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A Bayesian Inventory Model Using Real-Time Condition Monitoring Information

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Abstract

Lack of coordination between machinery fault diagnosis and inventory management for spare parts can lead to increased inventory costs as well as disruptions in production activity. In this paper, we develop a framework for incorporating real-time condition monitoring information into inventory management decisions for spare parts. We consider a manufacturer who periodically replenishes inventory for a machine part that is subject to deterioration. The deterioration process can be captured via condition monitoring and modeled using a Wiener process. The resulting degradation model can be used to derive the life distribution of a functioning part and to estimate the demand distribution for spare machine parts. This estimation is periodically updated, in a Bayesian manner, as additional information on part deterioration is obtained through condition monitoring. We develop an inventory model which incorporates this estimated and updated demand distribution. We use the model to demonstrate that the form of the optimal inventory control policy is a dynamic base-stock policy in which the optimal base-stock level is a function of some subset of the observed condition monitoring information. Adaptive inventory policies such as this can help manufacturers to increase machine availability and reduce inventory costs.

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1 Introduction

The management of complex mechanical systems can be greatly enhanced through the implementation of condition monitoring, i.e., the collection of real-time sensor information from a functioning device in order to monitor and learn about the condition of that device. Examples of condition monitoring techniques include vibration analysis, tribology (oil) analysis, and thermography (Moore and Starr, 2006). Condition monitoring is particularly useful for devices, e.g., machine parts, that are subject to deterioration and thus require periodic maintenance and/or replacement.

In order to use real-time sensor information to predict the remaining life of a device and to improve operational decisions such as replacement and repair, it is useful to identify and model a *degradation signal*, i.e., a quantity computed from sensor data that captures the current condition of the device and provides information on how that condition is likely to evolve (Nelson, 1990). For example, Gebraeel, *et al* (2005) develop degradation signal models for machine bearings. As bearings degrade, the vibration they emit exhibits an increasing trend. When this vibration reaches a specified threshold the bearing is considered to have failed. Thus, if properly constructed and estimated, a degradation signal based on bearing vibration can provide information on the remaining life of the bearing, as well as lead-time for maintenance planning and the procurement of spare parts. Similarly, oil analysis can be used to monitor machine condition and to identify abnormal wear. The most common form uses spectrometry to monitor the concentration of wear metals in the oil (Macin, *et al*, 2003). Generally, this concentration will increase with machine run time. Manufacturers often specify limits for these concentrations, indicating whether the machine condition is acceptable or abnormal (Evans, 2006). Thus, a degradation signal based on wear metal concentrations can provide useful information regarding machine wear, which can be used to manage preventive maintenance.

In practice, prognostics has drawn attention from the military, e.g., Greitzer and Ferryman (2001) discuss Naval applications, and heavy equipment manufacturers, such as Caterpillar, e.g., van de Voort, *et al* (2006), who own or produce complex mechanical systems. These systems include tanks, aircraft, ships, and earth-moving equipment. While there have been significant advances in condition monitoring technology, e.g., sensors, electronics and communications technologies, in recent years, as noted by Greitzer, *et al* (1999), the field “is still in a research and development phase, and implementing prognostics is a monumental task on several levels - the technical challenges involving hardware and sensor technologies, the analytical challenges involving predictive methods, and the logistical challenges centering on how to make use of prognostic information." Li (2001) and Gebraeel, *et al* (2005) consider the second-level challenge by studying how to incorporate real-time sensor information into the computation of remaining life distributions for functioning devices. In this paper, we extend this work and consider the third-level challenge. Specifically, we consider how this type of prognostic information can be used to improve spare parts inventory management.

2 Background and Literature Review

In this section, we provide necessary background on degradation signal modeling and briefly review the relevant literature. For a more complete overview of condition monitoring, including condition monitoring techniques and applications, please refer to Li (2001) and Gebraeel, *et al* (2005).

