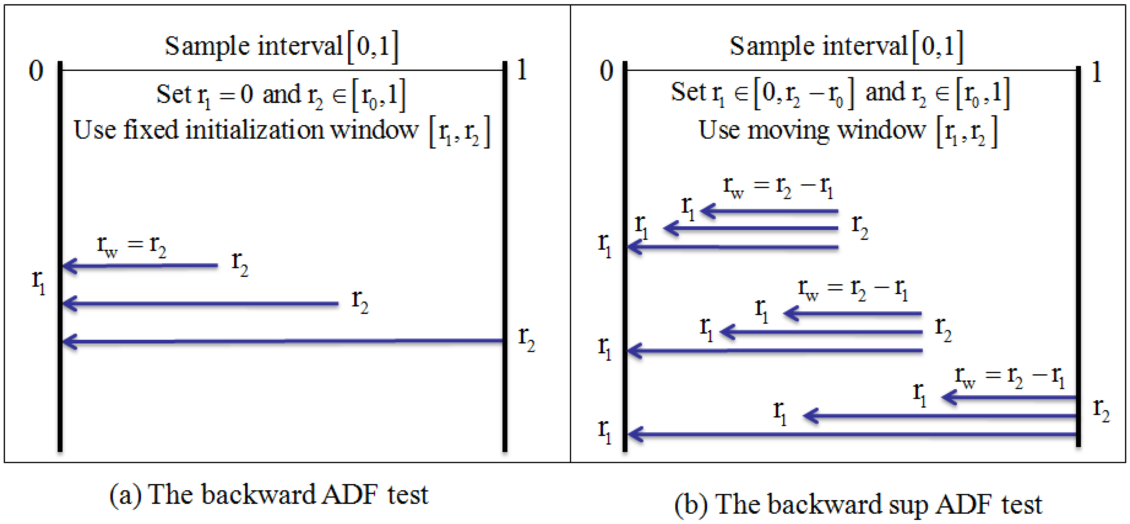

Figure 5: The sample sequences of the backward ADF test and the backward SADF test

where $cv_{r_2}^{\beta_T}$ is the $100\beta_T\%$ critical value of the backward ADF statistic based on $\lfloor Tr_2 \rfloor$ observations. The significance level β_T depends on the sample size T and we assume that $\beta_T \to 0$ as $T \to \infty$.

Instead of using the backward ADF statistic, the new strategy suggests making inferences on the explosiveness of observation $\lfloor Tr_2 \rfloor$ based on the backward sup ADF statistic, $BSADF_{r_2}(r_0)$. We define the origination date of a bubble as the first observation whose backward sup ADF statistic exceeds the critical value of the backward sup ADF statistic. The termination date of a bubble is calculated as the first observation after $\lfloor T\hat{r}_e \rfloor + \delta \log(T)$ whose backward sup ADF statistic falls below the critical value of the backward sup ADF statistic. We assume that the duration of the bubble is longer than $\delta \log(T)$, where δ is frequency dependent.[13] The (fractional) origination and termination points of a bubble (i.e. r_e and r_f) are calculated according to the following first crossing time equations:

$$\hat{r}_e = \inf_{r_2 \in [r_0,1]} \left\{ r_2 : BSADF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\},$$

[13]For instance, one may believe that the duration of bubbles should be longer than one year. Then, when the sample size is 30 years (360 months), δ is 0.7 for the yearly data and 5 for the monthly data.

$$\hat{r}_f = \inf_{r_2 \in [\hat{r}_e + \delta \log(T)/T, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) < scv_{r_2}^{\beta_T} \right\},$$

where $scv_{r_2}^{\beta_T}$ is the $100\beta_T\%$ critical value of the sup ADF statistic based on $\lfloor Tr_2 \rfloor$ observations. Analogously, the significance level β_T depends on the sample size T and it goes to zero as the sample size approaches infinity.

In addition, the SADF test can be viewed as a repeated implementation of the backward ADF test for each $r_2 \in [r_0, 1]$. The GSADF test is equivalent to a test which implements the backward sup ADF test repeatedly for each $r_2 \in [r_0, 1]$ and makes inferences based on the sup value of the backward sup ADF statistic sequence, $\{BSADF_{r_2}(r_0)\}_{r_2 \in [r_0, 1]}$. Hence, the SADF and GSADF statistics can respectively be rewritten as

$$SADF(r_0) = \sup_{r_2 \in [r_0, 1]} \{BADF_{r_2}\},$$
$$GSADF(r_0) = \sup_{r_2 \in [r_0, 1]} \{BSADF_{r_2}(r_0)\}.$$

Thus, the PWY date-stamping strategy corresponds to the SADF test and the new strategy corresponds to the GSADF test.

5.1 The null hypothesis: no bubbles

In order to derive the consistency properties of these date-stamping strategies, we first need to obtain the asymptotic distributions of the ADF statistic and the SADF statistic with $\lfloor Tr_2 \rfloor$ observations under the null hypothesis (3). We know that the backward ADF test with observation $\lfloor Tr_2 \rfloor$ is a special case of the GSADF test with $r_1 = 0$ and a fixed r_2 and the backward sup ADF test is a special case of the GSADF test with a fixed r_2 and $r_1 = r_2 - r_w$. Therefore, based on equation (5), we can derive the asymptotic distributions of these two statistics, namely

$$F_{r_2}(W) := \frac{\frac{1}{2} r_2 \left[W(r_2)^2 - r_2 \right] - \int_0^{r_2} W(r)\, dr W(r_2)}{r_2^{1/2} \left\{ r_2 \int_0^{r_2} W(r)^2 dr - \left[\int_0^{r_2} W(r)\, dr \right]^2 \right\}^{1/2}},$$

$$F_{r_2}^{r_0}(W) := \sup_{\substack{r_1 \in [0, r_2 - r_0] \\ r_w = r_2 - r_1}} \left\{ \frac{\frac{1}{2} r_w \left[W(r_2)^2 - W(r_1)^2 - r_w \right] - \int_{r_1}^{r_2} W(r)\, dr \left[W(r_2) - W(r_1) \right]}{r_w^{1/2} \left\{ r_w \int_{r_1}^{r_2} W(r)^2 dr - \left[\int_{r_1}^{r_2} W(r)\, dr \right]^2 \right\}^{1/2}} \right\}.$$

We, therefore, define $cv_{r_2}^{\beta_T}$ as the $100(1-\beta_T)\%$ quantile of $F_{r_2}(W)$ and $scv_{r_2}^{\beta_T}$ as the $100(1-\beta_T)\%$ quantile of $F_{r_2}^{r_0}(W)$. We know that $cv_{r_2}^{\beta_T} \to \infty$ and $scv_{r_2}^{\beta_T} \to \infty$ as $\beta_T \to 0$.

Notice that given $cv_{r_2}^{\beta_T} \to \infty$ and $scv_{r_2}^{\beta_T} \to \infty$, under the null hypothesis of no bubbles, the probabilities of (falsely) detecting the origination of bubble expansion and the termination of bubble collapse using the backward ADF statistic and the backward sup ADF statistic tend to zero, so that $\Pr\{\hat{r}_e \in [r_0, 1]\} \to 0$ and $\Pr\{\hat{r}_f \in [r_0, 1]\} \to 0$.

5.2 The alternative hypothesis: a single bubble

Consider the data generating process of Phillips and Yu (2009)

$$
\begin{aligned}
X_t &= X_{t-1} 1\{t < \tau_e\} + \delta_T X_{t-1} 1\{\tau_e \leq t \leq \tau_f\} \\
&\quad + \left(\sum_{k=\tau_f+1}^{t} \varepsilon_k + X_{\tau_f}^* \right) 1\{t > \tau_f\} + \varepsilon_t 1\{j \leq \tau_f\}, \qquad (11)
\end{aligned}
$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0,1)$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$, $X_{\tau_f}^* = X_{\tau_e} + X^*$ with $X^* = O_p(1)$, $\tau_e = \lfloor Tr_e \rfloor$ is the origination of bubble expansion and $\tau_f = \lfloor Tr_f \rfloor$ is the termination of bubble collapse. The pre-bubble period $N_0 = [1, \tau_e)$ is assumed to be a pure random walk process. The bubble expansion period $B = [\tau_e, \tau_f]$ is a mildly explosive process with expansion rate δ_T. The process then collapses to $X_{\tau_f}^*$, which equals X_{τ_e} plus a small perturbation, and continues its pure random walk path in the period $N_1 = (\tau_f, \tau]$.

Notice that there is only one bubble episode in the data generating process (11). Under this mechanism we have the following consistency results, whose proofs are collected in Appendix A.

Theorem 1 *Suppose $\hat{r}_e$ and $\hat{r}_f$ are obtained from the backward DF test based on the t statistic. Given an alternative hypothesis of mildly explosive behavior in model (11), if*

$$
\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0, \qquad (12)
$$

we have $\hat{r}_e \xrightarrow{p} r_e$ as $T \to \infty$; and if

$$
\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0 \qquad (13)
$$

and $\hat{r}_f > \hat{r}_e + \log(T)/T$, *we have* $\hat{r}_f \xrightarrow{p} r_f$ *as* $T \to \infty$.

Theorem 2 *Suppose* $\hat{r}_e$ *and* $\hat{r}_f$ *are obtained from the backward sup DF test based on the t statistic. Given an alternative hypothesis of mildly explosive behavior in model (11), if*

$$\frac{1}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0, \tag{14}$$

we have $\hat{r}_e \xrightarrow{p} r_e$ *as* $T \to \infty$*; and if*

$$\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0 \tag{15}$$

and $\hat{r}_f > \hat{r}_e + \delta \log(T)/T$, *we have* $\hat{r}_f \xrightarrow{p} r_f$ *as* $T \to \infty$.

These results show that both strategies consistently estimate the origination and termination points when there is only a single bubble episode in the sample period. Suppose $cv_{r_2}^{\beta_T} = O_p(T^{\gamma})$ and $scv_{r_2}^{\beta_T} = O_p(T^{\gamma_s})$. The regularity condition (12) in Theorem 1 implies that the order of magnitude (γ) of $cv_{r_2}^{\beta_T}$ needs to be greater than 0 and smaller than 1/2, namely $\gamma \in (0, 1/2)$. Condition (13) suggests that γ should fall between $(1-\alpha)/2$ and 1/2. Theorem 2 requires the order of magnitude (γ_s) of $scv_{r_2}^{\beta_T}$ to be greater than 0 and smaller than 1/2 to obtain the consistency of $\hat{r}_e$ and γ_s needs to satisfy the condition $\gamma_s \in \left(\frac{1-\alpha}{2}, 1/2\right)$ to ensure the consistency of $\hat{r}_f$.

5.3 The alternative hypothesis: two bubbles

Consider a data generating process with two bubble episodes:

$$\begin{aligned} X_t = {} & X_{t-1} 1\{t \in N_0\} + \delta_T X_{t-1} 1\{t \in B_1 \cup B_2\} + \left(\sum_{k=\tau_{1f}+1}^{t} \varepsilon_k + X_{\tau_{1f}}^* \right) 1\{t \in N_1\} \\ & + \left(\sum_{l=\tau_{2f}+1}^{t} \varepsilon_l + X_{\tau_{2f}}^* \right) 1\{t \in N_2\} + \varepsilon_t 1\{j \in N_0 \cup B_1 \cup B_2\}, \end{aligned} \tag{16}$$

where $N_0 = [1, \tau_{1e}), B_1 = [\tau_{1e}, \tau_{1f}], N_1 = (\tau_{1f}, \tau_{2e}), B_2 = [\tau_{2e}, \tau_{2f}]$ and $N_2 = (\tau_{2f}, \tau]$. $\tau_{1e} = \lfloor Tr_{1e} \rfloor$, $\tau_{1f} = \lfloor Tr_{1f} \rfloor$ are the origination and termination dates of the first bubble, $\tau_{2e} = \lfloor Tr_{2e} \rfloor$,

$\tau_{2f} = \lfloor Tr_{2f} \rfloor$ are the origination and termination dates of the second bubble and τ is the last observation of the sample. After the collapse of the first bubble, X_t continues its pure random walk path until $\tau_{2e} - 1$ and starts another expansion process at τ_{2e}. The expansion process lasts until τ_{2f} and collapses to a value of $X^*_{\tau_{2f}}$. It then continues its pure random walk path until the end of the sample period τ. We assume that the expansion duration of the first bubble is longer than that of the second bubble, namely $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$.

The date-stamping strategy of PWY suggests calculating r_{1e}, r_{1f}, r_{2e} and r_{2f} from the following equations (based on the ADF statistic):

$$\hat{r}_{1e} = \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BADF_{r_2} > cv_{r_2}^{\beta_T} \right\} \text{ and } \hat{r}_{1f} = \inf_{r_2 \in [\hat{r}_{1e} + \log(T)/T, 1]} \left\{ r_2 : BADF_{r_2} < cv_{r_2}^{\beta_T} \right\}, \tag{17}$$

$$\hat{r}_{2e} = \inf_{r_2 \in \left[\hat{r}_{1f}, 1\right]} \left\{ r_2 : BADF_{r_2} > cv_{r_2}^{\beta_T} \right\} \text{ and } \hat{r}_{2f} = \inf_{r_2 \in [\hat{r}_{2e} + \log(T)/T, 1]} \left\{ r_2 : BADF_{r_2} < cv_{r_2}^{\beta_T} \right\}, \tag{18}$$

where the duration of the bubble periods is restricted to be longer than $\log(T)$.

The new strategy recommends using the backward sup ADF test and calculating the origination and termination points according to the following equations:

$$\hat{r}_{1e} = \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\}, \tag{19}$$

$$\hat{r}_{1f} = \inf_{r_2 \in [\hat{r}_{1e} + \delta \log(T)/T, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) < scv_{r_2}^{\beta_T} \right\}, \tag{20}$$

$$\hat{r}_{2e} = \inf_{r_2 \in \left[\hat{r}_{1f}, 1\right]} \left\{ r_2 : BSADF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\}, \tag{21}$$

$$\hat{r}_{2f} = \inf_{r_2 \in [\hat{r}_{2e} + \delta \log(T)/T, 1]} \left\{ r_2 : BSADF_{r_2}(r_0) < scv_{r_2}^{\beta_T} \right\}. \tag{22}$$

An alternative implementation of the PWY procedure is to use that procedure sequentially, namely detect one bubble at a time. The dating criteria for the first bubble remains the same (i.e. equation (17)). Conditional on the first bubble having been found and terminated at $\hat{r}_{1f}$, the following dating criteria is used for a second bubble:

$$\hat{r}_{2e} = \inf_{r_2 \in \left(\hat{r}_{1f} + \varepsilon_T, 1\right]} \left\{ r_2 :_{\hat{r}_{1f}} BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} \text{ and } \hat{r}_{2f} = \inf_{r_2 \in [\hat{r}_{2e} + \log(T)/T, 1]} \left\{ r_2 :_{\hat{r}_{1f}} BDF_{r_2} < cv_{r_2}^{\beta_T} \right\}, \tag{23}$$

where ${}_{\hat{r}_{1f}}BDF_{r_2}$ is the ADF statistic calculated over $(\hat{r}_{1f}, r_2]$. Note that we need a few observations to initialize the procedure (i.e. $r_2 \in (\hat{r}_{1f} + \varepsilon_T, 1]$ for some $\varepsilon_T > 0$).[14]

We have the following asymptotic results for these dating estimates. Proofs of the theorems are given in Appendix B.

Theorem 3 *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$ and $\hat{r}_{2f}$ are obtained from the backward DF test based on the t statistic, (17) - (18). Given an alternative hypothesis of mildly explosive behavior of model (16) with $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$, if*

$$\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$ as $T \to \infty$; if

$$\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$$

and $\hat{r}_{1f} > \hat{r}_{1e} + \log(T)/T$, we have $\hat{r}_{1f} \xrightarrow{p} r_{1f}$ as $T \to \infty$; and $\hat{r}_{2e}$ and $\hat{r}_{2f}$ are not consistent estimators of r_{2e} and r_{2f}.

Theorem 4 *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$ and $\hat{r}_{2f}$ are obtained from the backward sup DF test based on the t statistic, (19) - (22). Given an alternative hypothesis of mildly explosive behavior of model (16) with $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$, if*

$$\frac{1}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$ as $T \to \infty$; if

$$\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0,$$

$\hat{r}_{1f} > \hat{r}_{1e} + \delta\log(T)/T$, we have $\hat{r}_{1f} \xrightarrow{p} r_{1f}$, $\hat{r}_{2e} \xrightarrow{p} r_{2e}$ and $\hat{r}_{2f} \xrightarrow{p} r_{2f}$ as $T \to \infty$.

[14] For example, $\varepsilon_T = \log T/T$ or $T^{-\delta}$ with some $\delta \in (0, 1)$.

Theorem 5 *Suppose $\hat{r}_{1e}$, $\hat{r}_{1f}$, $\hat{r}_{2e}$ and $\hat{r}_{2f}$ are obtained from the backward DF test based on the t statistic, (17) and (23). Given an alternative hypothesis of mildly explosive behavior of model (16) with $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$, if*

$$\frac{1}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0,$$

we have $\hat{r}_{1e} \xrightarrow{p} r_{1e}$ and $\hat{r}_{2e} \xrightarrow{p} r_{2e}$ as $T \to \infty$; if

$$\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0,$$

$\hat{r}_{1f} > \hat{r}_{1e} + \log(T)/T$, we have $\hat{r}_{1f} \xrightarrow{p} r_{1f}$ and $\hat{r}_{2f} \xrightarrow{p} r_{2f}$ as $T \to \infty$.

A restatement of Theorem 3 is useful. Suppose the sample period includes two bubble episodes and the duration of the first bubble is longer than the second. The strategy of PWY (corresponding to the SADF test) can consistently estimate the origination and termination of the first bubble but does not consistently estimate those of the second bubble. In contrast, Theorem 4 and Theorem 5 say that the new date-stamping strategy (corresponding to the GSADF test) and the alternative implementation of the PWY strategy can calculate the origination and termination of both bubbles consistently in this scenario.

We also analyze the consistency properties of these two date-stamping strategies when there are two bubbles and the duration of the first bubble is shorter than the second bubble. Under this circumstance, it turns out that all strategies consistently estimate the origination and termination dates of the two bubbles.

Theorem 3, 4 and 5 can be extended to a multiple bubbles scenario. Suppose there are N bubbles ($N > 2$). If the duration of the i^{th} bubble is longer than that of the j^{th} bubble, where $i, j \in \{1, 2, \cdots, N\}$ and $i < j$, then, the PWY strategy can consistently estimate the origination and termination dates of the i^{th} bubble but not those associated with the j^{th} bubble. In contrast, the new strategy and the alternative implementation of the PWY strategy can estimate dates associated with both bubbles consistently.

6 Empirical Application

The data comprise the real S&P 500 stock price index and the real S&P 500 stock price index dividend, both obtained from Robert Shiller's website. The data are sampled monthly over the period from January 1871 to December 2010, constituting 1,680 observations.

We apply the SADF test and the GSADF test to the price-dividend ratio (displayed earlier in Figure 1). Table 6 presents critical values for these two tests and these were obtained from $2,000$ Monte Carlo simulations with a sample size of $1,680$. In performing the ADF regressions and calculating critical values, the smallest window comprised 36 observations. From Table 6, the SADF and GSADF statistics for the full data series are 3.30 and 4.21. Both exceed their respective 1% right-tail critical values (i.e. $3.30 > 2.17$ and $4.21 > 3.31$), giving strong evidence that the S&P 500 price-dividend ratio had explosive subperiods. We conclude from both tests that there is evidence of bubbles in the S&P 500 stock market data.

Table 6: The SADF test and the GSADF test of the S&P 500 stock market

	SADF	GSADF
S&P500 Price-Dividend Ratio	3.30	4.21
Finite sample critical values		
90%	1.45	2.55
95%	1.70	2.80
99%	2.17	3.31

Note: Critical values of both tests are obtained from 2,000 Monte Carlo simulations with a sample size of 1,680. The smallest window has 36 observations.

To locate specific bubble periods, we compare the backward SADF statistic sequence with the 95% SADF critical value sequence, which is obtained as a by-product when simulating the critical values for the GSADF statistic. The top panel of Fig. 6 displays results for the date-stamping strategy over the period from January 1871 to December 1949 and the bottom panel displays results over the rest of the sample period. The identified exuberance and collapse periods include the explosive recovery phase following *the panic of 1873* (1878M07-1880M04), *the banking panic of 1907* (1907M09-1908M02), *the great crash episode* (1928M11-1929M09), *the postwar boom in 1954* (1954M09-1956M04), *the 1974 stock market crash* (1974M07-M12), *black*

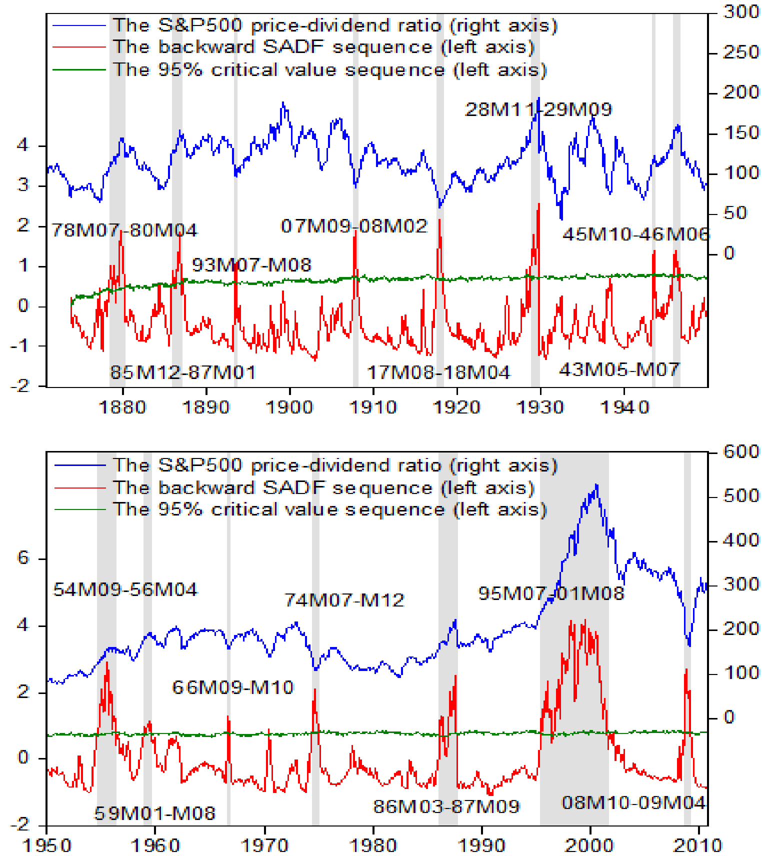

Figure 6: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the GSADF test

Monday in October 1987 (1986M03-1987M09), *the dot-com bubble* (1995M07-2001M08) and *the subprime mortgage crisis* (2008M10-2009M04). Notice that the new date-stamping strategy not only locates the explosive expansion periods but also identifies explosive collapse periods. Such market collapses have occured in the past when bubbles in other markets crashed and the collapse spread to the S&P 500 as, for instance, in *the banking panic of 1893* and *the subprime mortgage crisis.*

For comparison, we also plot the ADF statistic sequence against the 95% ADF critical value sequence. As seen in Fig. 7, the strategy of PWY (based on the SADF test) identifies only

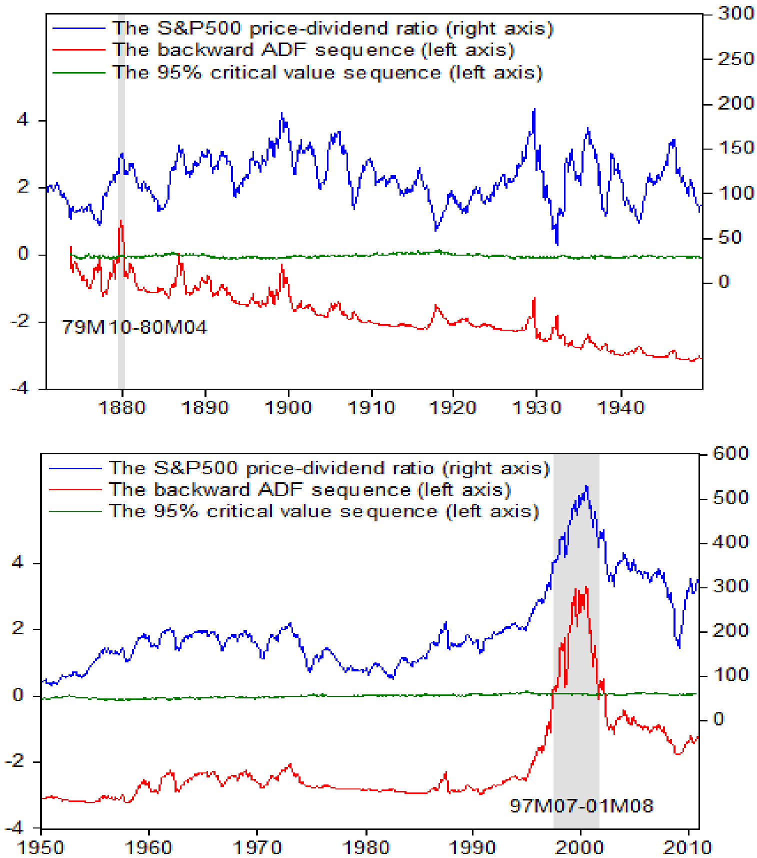

Figure 7: Date-stamping bubble periods in the S&P 500 price-dividend ratio: the SADF test.

two explosive periods – the recovery phase of *the panic of 1873* (1879M10-1880M04) and *the dot-com bubble* (1997M07-2001M08). In both cases, the estimated duration is shorter than that found by the GSADF dating strategy.

7 Conclusion and Implementation

The SADF test, which is also referred to as the forward recursive ADF test, implements the ADF test repeatedly on a sequence of forward expanding samples. The GSADF test can be viewed as a rolling window ADF test with a double-sup window selection criteria.[15] That is, we select a window size using the double-sup criteria and implement the ADF test repeatedly on a sequence of samples, which moves the window frame gradually toward the end of the sample. Experimenting on simulated asset prices reveals one of the shortcomings of the SADF test - its inability to find and locate bubbles when there are multiple collapsing episodes within the sample range. The GSADF test surmounts this problem and our simulation findings demonstrate that the GSADF test significantly improves discriminatory power in detecting bubbles.

The date-stamping strategy of PWY and the new date-stamping strategy are shown to have quite different behavior under the alternative of multiple bubbles. In particular, when the sample period includes two bubbles and the duration of the first bubble is longer than the second, the strategy of PWY fails to consistently estimate the timing of the second bubble while the new strategy consistently estimates and dates both bubbles.

We apply both SADF and GSADF tests, along with their date-stamping algorithms, to the S&P 500 price-dividend ratio from January 1871 to December 2010. Both tests find confirmatory evidence of bubble existence. The price-dividend ratio over this historical period contains many individual peaks and troughs, a trajectory that is similar to the multiple bubble scenario for which the PWY date-stamping strategy was found to be inconsistent. The empirical test results confirm the greater discriminatory power of the GSADF strategy found in the simulations and evidenced in the asymptotic theory. The new date-stamping strategy identifies all the well known historical episodes of banking crises and financial bubbles over this long period, whereas the SADF procedure locates only two episodes of exuberance and collapse.

To aid practitioners, we here provide a brief outline of the main steps involved in the empirical implementation of the new GSADF test and dating strategies. The Gauss and Matlab programs for implementing this algorithm are available for download from `https://sites.google.com/site/shupingshi/PrgGSADF.zip?attredirects=0\&d=1`.