2.1 Degradation Signal Models for Condition Monitoring

When monitoring a functioning device, it is useful to define and model a *degradation signal* which can be observed over time and used to define device failure. Generally, this degradation signal will be increasing as a function of time, representing increasing degradation. Lu and Meeker (1993) (L&M) present a random coefficient degradation signal model that contains fixed effects, used to model characteristics of an entire population of devices, and random effects, used to model heterogeneity across a population of devices: $y_{ij} = \eta(t_j, \phi, \Theta_i) + \epsilon_{ij}$, where y_{ij} is the degradation signal observed on device i at t_j, the time of the jth observation, ϕ is a vector of fixed-effect parameters, Θ_i is a vector of random-effect parameters for the ith device, and ϵ_{ij} is the independent Normal random error term observed at time t_j for device i, used to capture measurement error. The random-effects, Θ_i, follow a multivariate distribution which depends on some unknown parameters, which can be estimated using observed degradation signal values. The degradation signal model used in this paper is similar to the L&M model, but with a Brownian motion error process. Previous work considering Brownian motion models of degradation signals includes Gebraeel, *et al* (2005), Park and Padgett (2005), Whitmore and Schenkelberg (1997), Whitmore (1995), Doksum and Hoyland (1992).

2.1.1 Real-time Bayesian Updating of Brownian Degradation Signal Models

This paper builds on the methods developed in Li (2001) and Gebraeel, *et al* (2005). Among their models, the most relevant for us is the single parameter linear model with a Brownian motion error process (Li 2001).[1] Here, $Z(t) = \theta t + \epsilon(t)$ for $t \geq 0$, represents the signal process over time, where $\theta > 0$ is the *degradation* or *drift* parameter, and $\epsilon(t) = \sigma W(t)$, where W is a standard Brownian motion. We assume that θ, which measures the rate of deterioration of a device, varies across the population of devices and is unknown, i.e., requires estimation. Gebraeel, *et al* (2005) develop a Bayesian updating method to periodically estimate the distribution of θ using observed degradation signal values. They adopt a Normal prior distribution for θ. If $z_1, z_2, \ldots, z_k$ are degradation signal observations obtained from a functioning device at times $0 < t_1 < t_2 < \ldots < t_k$, then at time t_k the posterior distribution of θ can be derived using the prior distribution and these observed signal values. It is then easy to compute the predictive distribution of $Z(t_k + t)$ given $z_1, z_2, \ldots, z_k$, for $t > 0$, which can then be used to draw inferences regarding the remaining life of the device.

2.2 Predicting Device Failure Using Real-time Sensor Information

Degradation signal observations can be used to determine when a functioning device has failed and replacement should be initiated. "A common practice is to record a condition reading at a regular interval, and once the reading is higher than a pre-set critical level, the item monitored is declared faulty and repair or replacement may be initiated" (Wang 2000). If the device will be replaced when the signal value reaches a given threshold, then the remaining life of a functioning device is simply the time until the signal process reaches that threshold. This approach to find the distribution of remaining life has been used by L&M, Wang and Zhang (2005) and Gebraeel, *et al* (2005).

2.3 Bayesian Inventory Models

We develop an inventory model in which the distribution of demand is periodically updated based on newly obtained sensor readings. Related research considers Bayesian inventory models in which the distribution of demand is periodically updated based on newly obtained demand observations, e.g., Dvoretzky, *et al* (1952), Scarf (1959, 1960), Karlin (1960), Iglehart (1964), etc. However, this literature generally assumes exogenous demand and Bayesian updating of some demand distribution parameter using observed demand data. In contrast, we model demand endogenously by deriving the demand distribution from the life distribution for machine parts and we update the distribution

[1] Our results can be extended to other degradation signal forms, e.g., an exponential model.

of the degradation parameter using real-time sensor information.

3 Problem Description

We consider a system of $m \in \mathbb{N}$ machines, each of which uses a single part that is subject to deterioration. While in use, each of the parts on machine i, $i = 1, \ldots, m$, has an associated degradation signal, $Z^i(t)$, which is periodically monitored (at the start of each period). This signal takes the linear form with Brownian motion error terms, defined in Section 2.1.1, i.e., $Z^i(t) = \theta_i t + \epsilon_i(t), t \geq 0, \theta_i > 0$, where $\{Z^i(t) : t \geq 0\}$ represents the degradation signal process, θ_i is the degradation parameter for parts on machine i, and $\epsilon_i(t) = \sigma W_i(t)$, where W_i is a standard Brownian motion. The degradation parameter, θ_i, may differ for each machine, reflecting the fact that the degradation process may be affected by machine condition. We assume, however, that θ_i is the same for all parts used on machine i. The degradation parameter, θ_i, may known or unknown. When it is unknown, we follow the Bayesian approach in Li (2001) and Gebraeel, *et al* (2005) to update the probability distribution of θ_i. Let $\vec{\theta} = [\theta_1, \ldots, \theta_m]$ denote the array of the drift parameters for the m machines.