(i) Select a sample size (T_0) as a minimum sample size for the within-windows recursive

[15] First, we calculate the sup value of the ADF statistic over the feasible ranges of the window starting points for a fixed window size. Then, we calculate the sup value of the SADF statistic over the feasible range of window sizes.

regressions. In our empirical application to the S&P data we used $T_0 = 36$ (equivalent to 3 years) for which the ratio $r_0 = T_0/T = 36/1860$.

(ii) From observation T_0+i, where $i = 0, 1, \ldots, T-T_0$, gather a backward expanding sequence of samples with end point at observation $T_0 + i$ and start point selected from $\{i+1, i, \ldots, 1\}$.

(iii) Conduct a right-tailed ADF unit root test on the backward expanding sample sequence to obtain an ADF statistic sequence;

(iv) Calculate the maximum value of the ADF statistic sequence. This is the BSADF statistic.

(v) Repeat steps (iii) to (iv) for each $i = 0, 1, \ldots, T - T_0$ to obtain the BSADF statistic sequence.

(vi) Calculate the GSADF statistic, which is the maximum value of the BSADF statistic sequence, and test significance of this statistic against its critical values for inference about bubble existence.

(vii) If the GSADF test shows evidence of bubble existence, compare the BSADF statistic sequence obtained in (v) with the critical values for the sup ADF statistic to locate bubble episodes.

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APPENDIX A. The date-stamping strategies (a single bubble)

Notation and useful preliminary lemmas

We define the following notation:

- The bubble period $B = [\tau_e, \tau_f]$, where $\tau_e = \lfloor Tr_e \rfloor$ and $\tau_f = \lfloor Tr_f \rfloor$.
- The normal market periods $N_0 = [1, \tau_e)$ and $N_1 = [\tau_f + 1, \tau]$, where $\tau = \lfloor Tr \rfloor$ is the last observation of the sample.
- The starting point of the regression $\tau_1 = \lfloor Tr_1 \rfloor$, the ending point of the regression $\tau_2 = \lfloor Tr_2 \rfloor$, the regression sample size $\tau_w = \lfloor Tr_w \rfloor$ with $r_w = r_2 - r_1$ and observation $t = \lfloor Tp \rfloor$.
- $B(p) \equiv \sigma W(p)$, where W is a Wiener process.

We use the data generating process

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{for } t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t & \text{for } t \in B \\ X^*_{\tau_f} + \sum_{k=\tau_f+1}^{t} \varepsilon_k & \text{for } t \in N_1 \end{cases}, \tag{24}$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0,1)$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ and $X^*_{\tau_f} = X_{\tau_e} + X^*$ with $X^* = O_p(1)$. Under (24) we have the following lemmas.

Lemma 8.1 *Under the generating process (24),*
(1) *For* $t \in N_0$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p)$.
(2) *For* $t \in B$, $X_{t=\lfloor Tp \rfloor} = \delta_T^{t-\tau_e} X_{\tau_e} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-\tau_e} B(r_e)$.
(3) *For* $t \in N_1$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_f) + B(r_e)]$.

Proof. (1) For $t \in N_0$, X_t is a unit root process. We know that $T^{-1/2} X_{t=\lfloor Tp \rfloor} \xrightarrow{L} B(p)$ as $T \to \infty$. (2) For $t \in B$, the generating mechanism is

$$X_t = \delta_T X_{t-1} + \varepsilon_t = \delta_T^{t-\tau_e} X_{\tau_e} + \sum_{j=0}^{t-\tau_e-1} \delta_T^j \varepsilon_{t-j}.$$

Based on Phillips and Magdalinos (2007, lemma 4.2), we know that for $\alpha < 1$,

$$T^{-\alpha/2} \sum_{j=0}^{t-\tau_e-1} \delta_T^{-(t-\tau_e)+j} \varepsilon_{t-j} \xrightarrow{L} X_c \equiv N\left(0, \sigma^2/2c\right)$$

as $t-\tau_e \to \infty$. Furthermore, we know that $T^{-1/2}X_{\tau_e-1} \xrightarrow{L} B(r_e)$ and hence

$$\delta_T^{-(t-\tau_e)}T^{-1/2}X_t = T^{-1/2}X_{\tau_e-1} + T^{-(1-\alpha)/2}T^{-\alpha/2}\sum_{j=0}^{t-\tau_e-1}\delta_T^{-(t-\tau_e)+j}\varepsilon_{t-j} \xrightarrow{L} B(r_e).$$

This implies that the first term has a higher order than the second term and hence,

$$X_t = \delta_T^{t-\tau_e}X_{\tau_e}\left\{1+\frac{\sum_{j=0}^{t-\tau_e-1}\delta_T^j\varepsilon_{t-j}}{\delta_T^{t-\tau_e}X_{\tau_e}}\right\} = \delta_T^{t-\tau_e}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{1/2}\delta_T^{t-\tau_e}B(r_e).$$

(3) For $t \in N_1$,

$$X_t = \sum_{k=\tau_f+1}^{t}\varepsilon_k + X_{\tau_f}^* = \sum_{k=\tau_f+1}^{t}\varepsilon_k + X_{\tau_e} + X^* \sim_a T^{1/2}\left[B(p) - B(r_f) + B(r_e)\right]$$

due to the fact that $X_{\tau_e} \sim_a T^{1/2}B(r_e)$, $\sum_{k=\tau_f+1}^{t}\varepsilon_k \sim_a T^{1/2}\left[B(p)-B(r_f)\right]$ and $X^* = O_p(1)$.

■

Lemma 8.2 *Under the data generating process (24),*
(1) *For $\tau_1 \in N_0$ and $\tau_2 \in B$,*

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}\frac{1}{r_w c}B(r_e).$$

(2) *For $\tau_1 \in B$ and $\tau_2 \in N_1$,*

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j = \frac{T^{\alpha}\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}\frac{1}{r_w c}B(r_e).$$

(3) *For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,*

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j = X_{\tau_e}\frac{T^{\alpha}\delta_T^{\tau_f-\tau_e}}{\tau_w c}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\frac{1}{r_w c}B(r_e).$$

Proof. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$, we have

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j + \frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_2}X_j.$$

The first term is

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j = T^{1/2}\frac{\tau_e-\tau_1}{\tau_w}\left(\frac{1}{\tau_e-\tau_1}\sum_{j=\tau_1}^{\tau_e-1}\frac{X_j}{\sqrt{T}}\right) \sim_a T^{1/2}\frac{r_e-r_1}{r_w}\int_{r_1}^{r_e}B(s)\,ds. \qquad (25)$$

The second term is

$$\begin{aligned}
\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_2} X_j &= \frac{X_{\tau_e}}{\tau_w}\sum_{j=\tau_e}^{\tau_2}\delta_T^{j-\tau_e}\left\{1+o_p(1)\right\} \quad \text{from Lemma 8.1} \\
&= \frac{1}{\tau_w}\frac{\delta_T^{\tau_2-\tau_e+1}-1}{\delta_T-1}X_{\tau_e}\left\{1+o_p(1)\right\} \\
&= \frac{T^\alpha\delta_T^{\tau_2-\tau_e}+c\delta_T^{\tau_2-\tau_e}-T^\alpha}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \\
&= \frac{T^\alpha\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}\frac{1}{r_w c}B(r_e). \qquad (26)
\end{aligned}$$

Since

$$\frac{T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}}{T^{1/2}} = \frac{\delta_T^{\tau_2-\tau_e}}{T^{1-\alpha}} = \frac{e^{c(r_2-r_e)T^{1-\alpha}}}{T^{1-\alpha}} > 1,$$

$\tau_w^{-1}\sum_{j=\tau_e}^{\tau_2}X_j$ has a higher order than $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_e-1}X_j$ and hence

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_2-\tau_e}\frac{1}{r_w c}B(r_e).$$

(2) For $\tau_1 \in B$ and $\tau_2 \in N_1$, we have

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2} X_j = \frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_f} X_j + \frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2} X_j.$$

The first term is

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_f} X_j = \frac{T^\alpha\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\left\{1+o_p(1)\right\} \sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}\frac{1}{r_w c}B(r_e).$$

The second term is

$$\begin{aligned}
&\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2} X_j \\
&= \frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2}\left[\sum_{k=\tau_f+1}^{j}\varepsilon_k + X_{\tau_e}\right] \\
&= T^{1/2}\frac{\tau_2-\tau_f}{\tau_w}\left[\frac{1}{\tau_2-\tau_f}\sum_{j=\tau_f+1}^{\tau_2}\left(T^{-1/2}\sum_{k=\tau_f+1}^{j}\varepsilon_k\right)\right] + T^{1/2}\frac{\tau_2-\tau_f}{\tau_w}\left(T^{-1/2}X_{\tau_e}\right)
\end{aligned}$$

$$\sim_a T^{1/2}\frac{r_2-r_f}{r_w}\int_{r_f}^{r_2}\left[B(s)-B(r_f)\right]ds+T^{1/2}\frac{r_2-r_f}{r_w}B(r_e)$$
$$=T^{1/2}\frac{r_2-r_f}{r_w}\left\{\int_{r_f}^{r_2}\left[B(s)-B(r_f)\right]ds-B(r_e)\right\}. \tag{27}$$

We know that $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_f}X_j$ has a higher order than $\tau_w^{-1}\sum_{j=\tau_f+1}^{\tau_2}X_j$ and hence

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{T^\alpha\delta_T^{\tau_f-\tau_1}}{\tau_w c}X_{\tau_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_1}\frac{1}{r_w c}B(r_e).$$

(3) For $\tau_1\in N_0$ and $\tau_2\in N_1$,

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j+\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f}X_j+\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2}X_j.$$

Since

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_e-1}X_j\sim_a T^{1/2}\frac{r_e-r_1}{r_w}\int_{r_1}^{r_e}B(s)\,ds\quad\text{from (25)},$$
$$\frac{1}{\tau_w}\sum_{j=\tau_e}^{\tau_f}X_j=\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\frac{1}{r_w c}B(r_e),$$
$$\frac{1}{\tau_w}\sum_{j=\tau_f+1}^{\tau_2}X_j\sim_a T^{1/2}\frac{r_2-r_f}{r_w}\left\{\int_{r_f}^{r_2}\left[B(s)-B(r_f)\right]ds-B(r_e)\right\}\quad\text{from (27)},$$

it follows that $\tau_w^{-1}\sum_{j=\tau_e}^{\tau_f}X_j$ dominates $\tau_w^{-1}\sum_{j=\tau_1}^{\tau_e-1}X_j$ and $\tau_w^{-1}\sum_{j=\tau_f+1}^{\tau_2}X_j$ and hence

$$\frac{1}{\tau_w}\sum_{j=\tau_1}^{\tau_2}X_j=\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}\{1+o_p(1)\}\sim_a T^{\alpha-1/2}\delta_T^{\tau_f-\tau_e}\frac{1}{c r_w}B(r_e).$$

■

Lemma 8.3 *Define the centered quantity* $\tilde{X}_t=X_t-\tau_w^{-1}\sum_{j=\tau_1}^{\tau_2}X_j$.

(1) *For* $\tau_1\in N_0$ *and* $\tau_2\in B$,

$$\tilde{X}_t=\begin{cases}-\frac{T^\alpha\delta_T^{\tau_2-\tau_e}}{\tau_w c}X_{\tau_e}\{1+o_p(1)\} & \text{if } t\in N_0\\ \left[\delta_T^{t-\tau_e}-\frac{T^\alpha\delta_T^{\tau_2-\tau_e}}{\tau_w c}\right]X_{\tau_e}\{1+o_p(1)\} & \text{if } t\in B\end{cases}.$$

(2) *For* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c}\right] X_{\tau_e} \{1 + o_p(1)\} & \text{if } t \in B \\ -\frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c} X_{\tau_e} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}.$$

(3) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} \{1 + o_p(1)\} & \text{if } t \in N_0 \cup N_1 \\ \left[\delta_T^{t-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f - \tau_e}}{\tau_w c}\right] X_{\tau_e} \{1 + o_p(1)\} & \text{if } t \in B \end{cases}.$$

Proof. (1) Suppose $\tau_1 \in N_0$ and $\tau_2 \in B$. If $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_2 - \tau_e}}{\tau_w c} X_{\tau_e} \{1 + o_p(1)\}, \tag{28}$$

where the second term dominates the first term due to the fact that

$$\begin{aligned} X_t &\sim_a T^{1/2} B(p) \text{ from Lemma 8.1} \\ \frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j &\sim_a T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_e} \frac{1}{r_w c} B(r_e) \text{ from Lemma 8.2.} \end{aligned}$$

If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_e} - \frac{T^\alpha \delta_T^{\tau_2 - \tau_e}}{\tau_w c}\right] X_{\tau_e} \{1 + o_p(1)\}.$$

(2) Suppose $\tau_1 \in B$ and $\tau_2 \in N_1$. If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c}\right] X_{\tau_e} \{1 + o_p(1)\}.$$

If $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^\alpha \delta_T^{\tau_f - \tau_1}}{\tau_w c} X_{\tau_e} \{1 + o_p(1)\},$$

where the second term dominates the first term due to the fact that

$$\begin{aligned} X_{t=\lfloor Tp \rfloor} &\sim_a T^{1/2} \left[B(p) - B(r_f) + B(r_e)\right] \text{ from Lemma 8.1} \\ \frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j &\sim_a T^{\alpha - 1/2} \delta_T^{\tau_f - \tau_1} \frac{1}{r_w c} B(r_e) \text{ from Lemma 8.2.} \end{aligned}$$

(3) Suppose $\tau_1 \in N_0$ and $\tau_2 \in N_1$. If $t \in N_0$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\},$$

where the second term dominates the first term due to the fact that

$$\begin{aligned} X_{t=\lfloor Tp \rfloor} &\sim_a T^{1/2} B(p) \text{ from Lemma 8.1} \\ \frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j &\sim_a T^{\alpha - 1/2} \delta_T^{\tau_f - \tau_e} \frac{1}{r_w c} B(r_e) \text{ from Lemma 8.2.} \end{aligned}$$

If $t \in B$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = \left[\delta_T^{t - \tau_e} - \frac{T^{\alpha} \delta_T^{\tau_f - \tau_e}}{\tau_w c}\right] X_{\tau_e} \left\{1 + o_p(1)\right\}.$$

If $t \in N_1$,

$$\tilde{X}_t = X_t - \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} X_j = -\frac{T^{\alpha} \delta_T^{\tau_f - \tau_e}}{\tau_w c} X_{\tau_e} \left\{1 + o_p(1)\right\},$$

due to the fact that $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} \left[B(p) - B(r_f) + B(r_e)\right]$. ■

Lemma 8.4 *The sample variance terms involving $\tilde{X}_t$ behave as follows.*

(1) *For $\tau_1 \in N_0$ and $\tau_2 \in B$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_2 - \tau_e)}}{2c} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{1+\alpha} \delta_T^{2(\tau_2 - \tau_e)}}{2c} B(r_e)^2.$$

(2) *For $\tau_1 \in B$ and $\tau_2 \in N_1$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_f - \tau_e)}}{2c} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f - \tau_e)}}{2c} B(r_e)^2.$$

(3) *For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha} \delta_T^{2(\tau_f - \tau_e)}}{2c} X_{\tau_e}^2 \left\{1 + o_p(1)\right\} \sim_a \frac{T^{\alpha+1} \delta_T^{2(\tau_f - \tau_e)}}{2c} B(r_e)^2.$$

Proof. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e} \tilde{X}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2.$$

The first term is

$$\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e-1}\frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}}{\tau_w^2c^2}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \text{ from Lemma 8.3}$$

$$= \frac{\tau_e-\tau_1}{\tau_w^2c^2}T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \sim_a \frac{r_e-r_1}{r_w^2c}T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}B\left(r_e\right).$$

Given that

$$\sum_{j=\tau_e}^{\tau_2}\delta_T^{2(j-1-\tau_e)} = \frac{\delta_T^{2(\tau_2-\tau_e)}-\delta_T^{-2}}{\delta_T^2-1} = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}\left\{1+o_p\left(1\right)\right\}$$

$$\sum_{j=\tau_e}^{\tau_2}\delta_T^{j-1-\tau_e} = \frac{\delta_T^{\tau_2-\tau_e}-\delta_T^{-1}}{\delta_T-1} = \frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{c}\left\{1+o_p\left(1\right)\right\},$$

the second term is

$$\begin{aligned}
&\sum_{j=\tau_e}^{\tau_2}\tilde{X}_{j-1}^2 \\
&= \sum_{j=\tau_e}^{\tau_2}\left[\delta_T^{j-1-\tau_e}-\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_wc}\right]^2X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \\
&= \sum_{j=\tau_e}^{\tau_2}\left[\delta_T^{2(j-1-\tau_e)}-2\delta_T^{j-1-\tau_e}\frac{T^{\alpha}\delta_T^{\tau_2-\tau_e}}{\tau_wc}+\frac{T^{2\alpha}\delta_T^{2(\tau_2-\tau_e)}}{\tau_w^2c^2}\right]X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \\
&= \left[\frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}-2\frac{T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_e)}}{r_wc^2}+\frac{r_2-r_e+\frac{1}{T}}{r_w^2c^2}T^{2\alpha-1}\delta_T^{2(\tau_2-\tau_e)}\right]X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \\
&= \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \text{ (since } \alpha>2\alpha-1) \\
&\sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}B\left(r_e\right)^2.
\end{aligned}$$

Since $1+\alpha>2\alpha$, $\sum_{j=\tau_e}^{\tau_2}\tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_1}^{\tau_e}\tilde{X}_{j-1}^2$ and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2 = \frac{T^{\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\} \sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_e)}}{2c}B\left(r_e\right)^2.$$

(2) For $\tau_1\in B$ and $\tau_2\in N_1$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_f}\tilde{X}_{j-1}^2+\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}^2.$$

Since

$$\sum_{j=\tau_1}^{\tau_f}\tilde{X}_{j-1}^2=\sum_{j=\tau_1}^{\tau_f}\left[\delta_T^{j-1-\tau_e}-\frac{T^\alpha\delta_T^{\tau_f-\tau_1}}{\tau_w c}\right]^2 X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\quad\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}}{2c}B\left(r_e\right)^2,$$

$$\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}^2=\sum_{j=\tau_f+1}^{\tau_2}\frac{T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_1\right)}}{\tau_w^2c^2}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{r_2-r_f}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_1\right)}B\left(r_e\right)^2,$$

the quantity $\sum_{j=\tau_1}^{\tau_f}\tilde{X}_{j-1}^2$ dominates $\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}^2$ and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\frac{T^\alpha\delta_T^{2\left(\tau_f-\tau_e\right)}}{2c}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}}{2c}B\left(r_e\right)^2.$$

(3) For $\tau_1\in N_0$ and $\tau_2\in N_1$,

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^2+\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2+\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}^2.$$

Since

$$\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}^2=\sum_{j=\tau_1}^{\tau_e-1}\frac{T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_e\right)}}{\tau_w^2c^2}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_e\right)}\frac{r_e-r_1}{r_w^2c^2}B\left(r_e\right)^2,$$

$$\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2=\sum_{j=\tau_e}^{\tau_f}\left[\delta_T^{j-1-\tau_e}-\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{\tau_w c}\right]^2 X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}\frac{B\left(r_e\right)^2}{2c},$$

$$\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}^2=\sum_{j=\tau_f+1}^{\tau_2}\frac{T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_e\right)}}{\tau_w^2c^2}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{r_2-r_f}{r_w^2c^2}T^{2\alpha}\delta_T^{2\left(\tau_f-\tau_e\right)}B\left(r_e\right)^2,$$

the component $\sum_{j=\tau_e}^{\tau_f}\tilde{X}_{j-1}^2$ dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2=\frac{T^\alpha\delta_T^{2\left(\tau_f-\tau_e\right)}}{2c}X_{\tau_e}^2\left\{1+o_p\left(1\right)\right\}\sim_a\frac{T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}}{2c}B\left(r_e\right)^2.$$

■

Lemma 8.5 *The sample covariance of $\tilde{X}_t$ and ε_t behaves as follows.*

(1) *For $\tau_1\in N_0$ and $\tau_2\in B$,*

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j=\sum_{j=\tau_e}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j\left\{1+o_p\left(1\right)\right\}\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_e}X_cB\left(r_e\right).$$

(2) *For* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \left\{1 + o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

(3) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \left\{1 + o_p(1)\right\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

Proof. (1) For $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

The first term is

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j &= \sum_{j=\tau_1}^{\tau_e-1} -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c} X_{\tau_e}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{r_w c}\left(T^{-1/2}X_{\tau_e}\right)\left(T^{-1/2}\sum_{j=\tau_1}^{\tau_e-1}\varepsilon_j\right)\left\{1 + o_p(1)\right\} \\
&\sim_a -\frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{r_w c} B(r_e)\left[B(r_e) - B(r_1)\right].
\end{aligned}$$

The second term is

$$\begin{aligned}
&\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \\
&= \sum_{j=\tau_e}^{\tau_2}\left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_2-\tau_e}}{\tau_w c}\right] X_{\tau_e}\varepsilon_j \left\{1 + o_p(1)\right\} \\
&= \left[T^{\alpha/2}\delta_T^{\tau_2-\tau_e}\left(\frac{1}{T^{\alpha/2}}\sum_{j=\tau_e}^{\tau_2}\delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) - \frac{\delta_T^{\tau_2-\tau_e}}{T^{1/2-\alpha}r_w c}\left(\frac{1}{\sqrt{T}}\sum_{j=\tau_e}^{\tau_2}\varepsilon_j\right)\right] X_{\tau_e}\left\{1 + o_p(1)\right\} \\
&= T^{\alpha/2}\delta_T^{\tau_2-\tau_e}\left(T^{-\alpha/2}\sum_{j=\tau_e}^{\tau_2}\delta_T^{-(\tau_2-j+1)}\varepsilon_j\right) X_{\tau_e}\left\{1 + o_p(1)\right\} \ \ (\text{since } \alpha/2 > \alpha - 1/2) \\
&\sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_e} X_c B(r_e).
\end{aligned}$$

Since $(\alpha+1)/2 > \alpha$, $\sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j$ and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_e} X_c B(r_e).$$

(2) For $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

Since

$$\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c}\right] X_{\tau_e}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e),$$

$$\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_f+1}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c} X_{\tau_e}\varepsilon_j \{1+o_p(1)\} \sim_a -\frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{r_w c} B(r_e)\left[B(r_2)-B(r_f)\right],$$

the quantity $\sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j$ dominates $\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

(3) For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j.$$

Since

$$\sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_1}^{\tau_e-1} -\frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{\tau_w c} X_{\tau_e}\varepsilon_j \{1+o_p(1)\} \sim_a -\frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{r_w c} B(r_e)\left[B(r_e)-B(r_1)\right],$$

$$\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_f} \left[\delta_T^{j-1-\tau_e} - \frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{\tau_w c}\right] X_{\tau_e}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e),$$

$$\sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_f+1}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{\tau_w c} X_{\tau_e}\varepsilon_j \{1+o_p(1)\} \sim_a -\frac{T^\alpha \delta_T^{\tau_f-\tau_e}}{r_w c} B(r_e)\left[B(r_2)-B(r_f)\right],$$

and $(\alpha+1)/2 > \alpha$, the component $\sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j$ dominates the other two terms and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j \{1+o_p(1)\} \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e} X_c B(r_e).$$

■

Lemma 8.6

The sample covariance of $\tilde{X}_{j-1}$ *and* $X_j - \delta_T X_{j-1}$ *behaves as follows.*

(1) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a \frac{r_e - r_1}{r_w} T \delta_T^{\tau_2 - \tau_e} B\left(r_e\right) \int_{r_1}^{r_e} B\left(s\right) ds.$$

(2) *For* $\tau_1 \in B$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_f - \tau_e} X_c B\left(r_e\right).$$

(3) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T \delta_T^{\tau_f - \tau_e} B\left(r_e\right) \left[\frac{r_2 - r_f}{r_w} \int_{r_f}^{r_2} B\left(s\right) ds + \frac{r_e - r_1}{r_w} \int_{r_1}^{r_e} B\left(s\right) ds \right].$$

Proof. (1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\begin{aligned} \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) &= \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1} \left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1} \right) \\ &= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j - \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1} X_{j-1}. \end{aligned} \tag{29}$$

The first term is

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_2 - \tau_e} X_c B\left(r_e\right) \text{ from Lemma 8.5.}$$

The second term is

$$\begin{aligned} & \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1} X_{j-1} \\ &= \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e - 1} - \frac{T^\alpha \delta_T^{\tau_2 - \tau_e}}{\tau_w c} X_{\tau_e} X_{j-1} \left\{ 1 + o_p\left(1\right) \right\} \\ &= - \frac{\tau_e - \tau_1}{\tau_w} T \delta_T^{\tau_2 - \tau_e} \left(T^{-1/2} X_{\tau_e} \right) \left[\frac{1}{\tau_e - \tau_1} \sum_{j=\tau_1}^{\tau_e - 1} \left(T^{-1/2} X_{j-1} \right) \right] \left\{ 1 + o_p\left(1\right) \right\} \end{aligned}$$

$$\sim_a -\frac{r_e - r_1}{r_w} T\delta_T^{\tau_2-\tau_e} B\left(r_e\right) \int_{r_1}^{r_e} B\left(s\right) ds.$$

Since $\left(\alpha+1\right)/2 < 1$, the quantity $\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1}$ dominates $\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) = -\frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \sim_a \frac{r_e - r_1}{r_w} T\delta_T^{\tau_2-\tau_e} B\left(r_e\right) \int_{r_1}^{r_e} B\left(s\right) ds.$$

(2) When $\tau_1 \in B$ and $\tau_2 \in N$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) = \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha} \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1} X_{j-1}.$$

Since

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_f-\tau_e} X_c B\left(r_e\right) \text{ from Lemma 8.5,}$$

$$\frac{c}{T^\alpha} \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1} X_{j-1} = \frac{c}{T^\alpha} \sum_{j=\tau_f+1}^{\tau_2} -\frac{T^\alpha \delta_T^{\tau_f-\tau_1}}{\tau_w c} X_{\tau_e} X_{j-1} \left\{1 + o_p\left(1\right)\right\} \sim_a -\frac{r_2 - r_f}{r_w} T\delta_T^{\tau_f-\tau_1} B\left(r_e\right) \int_{r_f}^{r_2} B\left(s\right) ds,$$

and

$$\frac{T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e}}{T\delta_T^{\tau_f-\tau_1}} = \frac{\delta_T^{\tau_1-\tau_e}}{T^{(1-\alpha)/2}} = \frac{e^{c(r_1-r_e)T^{1-\alpha}}}{T^{(1-\alpha)/2}} > 1,$$

the component $\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$ dominates $\frac{c}{T^\alpha} \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1} X_{j-1}$ and hence

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) = \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \left\{1 + o_p\left(1\right)\right\} \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_f-\tau_e} X_c B\left(r_e\right).$$

(3) When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\begin{aligned}
\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\left(X_j - \delta_T X_{j-1}\right) &= \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}\varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) + \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1}\left(\varepsilon_j - \frac{c}{T^\alpha} X_{j-1}\right) \\
&= \sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j - \frac{c}{T^\alpha} \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} X_{j-1} - \frac{c}{T^\alpha} \sum_{j=\tau_f+1}^{\tau_2} \tilde{X}_{j-1} X_{j-1}.
\end{aligned}$$