A part is considered to have failed, and will be replaced, when the observed degradation signal exceeds a given threshold, B. We assume instantaneous part replacement (and a sufficient inventory of spare parts). A key issue is to derive the demand distribution for replacement parts. Since demand for parts in a given period is driven by part failures in that period, the demand distribution can be obtained using the remaining life distribution of each part. In a multi-period model, the age of the part being used on a given machine at the start of the period will be different in each period and for each machine. In addition, the signals observed at the start of each period will differ. Thus, even if the periods are of equal length, the demand distributions in each period will not be homogenous.

Given the demand distribution for each period, our goal is to determine the optimal method for controlling inventory of spare parts, assuming that inventory is ordered periodically to minimize the total expected cost for the remainder of the planning horizon. The planning horizon consists of $N < \infty$ periods of equal length $t_0 > 0$. We call the time interval $[(n-1)t_0, nt_0)$ the nth period, $n = 1, 2, \ldots, N$. Inventory replenishments are only allowed at the start of each of these N periods.

The order of events at the start of each period, assuming all machines begin operating at time 0, is: (1) Each machine is monitored to obtain a degradation signal value. (2) If the degradation parameter is unknown, the distribution of the parameter is updated using the degradation signal values obtained up to that time. (3) This distribution and the observed signal values are used to update the demand distribution for each machine. (4) The optimal inventory policy is updated.

4 Derivation of Demand Distribution Using Degradation Signal Model

As in any inventory control problem, the first step is to characterize the demand distribution. We will do so for two cases: (1) the degradation parameter, θ_i, is known and (2) θ_i is unknown.

4.1 Demand Distribution for Components with Known Drift Parameters

The demand for spare parts at machine i is driven by the failure of parts operating on machine i. Thus, we start by deriving a closed form expression for $F_i(\cdot)$, the cumulative distribution function (cdf) of the time between successive failures, e.g., between the $(k-1)$st and kth failures at machine i. Since a part is assumed to fail when its degradation signal reaches the threshold B, $F_i(\cdot)$ is the distribution of the time it takes for the degradation signal process from the kth part used on machine i to reach the failure threshold (barrier) B. To determine $F_i(\cdot)$, we use the concept of the first passage time for a Wiener process (Cox and Miller 1965). The first passage time, T, is defined as the time until the process, starting from 0, first reaches the absorbing barrier, B, where an absorbing barrier is a threshold such that, once the process crosses the threshold, the process is stopped. For a linear Wiener process, Z_i, with drift parameter θ_i and barrier B, let $T(\theta_i, B)$ denote the first passage time. Then the cdf of T is (Cox and Miller 1965, Section 5.7):

$$F_i(t) = P\left\{T(\theta_i, B) \le t\right\} = 1 - \Phi\left(\frac{B - \theta_i t}{\sigma\sqrt{t}}\right) + \exp\left\{\frac{2\theta_i B}{\sigma^2}\right\}\Phi\left(\frac{-B - \theta_i t}{\sigma\sqrt{t}}\right). \tag{1}$$

Next, we can find the distribution of the k-fold convolution of $F_i(\cdot)$, denoted $F_i^{(k)}(\cdot)$:

Proposition 4.1 $F_i^{(k)}(t) = P\{T(\theta_i, kB) \le t\}$.

Thus, the distribution of the time it takes for a single part to reach failure threshold kB is the same as the distribution of the time it takes for k successive parts to each reach failure threshold B.

We can now derive the pmf of the demand per period.

Proposition 4.2 *Let D_{in} represent the demand for parts from machine i during the period n, $n = 1, \ldots, N$. Let z_n^i represent the degradation signal value for the part operating on machine i, observed* *at the start of period n. Then the pmf of D_{in} can be written as follows:*

$$P\left\{D_{in}=k|\theta_i,z_n^i\right\}=P\left\{T(\theta_i,z_n^i,kB)\le t_0,T(\theta_i,z_n^i,(k+1)B)>t_0\right\}$$
$$=P\left\{T(\theta_i,z_n^i,kB)\le t_0\right\}-P\left\{T(\theta_i,z_n^i,(k+1)B)\le t_0\right\}, \qquad (2)$$

for $k\in\mathbb{N}\bigcup\{0\}$, where $P\left\{T(\theta_i,z_n^i,0B)\le t_0\right\}=1$, for any $z_n^i\in\mathbb{R}$ and

$$P\{T(\theta_i,B_1,B_2)\le t\}=1-\Phi\left(\frac{(B_2-B_1)-\theta_i t}{\sigma\sqrt{t}}\right)+\exp\left\{\frac{2\theta_i(B_2-B_1)}{\sigma^2}\right\}\Phi\left(\frac{-(B_2-B_1)-\theta_i t}{\sigma\sqrt{t}}\right).$$

Let D_n represent the demand for parts from a set of m machines during period n. Let $\vec{z}_n=[z_n^1,\ldots,z_n^m]$ denote the array of observed degradation signal values for the parts operating on all m machines at the start of period n. Then the pmf of D_n, given $\vec{\theta}$ and $\vec{z}_n$:

$$P\left\{D_n=k|\vec{\theta},\vec{z}_n\right\}=\sum_{(l_1,\ldots,l_m):\ \sum_{i=1}^m l_i=k}\left(\prod_{i=1}^m P\left\{D_{in}=l_i|\theta_i,z_n^i\right\}\right),\quad k\in\mathbb{N}\bigcup\{0\}. \qquad (3)$$

4.2 Demand Distribution for Components with Unknown Drift Parameters

We next derive the distribution of demand when the drift parameters are unknown. To estimate the unknown drift parameter, θ_i, we follow Li (2001). We treat θ_i as a random variable, denoted by Θ_i, with a Normal prior distribution, $N(\mu_{i1},\sigma_{i1}^2)$, $i=1,\ldots,m$. As we obtain degradation signal observations, we use these observations to update the distribution of Θ_i in a Bayesian manner. Since θ_i is the same for each part used on machine i, we use all of the degradation signal values that have been obtained from machine i up to that time. Specifically, at the beginning of period 1, with observed signal value $z_1^i=0$, Θ_i follows the prior distribution, $f_{\Theta_i|z_1^i}(\cdot)$. This distribution may be obtained from historical data or from information obtained from the part's manufacturer. At the beginning of period n, $n=1,2,\ldots,N$, we combine the observed degradation signal value, z_n^i, and the posterior distribution of Θ_i, updated at the start of period $n-1$, denoted by $f_{\Theta_i|z_1^i,\ldots,z_{n-1}^i}(\cdot)$, to derive a new updated distribution for Θ_i, denoted by $f_{\Theta_i|z_1^i,\ldots,z_n^i}(\cdot)$. Using this procedure, as in Li (2001), we can show that the updated (posterior) distribution of Θ_i, given $z_1^i,\ldots,z_n^i$, is Normal with mean, μ_{in}, and standard deviation, σ_{in}, where μ_{in} and σ_{in} are given in the Appendix.

Let $P\left\{D_{in}=k|z_1^i,\ldots,z_n^i\right\}$ denote the pmf of the demand at machine i in period n, given that θ_i is unknown. This probability function depends on all of the observed signal values up to and including the start of period n. Using the pmf for demand at machine i with known θ_i, and the posterior distribution of Θ_i, we have $P\left\{D_{in}=k|z_1^i,\ldots,z_n^i\right\}=\int_{-\infty}^{\infty}P\left\{D_{in}=k|\theta,z_n^i\right\}f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta$, for $k\in\mathbb{N}\bigcup\{0\}$, and $i=1,\ldots,m$. We can further simplify $P\left\{D_{in}=k|z_1^i,\ldots,z_n^i\right\}$:

For $k=0$:

$$\int_{-\infty}^{\infty}P\left\{D_{in}=0|\theta,z_n^i\right\}f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta=\int_{-\infty}^{\infty}P\{T(\theta,z_n^i,B)>t_0\}f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta. \tag{4}$$

For $k\geq 1$:

$$\begin{aligned}&\int_{-\infty}^{\infty}P\left\{D_{in}=k|\theta,z_n^i\right\}f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta\\ &=\int_{-\infty}^{\infty}\Big[P\{T(\theta,z_n^i,kB)\leq t_0\}-P\{T(\theta,z_n^i,(k+1)B)\leq t_0\}\Big]f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta.\end{aligned} \tag{5}$$