Since

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_f-\tau_e} X_c B\left(r_e\right) \text{ from Lemma 8.5,}$$

$$\frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_e-1}\tilde{X}_{j-1}X_{j-1}=\frac{c}{T^\alpha}\sum_{j=\tau_1}^{\tau_e-1}-\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}X_{j-1}\left\{1+o_p\left(1\right)\right\}\sim_a-\frac{r_e-r_1}{r_w}T\delta_T^{\tau_f-\tau_e}B\left(r_e\right)\int_{r_1}^{r_e}B\left(s\right)ds,$$

$$\frac{c}{T^\alpha}\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}X_{j-1}=\frac{c}{T^\alpha}\sum_{j=\tau_f+1}^{\tau_2}-\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{\tau_w c}X_{\tau_e}X_{j-1}\left\{1+o_p\left(1\right)\right\}\sim_a-\frac{r_2-r_f}{r_w}T\delta_T^{\tau_f-\tau_e}B\left(r_e\right)\int_{r_f}^{r_2}B\left(s\right)ds,$$

and $(\alpha+1)/2<1$, $\frac{c}{T^\alpha}\sum_{j=1}^{\tau_e-1}\tilde{X}_{j-1}X_{j-1}$ and $\frac{c}{T^\alpha}\sum_{j=\tau_f+1}^{\tau_2}\tilde{X}_{j-1}X_{j-1}$ dominate $\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\varepsilon_j$ and hence

$$\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)\sim_a T\delta_T^{\tau_f-\tau_e}B\left(r_e\right)\left[\frac{r_2-r_f}{r_w}\int_{r_f}^{r_2}B\left(s\right)ds+\frac{r_e-r_1}{r_w}\int_{r_1}^{r_e}B\left(s\right)ds\right].$$

■

Test asymptotics

The regression model used for the Dickey-Fuller test is

$$X_t=\alpha_{r_1,r_2}+\delta_{r_1,r_2}X_{t-1}+\varepsilon_t,\varepsilon_t\overset{iid}{\sim}N\left(0,\sigma^2_{r_2.r_w}\right).$$

First, we calculate the asymptotic distribution of the Dickey-Fuller statistic under the alternative hypothesis. Based on Lemma 8.4 and Lemma 8.6, we can obtain the limit distribution of $\hat{\delta}_{r_1,r_2}-\delta_T$. When $\tau_1\in N_0$ and $\tau_2\in B$,

$$\frac{T^\alpha\delta_T^{\tau_2-\tau_e}}{2c}\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)=\frac{\frac{1}{T\delta_T^{\tau_2-\tau_e}}\sum_{j=1}^{\tau}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)}{\frac{2c}{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_e)}}\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2}\xrightarrow{L}\frac{\left(r_e-r_1\right)\int_{r_1}^{r_e}B\left(s\right)ds}{r_wB\left(r_e\right)};$$

when $\tau_1\in B$ and $\tau_2\in N_1$,

$$\frac{T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e}}{2c}\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)=\frac{\frac{1}{T^{(\alpha+1)/2}\delta_T^{\tau_f-\tau_e}}\sum_{j=1}^{\tau}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)}{\frac{2c}{T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}}\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2}\xrightarrow{L}\frac{X_c}{B\left(r_e\right)};$$

when $\tau_1\in N_0$ and $\tau_2\in N_1$,

$$\frac{T^\alpha\delta_T^{\tau_f-\tau_e}}{2c}\left(\hat{\delta}_{r_1,r_2}-\delta_T\right)=\frac{\frac{1}{T\delta_T^{\tau_f-\tau_e}}\sum_{j=1}^{\tau}\tilde{X}_{j-1}\left(X_j-\delta_T X_{j-1}\right)}{\frac{2c}{T^{\alpha+1}\delta_T^{2\left(\tau_f-\tau_e\right)}}\sum_{j=\tau_1}^{\tau_2}\tilde{X}_{j-1}^2}$$

$$\xrightarrow{L} \frac{(r_2 - r_f) \int_{r_f}^{r_2} B(s)\, ds + (r_e - r_1) \int_{r_1}^{r_e} B(s)\, ds}{r_w B(r_e)}.$$

The asymptotic distribution of Dickey-Fuller coefficient statistic (denoted DF^z) is as follows. When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w \left(\hat{\delta}_{r_1,r_2} - 1\right) = \tau_w (\delta_T - 1) + \tau_w \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \\ &= \tau_w (\delta_T - 1) + o_p \left(2cr_w \frac{T^{1-\alpha}}{\delta_T^{\tau_2 - \tau_e}}\right) \\ &= r_w c T^{1-\alpha} + o_p(1) \to \infty; \end{aligned}$$

when $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w \left(\hat{\delta}_{r_1,r_2} - 1\right) = \tau_w (\delta_T - 1) + \tau_w \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \\ &= \tau_w (\delta_T - 1) + o_p \left(2cr_w \frac{T^{(1-\alpha)/2}}{\delta_T^{\tau_f - \tau_e}}\right) \\ &= r_w c T^{1-\alpha} + o_p(1) \to \infty; \end{aligned}$$

when $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\begin{aligned} DF^z_{r_1,r_2} &= \tau_w \left(\hat{\delta}_{r_1,r_2} - 1\right) = \tau_w (\delta_T - 1) + \tau_w \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \\ &= \tau_w (\delta_T - 1) + o_p \left(2cr_w \frac{T}{T^{\alpha} \delta_T^{\tau_f - \tau_e}}\right) \\ &= r_w c T^{1-\alpha} + o_p(1) \to \infty. \end{aligned}$$

Therefore, for all cases, we have $\hat{\delta}_{r_1,r_2} - 1 \sim_a T^{-\alpha} c$ or $T^{\alpha}\left(\hat{\delta}_{r_1,r_2} - 1\right) \xrightarrow{L} c$.

To obtain the asymptotic distribution of the Dickey-Fuller t-statistic, we need to estimate the standard error of $\hat{\delta}_{r_1,r_2}$. (1) When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$\begin{aligned} Var\left(\hat{\delta}_{r_1,r_2}\right) &= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\delta}_{r_1,r_2} \tilde{X}_{j-1}\right)^2 \\ &= \tau_w^{-1} \left[\sum_{j=\tau_1}^{\tau_e - 1} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1\right) \tilde{X}_{j-1}\right]^2 + \sum_{j=\tau_e}^{\tau_2} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tilde{X}_{j-1}\right]^2\right] \end{aligned}$$

$$= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}^2 + \left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2$$

$$- 2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}\varepsilon_j - 2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j$$

$$= \left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2 \sim_a \frac{2c}{T^\alpha} \frac{(r_e - r_1)^2}{r_w^3} \left[\int_{r_1}^{r_e} B(s)\, ds\right]^2.$$

The term $\left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2$ dominates the other terms due to the fact that

$$\left(\hat{\delta}_{r_1,r_2} - 1\right)^2 \tau_w{}^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}^2 = O_p\left(T^{-2\alpha}\right) O_p\left(T^{2\alpha - 1}\delta_T^{2(\tau_2 - \tau_e)}\right) = O_p\left(T^{-1}\delta_T^{2(\tau_2 - \tau_e)}\right),$$

$$\left(\hat{\delta}_{r_1,r_2} - \delta_T\right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}^2 = O_p\left(\frac{1}{T^{2\alpha}\delta_T^{2(\tau_2 - \tau_e)}}\right) O_p\left(T^\alpha \delta_T^{2(\tau_2 - \tau_e)}\right) = O_p\left(T^{-\alpha}\right),$$

$$2\left(\hat{\delta}_{r_1,r_2} - 1\right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_e - 1} \tilde{X}_{j-1}\varepsilon_j = O_p\left(T^{-\alpha}\right) O_p\left(\frac{\delta_T^{\tau_2 - \tau_e}}{T^{1-\alpha}}\right) = O_p\left(T^{-1}\delta_T^{\tau_2 - \tau_e}\right),$$

$$2\left(\hat{\delta}_{r_1,r_2} - \delta_T\right) \tau_w^{-1} \sum_{j=\tau_e}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j = O_p\left(\frac{1}{T^\alpha \delta_T^{\tau_2 - \tau_e}}\right) O_p\left(\frac{\delta_T^{\tau_2 - \tau_e}}{T^{(1-\alpha)/2}}\right) = O_p\left(T^{-(1+3\alpha)/2}\right).$$

(2) When $\tau_1 \in B$ and $\tau_2 \in N_1$, we know that

$$\tilde{X}_{\tau_f + 1} - \hat{\delta}_{r_1,r_2}\tilde{X}_{\tau_f} - \varepsilon_{\tau_f + 1}$$
$$= \left[X_{\tau_f+1} - \bar{X}\right] - \hat{\delta}_{r_1,r_2}\left[X_{\tau_f} - \bar{X}\right] - \varepsilon_{\tau_f+1}$$
$$= \left[\varepsilon_{\tau_f+1} + X_{\tau_e} + X^* - \bar{X}\right] - \hat{\delta}_{r_1,r_2}\left[X_{\tau_f} - \bar{X}\right] - \varepsilon_{\tau_f+1}$$
$$= X_{\tau_e} + X^* - \tilde{X}_{\tau_f} - \left[\hat{\delta}_{r_1,r_2} - 1\right]\tilde{X}_{\tau_f}$$
$$= O_p\left(T^{1/2}\right) + O_p(1) - O_p\left(T^{1/2}\delta_T^{\tau_f - \tau_e}\right) - O_p\left(T^{1/2-\alpha}\delta_T^{\tau_f - \tau_e}\right)$$
$$= -\tilde{X}_{\tau_f} = -\delta_T^{\tau_f - \tau_e} X_{\tau_e}\left\{1 + o_p(1)\right\} \text{ from Lemma 8.3.}$$

The variance of $\hat{\delta}_{r_1,r_2}$ is

$$Var\left(\hat{\delta}_{r_1,r_2}\right) = \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\delta}_{r_1,r_2}\tilde{X}_{j-1}\right)^2$$

$$= \tau_w^{-1} \left\{ \sum_{j=\tau_f+2}^{\tau_2} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tilde{X}_{j-1} \right]^2 + \sum_{j=\tau_1}^{\tau_f} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \tilde{X}_{j-1} \right]^2 \right.$$

$$\left. + \left[\tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2} \tilde{X}_{\tau_f}^2 - \varepsilon_{\tau_f+1} + \varepsilon_{\tau_f+1} \right]^2 \right\}$$

$$= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2} - 1 \right)^2 \tau_w{}^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}^2 + \left(\hat{\delta}_{r_1,r_2} - \delta_T \right)^2 \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2$$

$$- 2 \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tau_w^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j - 2 \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1} \varepsilon_j + \tau_w^{-1} \tilde{X}_{\tau_f}^2$$

$$= \tau_w^{-1} \tilde{X}_{\tau_f}^2 = \tau_w^{-1} \delta_T^{2\left(\tau_f - \tau_e\right)} X_{\tau_e}^2 \left\{ 1 + o_p(1) \right\} \sim_a \frac{1}{r_w} \delta_T^{2\left(\tau_f - \tau_e\right)} B\left(r_e\right)^2 .$$

The term $\tau_w^{-1} \tilde{X}_{\tau_f}^2$ dominates the other terms due to the fact that

$$\left(\hat{\delta}_{r_1,r_2} - 1 \right)^2 \tau_w{}^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}^2 = O_p \left(T^{-1} \delta_T^{2\left(\tau_f - \tau_1\right)} \right),$$

$$\left(\hat{\delta}_{r_1,r_2} - \delta_T \right)^2 \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1}^2 = O_p\left(T^{-1}\right),$$

$$2 \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tau_w^{-1} \sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j = O_p \left(T^{-1} \delta_T^{\tau_f - \tau_1} \right),$$

$$2 \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \tau_w^{-1} \sum_{j=\tau_1}^{\tau_f} \tilde{X}_{j-1} \varepsilon_j = O_p\left(T^{-1}\right),$$

$$\tau_w^{-1} \tilde{X}_{\tau_f}^2 = O_p \left(\delta_T^{2\left(\tau_f - \tau_e\right)} \right).$$

(3) When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2} \tilde{X}_{\tau_f} - \varepsilon_{\tau_f+1}$$

$$= X_{\tau_e} + X^* - \tilde{X}_{\tau_f} - \left[\hat{\delta}_{r_1,r_2} - 1 \right] \tilde{X}_{\tau_f}$$

$$= O_p\left(T^{1/2}\right) + o_p(1) - O_p\left(T^{1/2} \delta_T^{\tau_f - \tau_e}\right) - O_p\left(T^{1/2-\alpha} \delta_T^{\tau_f - \tau_e}\right)$$

$$= -\tilde{X}_{\tau_f} = -\delta_T^{\tau_f - \tau_e} X_{\tau_e} \left\{ 1 + o_p(1) \right\} \text{ from Lemma 8.3.}$$

The variance of $\hat{\delta}_{r_1,r_2}$ is

$$Var\left(\hat{\delta}_{r_1,r_2}\right)$$

$$
\begin{aligned}
&= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \left(\tilde{X}_j - \hat{\delta}_{r_1,r_2} \tilde{X}_{j-1} \right)^2 \\
&= \tau_w^{-1} \left\{ \sum_{j=\tau_f+2}^{\tau_2} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tilde{X}_{j-1} \right]^2 + \sum_{j=\tau_1}^{\tau_e-1} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tilde{X}_{j-1} \right]^2 \right. \\
&\left. + \sum_{j=\tau_e}^{\tau_f} \left[\varepsilon_j - \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \tilde{X}_{j-1} \right]^2 + \tilde{X}_{\tau_f+1} - \hat{\delta}_{r_1,r_2} \tilde{X}_{\tau_f}^2 \right\}^2 \\
&= \tau_w^{-1} \sum_{j=\tau_1}^{\tau_2} \varepsilon_j^2 + \left(\hat{\delta}_{r_1,r_2} - 1 \right)^2 \tau_w{}^{-1} \left[\sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}^2 + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^2 \right] + \left(\hat{\delta}_{r_1,r_2} - \delta_T \right)^2 \tau_w^{-1} \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2 \\
&- 2 \left(\hat{\delta}_{r_1,r_2} - 1 \right) \tau_w^{-1} \left[\sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} \varepsilon_j \right] - 2 \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \tau_w^{-1} \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1} \varepsilon_j + \tau_w^{-1} \tau_f^2 \\
&= \tau_w^{-1} \tilde{X}_{\tau_f}^2 = \frac{\delta_T^{2\left(\tau_f - \tau_e\right)}}{\tau_w} X_{\tau_e}^2 \left\{ 1 + o_p(1) \right\} \sim_a \frac{\delta_T^{2\left(\tau_f - \tau_e\right)}}{r_w} B\left(r_e\right)^2 .
\end{aligned}
$$

The term $\tau_w^{-1} \tilde{X}_{\tau_f}^2$ dominates the other terms due to the fact that

$$
\begin{aligned}
\left(\hat{\delta}_{r_1,r_2} - 1 \right)^2 \frac{1}{\tau_w} \left[\sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1}^2 + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1}^2 \right] &= O_p\left(\frac{\delta_T^{2\left(\tau_f-\tau_e\right)}}{T} \right), \\
\left(\hat{\delta}_{r_1,r_2} - \delta_T \right)^2 \frac{1}{\tau_w} \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1}^2 &= O_p\left(\frac{1}{T^\alpha} \right), \\
2 \left(\hat{\delta}_{r_1,r_2} - 1 \right) \frac{1}{\tau_w} \left[\sum_{j=\tau_f+2}^{\tau_2} \tilde{X}_{j-1} \varepsilon_j + \sum_{j=\tau_1}^{\tau_e-1} \tilde{X}_{j-1} \varepsilon_j \right] &= O_p\left(\frac{\delta_T^{\tau_f-\tau_e}}{T} \right), \\
2 \left(\hat{\delta}_{r_1,r_2} - \delta_T \right) \frac{1}{\tau_w} \sum_{j=\tau_e}^{\tau_f} \tilde{X}_{j-1} \varepsilon_j &= O_p\left(\frac{1}{T^{(1+\alpha)/2}} \right), \\
\tau_w^{-1} \tilde{X}_{\tau_f}^2 &= O_p\left(\delta_T^{2\left(\tau_f-\tau_e\right)} \right).
\end{aligned}
$$

The asymptotic distribution of the DF t-statistic can be calculated as follows. When $\tau_1 \in N_0$ and $\tau_2 \in B$,

$$
DF_{r_1,r_2}^t = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\delta}_{r_1,r_2} - 1 \right) \sim_a \frac{T^{1/2} \delta_T^{\tau_2-\tau_e} r_w^{3/2} B\left(r_e\right)}{2\left(r_e - r_1\right) \int_{r_1}^{r_e} B(s)\, ds} \to \infty.
$$

When $\tau_1 \in B$ and $\tau_2 \in N_1$,

$$DF^t_{r_1,r_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^2_{j-1}}{\hat{\sigma}^2}\right)^{1/2} \left(\hat{\delta}_{r_1,r_2} - 1\right) \sim_a \left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to \infty.$$

When $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$DF^t_{r_1,r_2} = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}^2_{j-1}}{\hat{\sigma}^2}\right)^{1/2} \left(\hat{\delta}_{r_1,r_2} - 1\right) \sim_a \left(\frac{1}{2} c r_w\right)^{1/2} T^{(1-\alpha)/2} \to \infty.$$

The date-stamping strategy of PWY

The origination of the bubble expansion and the termination of the bubble collapse based on the backward DF test are identified as

$$\hat{r}^e = \inf_{r_2 \in [r_0,1]} \left\{ r_2 : BDF_{r_2} > cv^{\beta_T}_{r_2} \right\} \text{ and } \hat{r}^f = \inf_{r_2 \in [\hat{r}_e + \log(T)/T, 1]} \left\{ r_2 : BDF_{r_2} < cv^{\beta_T}_{r_2} \right\}.$$

We know that when $\beta_T \to 0$, $cv^{\beta_T}_{r_2} \to \infty$.

The asymptotic distributions of the backward DF statistic under the alternative hypothesis are

$$BDF_{r_2} \sim_a \begin{cases} F_{r_2}(W) & \text{if } r_2 \in N_0 \\ \frac{T^{1/2} \delta_T^{\tau_2 - \tau_e} r_w^{3/2} B(r_e)}{2(r_e - r_1) \int_{r_1}^{r_e} B(s) ds} \to \infty & \text{if } r_2 \in B \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w\right)^{1/2} \to \infty & \text{if } r_2 \in N_1 \end{cases}.$$

It is obvious that if $r_2 \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{ BDF_{r_2} > cv^{\beta_T}_{r_2} \right\} = \Pr\left\{ F_{r_2}(W) = \infty \right\} = 0.$$

If $r_2 \in B$, $\lim_{T\to\infty} \Pr\left\{ BDF_{r_2} > cv^{\beta_T}_{r_2} \right\} = 1$ provided that $\frac{cv^{\beta_T}_{r_2}}{T^{1/2} \delta_T^{\tau_2 - \tau_e}} \to 0$. It implies that provided $\frac{cv^{\beta_T}_{r_2}}{T^{1/2}} \to 0$, $\lim_{T\to\infty} \Pr\left\{ BDF_{r_2} > cv^{\beta_T}_{r_2} \right\} = 1$ for any $r_2 \in B$. If $r_2 \in N_1$, $\lim_{T\to\infty} \Pr\left\{ BDF_{r_2} > cv^{\beta_T}_{r_2} \right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{cv^{\beta_T}_{r_2}} \to 0$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{ \hat{r}_e > r_e + \eta \right\} \to 0 \text{ and } \Pr\left\{ \hat{r}_f < r_f - \gamma \right\} \to 0$$

due to the fact that $\Pr\left\{BDF_{r_e+a_\eta} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BDF_{r_f-a_\gamma} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_e < r_e\} \to 0$ and $\Pr\{\hat{r}_f > r_f\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$), we deduce that $\Pr\{|\hat{r}_e - r_e| > \eta\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\{|\hat{r}_f - r_f| > \gamma\} \to 0$, provided that

$$\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0.$$

Therefore, $\hat{r}_e$ and $\hat{r}_f$ are consistent estimators of r_e and r_f.

The new date-stamping strategy

The origination of the bubble expansion and the termination of the bubble collapse based on the backward sup DF test are identified as

$$\hat{r}^e = \inf_{r_2 \in [r_0,1]} \left\{r_2 : BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T}\right\},$$
$$\hat{r}^f = \inf_{r_2 \in [\hat{r}_e + \delta \log(T)/T, 1]} \left\{r_2 : BSDF_{r_2}(r_0) < scv_{r_2}^{\beta_T}\right\}.$$

We know that when $\beta_T \to 0$, $scv_{r_2}^{\beta_T} \to \infty$.

The asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are

$$BSDF_{r_2}(r_0) \sim_a \begin{cases} F_{r_2}^{r_0}(W) & \text{if } r_2 \in N_0 \\ T^{1/2}\delta_T^{\tau_2-\tau_e} \sup_{r_1\in[0,r_2-r_0]}\left\{\frac{r_w^{3/2}B(r_e)}{2(r_e-r_1)\int_{r_1}^{r_e}B(s)ds}\right\} & \text{if } r_2 \in B \\ T^{(1-\alpha)/2}\sup_{r_1\in[0,r_2-r_0]}\left\{\left(\frac{1}{2}cr_w\right)^{1/2}\right\} \to \infty & \text{if } r_2 \in N_1 \end{cases}.$$

It is obvious that if $r_2 \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T}\right\} = \Pr\left\{F_{r_2}^{r_0}(W) = \infty\right\} = 0.$$

If $r_2 \in B$, $\lim_{T\to\infty} \Pr\left\{BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T}\right\} = 1$ provided that $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}\delta_T^{\tau_2-\tau_e}} \to 0$. It implies that provided $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T\to\infty} \Pr\left\{BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T}\right\} = 1$ for any $r_2 \in B$. If $r_2 \in N_1$, $\lim_{T\to\infty} \Pr\left\{BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T}\right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{\hat{r}_e > r_e + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_f < r_f - \gamma\right\} \to 0,$$

since $\Pr\left\{BSDF_{r_e+a_\eta}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BSDF_{r_f-a_\gamma}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\left\{\hat{r}_e < r_e\right\} \to 0$ and $\Pr\left\{\hat{r}_f > r_f\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0$), we deduce that $\Pr\left\{\left|\hat{r}_e - r_e\right| > \eta\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\left\{\left|\hat{r}_f - r_f\right| > \gamma\right\} \to 0$, provided that

$$\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0.$$

Therefore, $\hat{r}_e$ and $\hat{r}_f$ are consistent estimators of r_e and r_f.

APPENDIX B. Date-stamping strategies (two bubbles)

Notation and lemmas

- The two bubble periods are $B_1 = [\tau_{1e}, \tau_{1f}]$ and $B_2 = [\tau_{2e}, \tau_{2f}]$, where $\tau_{1e} = \lfloor Tr_{1e} \rfloor$, $\tau_{1f} = \lfloor Tr_{1f} \rfloor$, $\tau_{2e} = \lfloor Tr_{2e} \rfloor$ and $\tau_{2f} = \lfloor Tr_{2f} \rfloor$.
- The normal periods are $N_0 = [1, \tau_{1e})$, $N_1 = (\tau_{1f}, \tau_{2e})$, $N_2 = (\tau_{2f}, \tau]$, where $\tau = \lfloor Tr \rfloor$ is the last observation of the sample.
- We assume that $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$.

We use the data generating process

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{for } t \in N_0 \\ \delta_T X_{t-1} + \varepsilon_t & \text{for } t \in B_i \text{ with } i = 1, 2 \\ X^*_{\tau_{if}} + \sum_{k=\tau_{if}+1}^{t} \varepsilon_k & \text{for } t \in N_i \text{ with } i = 1, 2 \end{cases}, \tag{30}$$

where $\delta_T = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0, 1)$, $\varepsilon_t \overset{iid}{\sim} N\left(0, \sigma^2\right)$ and $X^*_{\tau_{if}} = X_{\tau_{ie}} + X^*$ with $X^* = O_p(1)$ for $i = 1, 2$. We have the following lemmas.

Lemma 8.7 *Under the data generating process (30),*

(1) *For* $t \in N_0$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} B(p)$.

(2) *For* $t \in B_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tp \rfloor} = \delta_T^{t-\tau_{ie}} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{1/2} \delta_T^{t-\tau_{ie}} B(r_{ie})$.

(3) *For* $t \in N_i$ *with* $i = 1, 2$, $X_{t=\lfloor Tp \rfloor} \sim_a T^{1/2} [B(p) - B(r_{if}) + B(r_{ie})]$.

Lemma 8.8 *Under the data generating process (30),*

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_2 - \tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_2 - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{if} - \tau_1}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_1} \frac{1}{r_w c} B(r_{ie}).$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = X_{\tau_{ie}} \frac{T^\alpha \delta_T^{\tau_{if} - \tau_{ie}}}{\tau_w c} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{if} - \tau_{ie}} \frac{1}{r_w c} B(r_{ie}).$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_{1e}} \frac{1}{r_w c} B(r_{1e}).$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_1}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_1} \frac{1}{r_w c} B(r_{1e}) & \text{if } \tau_{1f} - \tau_1 > \tau_2 - \tau_{2e} \\ \frac{T^\alpha \delta_T^{\tau_{2f} - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{2f} - \tau_{2e}} \frac{1}{r_w c} B(r_{2e}) & \text{if } \tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e} \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \begin{cases} \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_1}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_1} \frac{1}{r_w c} B(r_1) & \text{if } \tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e} \\ \frac{T^\alpha \delta_T^{\tau_{2f} - \tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{2f} - \tau_{2e}} \frac{1}{r_w c} B(r_{2e}) & \text{if } \tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\frac{1}{\tau_w} \sum_{j=\tau_1}^{\tau_2} X_j = \frac{T^\alpha \delta_T^{\tau_{1f} - \tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} \sim_a T^{\alpha - 1/2} \delta_T^{\tau_{1f} - \tau_{1e}} \frac{1}{r_w c} B(r_{1e}).$$

Lemma 8.9 *Define the centered quantity* $\tilde{X}_t = X_t - \tau_w^{-1}\sum_{j=\tau_1}^{\tau_2} X_j$.

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_2-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in N_{i-1} \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in B_i \end{cases}.$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{if}-\tau_1}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in B_i \\ -\frac{T^\alpha \delta_T^{\tau_{if}-\tau_1}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in N_i \end{cases}.$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c} X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in N_{i-1} \cup N_i \\ \left[\delta_T^{t-\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{if}-\tau_{ie}}}{\tau_w c}\right] X_{\tau_{ie}} \{1 + o_p(1)\} & \text{if } t \in B_i \end{cases}.$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} & \text{if } t \in N_i \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right] \{1 + o_p(1)\} & \text{if } t \in B_i,\ i = 1, 2, \end{cases}.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$, *if* $\tau_{1f} - \tau_1 > \tau_2 - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right] \{1 + o_p(1)\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq \tau_2 - \tau_{2e}$

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right] \{1 + o_p(1)\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_2-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}} \{1 + o_p(1)\} & \text{if } t \in N_1 \end{cases}.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}}\right] \{1 + o_p(1)\} & \text{if } t \in B_i,\ i = 1, 2, \\ -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_1}}{\tau_w c} X_{\tau_{1e}} \{1 + o_p(1)\} & \text{if } t \in N_i,\ i = 1, 2, \end{cases}$$

and if $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$,

$$\tilde{X}_t = \begin{cases} \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\right]\{1+o_p(1)\} & \text{if } t \in B_i,\ i=1,2, \\ -\frac{T^\alpha \delta_T^{\tau_{2f}-\tau_{2e}}}{\tau_w c} X_{\tau_{2e}}\{1+o_p(1)\} & \text{if } t \in N_i,\ i=1,2, \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\tilde{X}_t = \begin{cases} -\frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\{1+o_p(1)\} & \text{if } t \in N_i,\ i=1,2, \\ \left[\delta_T^{t-\tau_{ie}} X_{\tau_{ie}} - \frac{T^\alpha \delta_T^{\tau_{1f}-\tau_{1e}}}{\tau_w c} X_{\tau_{1e}}\right]\{1+o_p(1)\} & \text{if } t \in B_i,\ i=1,2, \end{cases}.$$

Lemma 8.10 *The sample variance of* $\tilde{X}_t$ *has the following limit form:*