In (5), for $k\geq 1$,

$$\begin{aligned}\int_{-\infty}^{\infty}&P\{T(\theta,z_n^i,kB)\leq t_0\}f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta=\int_{-\infty}^{\infty}\left(\int_0^{t_0}f_{T(\theta,z_n^i,kB)}(t)dt\right)f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta\\ &=\int_0^{t_0}\left(\int_{-\infty}^{\infty}f_{T(\theta,z_n^i,kB)}(t)f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta)d\theta\right)dt \qquad (6)\\ &=\int_0^{t_0}\left(\int_{-\infty}^{\infty}\left(\frac{kB-z_n^i}{t}\right)\phi\left(\frac{kB-z_n^i-\theta t}{\sigma\sqrt{t}}\right)\phi\left(\frac{\theta-\mu_{in}}{\sigma_{in}}\right)d\theta\right)dt \qquad (7)\\ &=\int_0^{t_0}\left(\frac{kB-z_n^i}{\sqrt{2\pi t^3(\sigma_{in}^2+\sigma^2)}}\right)\exp\left\{-\frac{\mu_{in}^2}{2\sigma_{in}^2}-\frac{(kB-z_n^i)^2}{2\sigma^2 t}+\frac{[(kB-z_n^i)\sigma_{in}^2+\mu_{in}\sigma^2]^2}{2\sigma^2\sigma_{in}^2(\sigma_{in}^2 t+\sigma^2)}\right\}dt,\end{aligned}$$

where $\phi(w)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{w^2}{2}\right\}$, (6) is obtained by switching integrals, following the Fubini Theorem, and $f_{T(\theta,z_n^i,kB)}(t)$ is obtained by differentiating $P\{T(\theta,z_n^i,kB)\leq t\}$ in t.

Finally, we can obtain the distribution of the demand for parts in period n for a system of m machines, D_n, using the same approach as in (16).

5 Finding the Optimal Inventory Control Policy

To find an optimal inventory policy, we develop a dynamic programming (DP) formulation for this problem. In this formulation, the cost-to-go function depends on the signal values that have been observed through a given time. Thus, the optimal policy will also depend on these signal values. In addition, when we take the expectation of the cost-to-go functions, we must take into account the future signal values, which are random variables that will be observed after the replenishment decision is made. These random variables depend on the ages of the parts in use at the start of future planning periods, which are also random. Therefore, taking the expectation is not straightforward.

We first consider the case in which the degradation parameters are known, and then consider the case in which these parameters are estimated using observed degradation signal values.

5.1 Inventory Model with Known Drift Parameters $\{\theta_i : i = 1, \ldots, m\}$

At the start of each period n, we perform condition monitoring and acquire a degradation signal value z_n^i from each machine i. This value is used to update the demand distribution, as described in Section 4.1, and then to make our inventory decisions.

Let $C_n^i(x^i|\theta_i, z_n^i)$ denote the minimum expected inventory cost for machine i for the remainder of the horizon, i.e., for periods $n, n+1, \ldots, N$, given observed degradation signal value z_n^i, where x^i is the on-hand inventory level for machine i at the start of period n. This cost function depends only on the most recently observed degradation signal value, z_n^i, and not on the previously observed values, $z_1^i, \ldots, z_{n-1}^i$, since only z_n^i impacts the future demand distributions. We assume that the unit ordering, holding and penalty costs for the parts used on each machine are the same, and we denote them by $c > 0$, $0 < h < c$, and $p > h$, respectively.[2] We let $\alpha \in (0, 1)$ denote the discount factor for each period. We then have the following dynamic programming formulation for $C_n^i(x^i|\theta_i, z_n^i)$:

$$\begin{cases} C_n^i(x^i|\theta_i, z_n^i) = \min\limits_{y \geq x^i} \left\{ c(y - x^i) + L_n^i(y|\theta_i, z_n^i) + \alpha \sum\limits_{k=0}^{\infty} E\left[C_{n+1}^i(y-k|\theta_i, Z_{n+1}^i) 1_{\{D_{in}=k|\theta_i, z_n^i\}} \right] \right\} \\ \qquad\qquad\qquad\quad = \min\limits_{y \geq x^i} G_n^i(x^i, y|\theta_i, z_n^i), \\ C_{N+1}^i(x|\theta_i, z_{N+1}^i) = 0, \quad \text{for } x \in Z, z_{N+1}^i \in \mathbb{R}, \end{cases} \tag{8}$$