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_2-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1+o_p(1)\} \sim_a \frac{T^{1+\alpha}\delta_T^{2(\tau_2-\tau_{ie})}}{2c} B(r_{ie})^2.$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} B(r_{ie})^2.$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} X_{\tau_{ie}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{if}-\tau_{ie})}}{2c} B(r_{ie})^2.$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} B(r_{1e})^2.$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^\alpha \delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} B(r_{1e})^2.$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, *if* ,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \begin{cases} \frac{T^\alpha \delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} X_{\tau_{1e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{1f}-\tau_{1e})}}{2c} B(r_{1e})^2 & \text{if } \tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e} \\ \frac{T^\alpha \delta_T^{2(\tau_{2f}-\tau_{2e})}}{2c} X_{\tau_{2e}}^2 \{1+o_p(1)\} \sim_a \frac{T^{\alpha+1}\delta_T^{2(\tau_{2f}-\tau_{2e})}}{2c} B(r_{2e})^2 & \text{if } \tau_{1f}-\tau_1 \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2 = \frac{T^{\alpha}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} X_{\tau_{1e}}^2 \left\{1 + o_p\left(1\right)\right\} \sim_a \frac{T^{\alpha+1}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)}}{2c} B\left(r_{1e}\right)^2.$$

Lemma 8.11 *The sample covariance of* $\tilde{X}_t$ *and* ε_t *has the following limit form:*

(1) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in B_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_2-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(2) *For* $\tau_1 \in B_i$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(3) *For* $\tau_1 \in N_{i-1}$ *and* $\tau_2 \in N_i$ *with* $i = 1, 2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}} X_c B\left(r_{ie}\right).$$

(4) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in N_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).$$

(5) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).$$

(6) *For* $\tau_1 \in B_1$ *and* $\tau_2 \in N_2$, ,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a \begin{cases} T^{(1+\alpha)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right) & \text{if } \tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e} \\ T^{(\alpha+1)/2}\delta_T^{\tau_{2f}-\tau_{2e}} X_c B\left(r_{2e}\right) & \text{if } \tau_{1f}-\tau_1 \leq \tau_{2f}-\tau_{2e} \end{cases}.$$

(7) *For* $\tau_1 \in N_0$ *and* $\tau_2 \in B_2$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}\varepsilon_j \sim_a T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}} X_c B\left(r_{1e}\right).$$

Lemma 8.12 *The sample covariance of $\tilde{X}_{j-1}$ and $X_j - \delta_T X_{j-1}$ has the following limit form:*

(1) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a \frac{r_{ie} - r_1}{r_w} T \delta_T^{\tau_2 - \tau_{ie}} B\left(r_{ie} \right) \int_{r_1}^{r_{ie}} B\left(s \right) ds.$$

(2) *For $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_{if} - \tau_{ie}} X_c B\left(r_{ie} \right).$$

(3) *For $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T \delta_T^{\tau_{if} - \tau_{ie}} B\left(r_{ie} \right) \left[\frac{r_2 - r_{if}}{r_w} \int_{r_{if}}^{r_2} B\left(s \right) ds + \frac{r_{ie} - r_1}{r_w} \int_{r_1}^{r_{ie}} B\left(s \right) ds \right].$$

(4) *For $\tau_1 \in N_0$ and $\tau_2 \in N_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T \delta_T^{\tau_{1f} - \tau_{1e}} B\left(r_{1e} \right) \left[\frac{r_{2e} - r_{1f}}{r_w} \int_{r_{1f}}^{r_{2e}} B\left(s \right) ds \right.$$
$$\left. + \frac{r_{2e} - r_{1f}}{r_w} \int_{r_{1f}}^{r_{2e}} B\left(s \right) ds + \frac{r_2 - r_{2f}}{r_w} \int_{r_{2f}}^{r_2} B\left(s \right) ds \right].$$

(5) *For $\tau_1 \in B_1$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T^{(\alpha+1)/2} \delta_T^{\tau_{1f} - \tau_{1e}} X_c B\left(r_{1e} \right).$$

(6) *For $\tau_1 \in B_1$ and $\tau_2 \in N_2$, if $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a \left[\frac{r_{2e} - r_{1f}}{r_w} \int_{r_{1f}}^{r_{2e}} B\left(s \right) ds + \frac{r_2 - r_{2f}}{r_w} \int_{r_{2f}}^{r_2} B\left(s \right) ds \right] T \delta_T^{\tau_{1f} - \tau_1} B\left(r_{1e} \right)$$

and if $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$,

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a \left[\frac{r_{2e} - r_{1f}}{r_w} \int_{r_{1f}}^{r_{2e}} B\left(s \right) ds + \frac{r_2 - r_{2f}}{r_w} \int_{r_{2f}}^{r_2} B\left(s \right) ds \right] T \delta_T^{\tau_{2f} - \tau_{2e}} B\left(r_{2e} \right).$$

(7) *For $\tau_1 \in N_0$ and $\tau_2 \in B_2$,*

$$\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1} \left(X_j - \delta_T X_{j-1} \right) \sim_a T \delta_T^{\tau_{1f} - \tau_{1e}} B\left(r_{1e} \right) \left[\frac{r_{1e} - r_1}{r_w} \int_{r_1}^{r_{1e}} B\left(s \right) ds + \frac{r_{2e} - r_{1f}}{r_w} \int_{r_{1f}}^{r_{2e}} B\left(s \right) ds \right].$$

We refer to Appendix A for the proof of Lemma 8.7, Lemma 8.8, Lemma 8.9, Lemma 8.10, Lemma 8.11 and Lemma 8.12. A more detailed proof of Appendix B is available online at `https://sites.google.com/site/shupingshi/TN_GSADF.pdf?attredirects=0&d=1`.

Test asymptotics

The regression model for the Dickey-Fuller test is

$$X_t = \alpha_{r_1,r_2} + \delta_{r_1,r_2} X_{t-1} + \varepsilon_t, \varepsilon_t \overset{iid}{\sim} N\left(0, \sigma^2_{r_2.r_w}\right).$$

First, we calculate the asymptotic distribution of the Dickey-Fuller statistic under the alternative hypothesis. Based on Lemma 8.10 and Lemma 8.12, we can obtain the limit distribution of $\hat{\delta}_{r_1,r_2} - \delta_T$. When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$\frac{T^{\alpha}\delta_T^{\tau_2-\tau_{ie}}}{2c}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_{ie} - r_1)\int_{r_1}^{r_{ie}} B(s)\,ds}{r_w B(r_{ie})};$$

when $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\frac{T^{(\alpha+1)/2}\delta_T^{\tau_{if}-\tau_{ie}}}{2c}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{X_c}{B(r_{ie})};$$

when $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1, 2$,

$$\frac{T^{\alpha}\delta_T^{\tau_{if}-\tau_{ie}}}{2c}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_2 - r_{if})\int_{r_{if}}^{r_2} B(s)\,ds + (r_{ie} - r_1)\int_{r_1}^{r_{ie}} B(s)\,ds}{r_w B(r_{ie})};$$

when $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{2c}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_{2e} - r_{1f})\int_{r_{1f}}^{r_{2e}} B(s)\,ds + (r_{2e} - r_{1f})\int_{r_{1f}}^{r_{2e}} B(s)\,ds + (r_2 - r_{2f})\int_{r_{2f}}^{r_2} B(s)\,ds}{r_w B(r_{1e})};$$

when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_{1e}}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} 2cX_c B(r_{1e})^{-1};$$

when $\tau_1 \in B_1$ and $\tau_2 \in N_2$, if $\tau_{1f} - \tau_1 > \tau_{2f} - \tau_{2e}$,

$$T^{(\alpha+1)/2}\delta_T^{\tau_{1f}-\tau_1}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_{2e} - r_{1f})\int_{r_{1f}}^{r_{2e}} B(s)\,ds + (r_2 - r_{2f})\int_{r_{2f}}^{r_2} B(s)\,ds}{r_w B(r_{1e})}$$

and if $\tau_{1f} - \tau_1 \leq \tau_{2f} - \tau_{2e}$,

$$T^{(\alpha+1)/2}\delta_T^{\tau_{2f}-\tau_{2e}}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_{2e} - r_{1f})\int_{r_{1f}}^{r_{2e}} B(s)\,ds + (r_2 - r_{2f})\int_{r_{2f}}^{r_2} B(s)\,ds}{r_w B(r_{2e})};$$

when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$\frac{T^{\alpha}\delta_T^{\tau_{1f}-\tau_{1e}}}{2c}\left(\hat{\delta}_T - \delta_T\right) \xrightarrow{L} \frac{(r_{1e}-r_1)\int_{r_1}^{r_{1e}} B(s)\,ds + (r_{2e}-r_{1f})\int_{r_{1f}}^{r_{2e}} B(s)\,ds}{r_w B(r_{1e})}.$$

The asymptotic distribution of Dickey-Fuller coefficient statistic is

$$DF_{r_1,r_2}^{z} = r_w c T^{1-\alpha} + o_p(1) \to \infty.$$

for all cases, which implies that $\hat{\delta}_{r_1,r_2} - 1 \sim_a T^{-\alpha}c$ or $T^{\alpha}\left(\hat{\delta}_{r_1,r_2} - 1\right) \xrightarrow{L} c$.

To obtain the asymptotic distribution of the Dickey-Fuller t-statistic, we need to estimate the standard error of $\hat{\delta}_{r_1,r_2}$. When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1,2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \frac{2c}{T^{\alpha}}\frac{(r_{ie}-r_1)^2}{r_w^3}\left[\int_{r_1}^{r_{ie}} B(s)\,ds\right]^2;$$

when $\tau_1 \in B_i$ and $\tau_2 \in N_i$ with $i = 1,2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} r_w^{-1} B(r_{ie})^2;$$

when $\tau_1 \in N_{i-1}$ and $\tau_2 \in N_i$ with $i = 1,2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \delta_T^{2\left(\tau_{if}-\tau_{ie}\right)} r_w^{-1} B(r_{ie})^2;$$

when $\tau_1 \in N_0$ and $\tau_2 \in N_2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} r_w^{-1} B(r_{1e})^2;$$

when $\tau_1 \in B_1$ and $\tau_2 \in B_2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} r_w^{-1} B(r_{1e})^2;$$

when $\tau_1 \in B_1$ and $\tau_2 \in N_2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \begin{cases} T\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{2e})^2 & \text{if } \tau_{1f}-\tau_1 > \tau_{2f}-\tau_{2e} \\ T\delta_T^{2\left(\tau_{2f}-\tau_{2e}\right)} B(r_{2e}) & \text{if } \tau_{1f}-\tau_1 \le \tau_{2f}-\tau_{2e} \end{cases};$$

when $\tau_1 \in N_0$ and $\tau_2 \in B_2$,

$$Var\left(\hat{\delta}_{r_1,r_2}\right) \sim_a \frac{1}{r_w}\delta_T^{2\left(\tau_{1f}-\tau_{1e}\right)} B(r_{1e})^2.$$

The asymptotic distributions of the DF t-statistic can be calculated as follows. When $\tau_1 \in N_{i-1}$ and $\tau_2 \in B_i$ with $i = 1, 2$,

$$DF_{r_1,r_2}^t \sim_a T^{1/2} \delta_T^{\tau_2 - \tau_{ie}} \frac{r_w^{3/2} B(r_{ie})}{2(r_{ie} - r_1) \int_{r_1}^{r_{ie}} B(s)\, ds} \to \infty;$$

and for all other cases considered,

$$DF_{r_1,r_2}^t = \left(\frac{\sum_{j=\tau_1}^{\tau_2} \tilde{X}_{j-1}^2}{\hat{\sigma}^2} \right)^{1/2} \left(\hat{\delta}_{r_1,r_2} - 1 \right) \sim_a \left(\frac{1}{2} c r_w \right)^{1/2} T^{(1-\alpha)/2} \to \infty.$$

The date-stamping strategy of PWY

The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the backward DF test are identified as

$$\begin{aligned}
\hat{r}_{1e} &= \inf_{r_2 \in [r_0, 1]} \left\{ r_2 : BDF_{r_2} > cv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{1f} &= \inf_{r_2 \in [\hat{r}_{1e} + \log(T)/T, 1]} \left\{ r_2 : BDF_{r_2} < cv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{2e} &= \inf_{r_2 \in \left(\hat{r}_{1f}, 1 \right]} \left\{ r_2 : BDF_{r_2} > cv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{2f} &= \inf_{r_2 \in [\hat{r}_{2e} + \log(T)/T, 1]} \left\{ r_2 : BDF_{r_2} < cv_{r_2}^{\beta_T} \right\}.
\end{aligned}$$

We know that when $\beta_T \to 0$, $cv_{r_2}^{\beta_T} \to \infty$.

The asymptotic distributions of the backward DF statistic under the alternative hypothesis are (given $r_1 \in N_0$ and $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$)

$$BDF_{r_2} \sim_a \begin{cases} F_{r_2}(W) & \text{if } r_2 \in N_0 \\ T^{1/2} \delta_T^{\tau_2 - \tau_{1e}} \frac{r_w^{3/2} B(r_{1e})}{2(r_{1e} - r_1) \int_{r_1}^{r_{1e}} B(s) ds} & \text{if } r_2 \in B_1 \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w \right)^{1/2} & \text{if } r_2 \in N_1 \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w \right)^{1/2} & \text{if } r_2 \in B_2 \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w \right)^{1/2} & \text{if } r_2 \in N_2 \end{cases}.$$

It is obvious that if $r_2 \in N_0$,

$$\lim_{T \to \infty} \Pr \left\{ BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = \Pr \left\{ F_{r_2}(W) = \infty \right\} = 0.$$

If $r_2 \in B_1$, $\lim_{T\to\infty} \Pr\left\{BDF_{r_2} > cv_{r_2}^{\beta_T}\right\} = 1$ provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2}\delta_T^{\tau_2-\tau_{1e}}} \to 0$. It implies that provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T\to\infty} \Pr\left\{BDF_{r_2} > cv_{r_2}^{\beta_T}\right\} = 1$ for any $r_2 \in B_1$. If $r_2 \in N_1 \cup N_2$, $\lim_{T\to\infty} \Pr\left\{BDF_{r_2} > cv_{r_2}^{\beta_T}\right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$. If $r_2 \in B_2$, $\lim_{T\to\infty} \Pr\left\{BDF_{r_2} > cv_{r_2}^{\beta_T}\right\} = 1$ provided that $\frac{cv_{r_2}^{\beta_T}}{T^{(1-\alpha)/2}} \to 0$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{\hat{r}_{1e} > r_{1e} + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

due to the fact that $\Pr\left\{BDF_{r_{1e}+a_\eta} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BDF_{r_{1f}-a_\gamma} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\left\{\hat{r}_{1e} < r_{1e}\right\} \to 0$ (given $\frac{1}{cv_{r_2}^{\beta_T}}$) and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$), we deduce that $\Pr\left\{|\hat{r}_{1e} - r_{1e}| > \eta\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\left\{|\hat{r}_{1f} - r_{1f}| > \gamma\right\} \to 0$, provided that

$$\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0.$$

We can see that the date-stamping strategy can consistently estimate r_{1e} and r_{1f}.