[2]Recall that we assume that a failed part is replaced immediately. Thus, we assume that, if there is no on-hand inventory when a part fails, a part can be obtained immediately from an outside source for a premium, p.

where

$$L_n^i(y|\theta_i, z_n^i) = \begin{cases} h\sum_{k=0}^{y}(y-k)P\left\{D_{in}=k|\theta_i, z_n^i\right\} + p\sum_{k=y}^{\infty}(k-y)P\left\{D_{in}=k|\theta_i, z_n^i\right\} & y \geq 0, \\ p\sum_{k=0}^{\infty}(k-y)P\left\{D_{in}=k|\theta_i, z_n^i\right\} & y < 0, \end{cases} \tag{9}$$

and

$$E\left[C_{n+1}^i(y-k|\theta_i, Z_{n+1}^i)1_{\{D_{in}=k|\theta_i,z_n^i\}}\right] = E\left[C_{n+1}^i\left(y-k|\theta_i, Z^i(A^i(nt_0))\right)1_{\{D_{in}=k|\theta_i,z_n^i\}}\right], \tag{10}$$

here $1_{\{D_{in}=k|\theta_i,z_n^i\}}$ is an indicator function and the expectation is taken over the random variable Z_{n+1}^i under the condition $D_{in} = k$, where Z_{n+1}^i represents the degradation signal value obtained at time nt_0, i.e., the start of period $n+1$. Since $Z_{n+1}^i = Z^i(A^i(nt_0))$, where $A^i(nt_0)$ is the age of the part in use on machine i at the start of period $n+1$, there are two sources of randomness: the age of the part, $A^i(nt_0)$, at the start of period and the noise in the degradation signal process, $W^i(\cdot)$. Given this DP formulation, we can now prove the following:

Proposition 5.1 *An order-up-to inventory policy with order-up-to level $\bar{x}_n^i(\theta_i, z_n^i)$ is optimal for period n, for all $n = 1, 2, \ldots, N$, where $\bar{x}_n^i(\theta_i, z_n^i)$ denotes the value of y minimizing the function $G_n^i(x^i, y|\theta_i, z_n^i)$, as defined in (8).*

The optimal order-up-to level for this problem, $\bar{x}_n^i(\theta_i, z_n^i)$, depends only on the most recent signal observation, z_n^i, and the degradation parameter, θ_i. Intuitively, if θ_i, the rate of degradation, increases, the life of the part is likely to decrease and thus the demand in period n is likely to be higher. If z_n^i increases, i.e., if the part in use at the start of period n is closer to failure, the demand in period n is likely to be higher. Thus, an increase in either θ_i or z_n^i should lead to an increase in the order-up-to level for period n. Formally, we can prove this result only for the final period:

Proposition 5.2 *$\bar{x}_N^i(\theta_i, z_N^i)$ is a non-decreasing function of θ_i and z_N^i.*

Next, we consider jointly managing the inventory for a system consisting of m machines. We assume that the system will use first come, first served inventory allocation to each machine. This policy is feasible since the probability that any two or more parts fail at the same time from different machines is zero. As for the single machine case, we have the following result:

Proposition 5.3 *An order-up-to inventory policy with order-up-to level $\bar{x}_n(\vec{\theta}, \vec{z}_n)$ is optimal for period n, for all $n = 1, 2, \ldots, N$, where $\bar{x}_n(\vec{\theta}, \vec{z}_n)$ denotes the value of y minimizing the function $G_n(x, y|\vec{\theta}, \vec{z}_n)$, defined in (24).*

Notice that the optimal order-up-to level for this problem, $\bar{x}_n(\vec{\theta}, \vec{z}_n)$, depends on the most recent degradation signal observation, z_n^i, obtained from each machine, as well as the degradation parameter, θ_i, for each machine, $i = 1, \ldots, m$.