For any $\phi, \kappa > 0$,

$$\Pr\left\{\hat{r}_{2e} > r_{2e} + \phi\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{2f} < r_{2f} - \kappa\right\} \to 0,$$

due to the fact that $\Pr\left\{BDF_{r_{2e}+a_\phi} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\phi < \phi$ and $\Pr\left\{BDF_{r_{2f}-a_\kappa} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\kappa < \kappa$. Since $\phi, \kappa > 0$ is arbitrary and $\Pr\left\{r_{1f} < \hat{r}_{2e} < r_{2e}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$) and $\Pr\left\{\hat{r}_{2f} > r_{2f}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}}$), we deduce that $\Pr\left\{|\hat{r}_{2e} - r_{2e}| > \eta\right\} \to 0$ and $\Pr\left\{|\hat{r}_{2f} - r_{2f}| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{(1-\alpha)/2}} \to 0.$$

Since $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}}$ and $\frac{cv_{r_2}^{\beta_T}}{T^{(1-\alpha)/2}}$ can not converge to zero simultaneously, the strategy **can not** estimate r_{2e} and r_{2f} consistently when $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$.

The new date-stamping strategy

The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the backward sup DF test are identified as

$$\begin{aligned}
\hat{r}_{1e} &= \inf_{r_2 \in [r_0,1]} \left\{ r_2 : BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{1f} &= \inf_{r_2 \in [\hat{r}_{1e}+\delta \log(T)/T,1]} \left\{ r_2 : BSDF_{r_2}(r_0) < scv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{2e} &= \inf_{r_2 \in \left(\hat{r}_{1f},1\right]} \left\{ r_2 : BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\}, \\
\hat{r}_{2f} &= \inf_{r_2 \in [\hat{r}_{2e}+\delta \log(T)/T,1]} \left\{ r_2 : BSDF_{r_2}(r_0) < scv_{r_2}^{\beta_T} \right\}.
\end{aligned}$$

We know that when $\beta_T \to 0$, $scv_{r_2}^{\beta_T} \to \infty$.

The asymptotic distributions of the backward sup DF statistic under the alternative hypothesis are (given $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$)

$$BSDF_{r_2}(r_0) \sim_a \begin{cases} F_{r_2}^{r_0}(W) & \text{if } r_2 \in N_0 \\ T^{1/2}\delta_T^{\tau_2-\tau_{1e}} \sup\limits_{r_1 \in [0,r_2-r_0]} \left\{ \frac{r_w^{3/2} B(r_{1e})}{2(r_{1e}-r_1)\int_{r_1}^{r_{1e}} B(s)ds} \right\} & \text{if } r_2 \in B_1 \\ T^{(1-\alpha)/2} \sup\limits_{r_1 \in [0,r_2-r_0]} \left(\frac{1}{2}cr_w\right)^{1/2} & \text{if } r_2 \in N_1 \\ T^{1/2}\delta_T^{\tau_2-\tau_{2e}} \sup\limits_{r_1 \in [0,r_2-r_0]} \left\{ \frac{r_w^{3/2} B(r_{2e})}{2(r_{2e}-r_1)\int_{r_1}^{r_{2e}} B(s)ds} \right\} & \text{if } r_2 \in B_2 \\ T^{(1-\alpha)/2} \sup\limits_{r_1 \in [0,r_2-r_0]} \left(\frac{1}{2}cr_w\right)^{1/2} & \text{if } r_2 \in N_2 \end{cases}.$$

It is obvious that if $r_2 \in N_0$,

$$\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = \Pr\left\{ F_{r_2}^{r_0}(W) = \infty \right\} = 0.$$

If $r_2 \in B_1$, $\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = 1$ provided that $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}\delta_T^{\tau_2-\tau_{1e}}} \to 0$. It implies that provided that $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = 1$ for any $r_2 \in B_1$. If $r_2 \in B_2$, $\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = 1$ provided that $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}\delta_T^{\tau_2-\tau_{2e}}} \to 0$. It implies that provided that $\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = 1$ for any $r_2 \in B_2$. If $r_2 \in N_1 \cup N_2$, $\lim_{T\to\infty} \Pr\left\{ BSDF_{r_2}(r_0) > scv_{r_2}^{\beta_T} \right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr\left\{\hat{r}_{1e} > r_{1e} + \eta\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{1f} < r_{1f} - \gamma\right\} \to 0,$$

since $\Pr\left\{BSDF_{r_{1e}+a_\eta}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BSDF_{r_{1f}-a_\gamma}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\left\{\hat{r}_{1e} < r_{1e}\right\} \to 0$ (given $\frac{1}{scv_{r_2}^{\beta_T}}$) and $\Pr\left\{\hat{r}_{1f} > r_{1f}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0$), we deduce that $\Pr\left\{\left|\hat{r}_{1e} - r_{1e}\right| > \eta\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\left\{\left|\hat{r}_{1f} - r_{1f}\right| > \gamma\right\} \to 0$, provided that

$$\frac{scv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0;$$

For any $\phi, \kappa > 0$,

$$\Pr\left\{\hat{r}_{2e} > r_{2e} + \phi\right\} \to 0 \text{ and } \Pr\left\{\hat{r}_{2f} < r_{2f} - \kappa\right\} \to 0,$$

since $\Pr\left\{BSDF_{r_{2e}+a_\phi}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\phi < \phi$ and $\Pr\left\{BSDF_{r_{2f}-a_\kappa}(r_0) > scv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\kappa < \kappa$. Since $\phi, \kappa > 0$ is arbitrary and $\Pr\left\{\hat{r}_{2e} < r_{2e}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} \to 0$) and $\Pr\left\{\hat{r}_{2f} > r_{2f}\right\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}}$), we deduce that $\Pr\left\{\left|\hat{r}_{2e} - r_{2e}\right| > \eta\right\} \to 0$ and $\Pr\left\{\left|\hat{r}_{2f} - r_{2f}\right| > \gamma\right\} \to 0$ as $T \to \infty$, provided that

$$\frac{T^{(1-\alpha)/2}}{scv_{r_2}^{\beta_T}} + \frac{scv_{r_2}^{\beta_T}}{T^{1/2}} \to 0.$$

Therefore, the date-stamping strategy based on the generalized sup ADF test can consistently estimate r_{1e}, r_{1f}, r_{2e} and r_{2f}.

Sequential implementation of the date-stamping strategy of PWY

The origination of the bubble expansion r_{1e}, r_{2e} and the termination of the bubble collapse r_{1f}, r_{2f} based on the backward DF test are identified as

$$\hat{r}_{1e} = \inf_{r_2 \in [r_0, 1]} \left\{r_2 : BDF_{r_2} > cv_{r_2}^{\beta_T}\right\},$$

$$\hat{r}_{1f} = \inf_{r_2 \in [\hat{r}_{1e} + \log(T)/T, 1]} \left\{ r_2 : BDF_{r_2} < cv_{r_2}^{\beta_T} \right\},$$
$$\hat{r}_{2e} = \inf_{r_2 \in \left(\hat{r}_{1f} + \varepsilon_T, 1 \right]} \left\{ r_2 :_{\hat{r}_{1f}} BDF_{r_2} > cv_{r_2}^{\beta_T} \right\},$$
$$\hat{r}_{2f} = \inf_{r_2 \in [\hat{r}_{2e} + \log(T)/T, 1]} \left\{ r_2 :_{\hat{r}_{1f}} BDF_{r_2} < cv_{r_2}^{\beta_T} \right\}.$$

where ${}_{\hat{r}_{1f}}BDF_{r_2}$ is the backward DF statistic calculate over $(\hat{r}_{1f}, r_2]$. We know that when $\beta_T \to 0$, $cv_{r_2}^{\beta_T} \to \infty$.

The asymptotic distributions of the backward DF statistic under the alternative hypothesis are (given $\tau_{1f} - \tau_{1e} > \tau_{2f} - \tau_{2e}$)

$$BDF_{r_2} \sim_a \begin{cases} F_{r_2}(W) & \text{if } r_1 \in N_0 \text{ and } r_2 \in N_0 \\ T^{1/2} \delta_T^{\tau_2 - \tau_{1e}} \frac{r_w^{3/2} B(r_{1e})}{2(r_{1e} - r_1) \int_{r_1}^{r_{1e}} B(s) ds} & \text{if } r_1 \in N_0 \text{ and } r_2 \in B_1 \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w \right)^{1/2} & \text{if } r_1 \in N_0 \text{ and } r_2 \in N_1 \end{cases}$$

and

$${}_{\hat{r}_{1f}}BDF_{r_2} \sim_a \begin{cases} F_{r_2}(W) & \text{if } r_2 \in N_1 \\ T^{1/2} \delta_T^{\tau_2 - \tau_{2e}} \frac{r_w^{3/2} B(r_{ie})}{2(r_{ie} - r_1) \int_{r_1}^{r_{ie}} B(s) ds} & \text{if } r_2 \in B_2 \\ T^{(1-\alpha)/2} \left(\frac{1}{2} c r_w \right)^{1/2} & \text{if } r_2 \in N_2 \end{cases}.$$

It is obvious that if $r_2 \in N_0$,

$$\lim_{T \to \infty} \Pr \left\{ BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = \Pr \left\{ F_{r_2}(W) = \infty \right\} = 0.$$

If $r_2 \in B_1$, $\lim_{T \to \infty} \Pr \left\{ BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 1$ provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2} \delta_T^{\tau_2 - \tau_{1e}}} \to 0$. So, provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T \to \infty} \Pr \left\{ BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 1$ for any $r_2 \in B_1$. If $r_2 \in N_1$, $\lim_{T \to \infty} \Pr \left\{ BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$ and $\lim_{T \to \infty} \Pr \left\{ {}_{\hat{r}_{1f}}BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = \Pr \left\{ F_{r_2}(W) = \infty \right\} = 0$. If $r_2 \in B_2$, $\lim_{T \to \infty} \Pr \left\{ {}_{\hat{r}_{1f}}BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 1$ provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2} \delta_T^{\tau_2 - \tau_{2e}}} \to 0$. It implies that provided that $\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$, $\lim_{T \to \infty} \Pr \left\{ {}_{\hat{r}_{1f}}BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 1$ for any $r_2 \in B_2$. If $r_2 \in N_2$, $\lim_{T \to \infty} \Pr \left\{ {}_{\hat{r}_{1f}}BDF_{r_2} > cv_{r_2}^{\beta_T} \right\} = 0$ provided that $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$.

It follows that for any $\eta, \gamma > 0$,

$$\Pr \left\{ \hat{r}_{1e} > r_{1e} + \eta \right\} \to 0 \text{ and } \Pr \left\{ \hat{r}_{1f} < r_{1f} - \gamma \right\} \to 0,$$

since $\Pr\left\{BDF_{r_{1e}+a_\eta} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\eta < \eta$ and $\Pr\left\{BDF_{r_{1f}-a_\gamma} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\gamma < \gamma$. Since $\eta, \gamma > 0$ is arbitrary and $\Pr\{\hat{r}_{1e} < r_{1e}\} \to 0$ (given $\frac{1}{cv_{r_2}^{\beta_T}}$) and $\Pr\{\hat{r}_{1f} > r_{1f}\} \to 0$ (given $\frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0$), we deduce that $\Pr\{|\hat{r}_{1e} - r_{1e}| > \eta\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\{|\hat{r}_{1f} - r_{1f}| > \gamma\} \to 0$, provided that

$$\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0.$$

Thus, this date-stamping strategy consistently estimates r_{1e} and r_{1f}.

For any $\phi, \kappa > 0$,

$$\Pr\{\hat{r}_{2e} > r_{2e} + \phi\} \to 0 \text{ and } \Pr\{\hat{r}_{2f} < r_{2f} - \kappa\} \to 0,$$

since $\Pr\left\{{}_{\hat{r}_{1f}}BDF_{r_{2e}+a_\phi} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\phi < \phi$ and $\Pr\left\{{}_{\hat{r}_{1f}}BDF_{r_{2f}-a_\kappa} > cv_{r_2}^{\beta_T}\right\} \to 1$ for all $0 < a_\kappa < \kappa$. Since $\phi, \kappa > 0$ is arbitrary and $\Pr\{r_{1f} < \hat{r}_{2e} < r_{2e}\} \to 0$ and $\Pr\{\hat{r}_{2f} > r_{2f}\} \to 0$, we deduce that $\Pr\{|\hat{r}_{2e} - r_{2e}| > \eta\} \to 0$ as $T \to \infty$, provided that

$$\frac{1}{cv_{r_2}^{\beta_T}} + \frac{cv_{r_2}^{\beta_T}}{T^{1/2}} \to 0$$

and $\Pr\{|\hat{r}_{2f} - r_{2f}| > \gamma\} \to 0$, provided that

$$\frac{cv_{r_2}^{\beta_T}}{T^{1/2}} + \frac{T^{(1-\alpha)/2}}{cv_{r_2}^{\beta_T}} \to 0.$$

Therefore, the alternative sequential implementation of the PWY procedure consistently estimates r_{2e} and r_{2f}.