5.2 Inventory Model with Unknown Drift Parameters $\{\theta_i : i = 1, \ldots, m\}$

Next, we extend the results in Section 5.1 to the case in which the degradation parameters, θ_i, $i = 1, \ldots, m$, are unknown and thus we use the Bayesian updating approach discussed in Section 4.2. The DP formulation (and associated notation) for this model is quite similar to that presented in Section 5.1. Thus, we will present the formulation with little discussion, except to point out the key differences between the two. For machine i, we have:

$$\begin{cases} C_n^i(x^i|z_1^i, \ldots, z_n^i) = \min\limits_{y \geq x^i} \Big\{ c(y - x^i) + L_n^i(y|z_1^i, \ldots, z_n^i) \\ \qquad\qquad + \alpha \sum\limits_{k=0}^{\infty} E\left[C_{n+1}^i(y - k|z_1^i, \ldots, z_n^i, Z_{n+1}^i) 1_{\{D_{in}=k|z_1^i, \ldots, z_n^i\}} \right] \Big\} \\ \qquad = \min\limits_{y \geq x^i} G_n^i(x^i, y|z_1^i, \ldots, z_n^i), \\ C_{N+1}^i(x|z_1^i, \ldots, z_{N+1}^i) = 0, \quad x \in Z, z_j^i \in \mathbb{R}, j = 1, \ldots, N+1. \end{cases} \tag{11}$$

The key difference between this formulation and that in Section 5.1 is that, when θ_i is known, the cost functions and expectations are conditioned on both θ_i, the known degradation parameter, and z_n^i, the most recent degradation signal observation. However, when θ_i is unknown, its posterior distribution will depend on the entire history of degradation signal observations from machine i. Thus, here the cost functions and expectations are conditioned on $z_1^i, \ldots, z_n^i$. In addition, in (11), the expectation is taken over $Z_{n+1}^i = Z^i(A^i(nt_0))$ and Θ_i. In Section 5.1, this expectation was taken only over $Z^i(A^i(nt_0))$. We can now specify the form of the optimal inventory policy for this model.

Proposition 5.4 *An order-up-to inventory policy with order-up-to level $\bar{x}_n^i(z_1^i, \ldots, z_n^i)$ is optimal for period n, for all $n = 1, 2, \ldots, N$, where $\bar{x}_n^i(z_1^i, \ldots, z_n^i)$ denotes the value of y minimizing the function $G_n^i(x^i, y|z_1^i, \ldots, z_n^i)$, defined in (11).*

When θ_i is unknown, the order-up-to level, $\bar{x}_n^i(z_1^i, \ldots, z_n^i)$, depends on all degradation signal observations obtained from machine i, as well as the ages of the parts when the signals were observed.

Finally, we consider jointly managing inventory for the m machines, assuming all degradation parameters are unknown.

Proposition 5.5 *An order-up-to inventory policy with order-up-to level $\bar{x}_n(\vec{z}_1, \ldots, \vec{z}_n)$ is optimal for period n, for all $n = 1, 2, \ldots, N$, where $\bar{x}_n(\vec{z}_1, \ldots, \vec{z}_n)$ denotes the value of y minimizing the function $G_n(x, y|\vec{z}_1, \ldots, \vec{z}_n)$, defined in (30).*

5.3 Comparison of Models with Known vs Unknown Drift Parameters

We next show that, as the number of the planning periods, N, increases to infinity, the minimum cost for the last period, given that θ_i is unknown but periodically updated, converges to the minimum cost of the last period, given that θ_i is known, for $i = 1, \ldots, m$. Let Θ_{in} denote a Normal random variable with mean μ_{in} and variance σ_{in}^2, the posterior mean and variance of Θ_i, as given in Section 4.2. Note that μ_{in} and σ_{in}^2 are functions of $Z_1^i, \ldots, Z_n^i$. We can now state our first convergence result:

Proposition 5.6 *Given the Bayesian updating procedure described in Section 4.2:*

$$\mu_{in} \overset{n\to\infty}{\to} \theta_i, \quad \textit{almost surely (a.s.)}, \tag{12}$$

$$\sigma_{in}^2 \overset{n\to\infty}{\to} 0, \quad \textit{a.s.}, \tag{13}$$

which implies $\Theta_{in} \overset{n\to\infty}{\to} \theta_i$, in distribution.

We use this result to prove convergence of the cost functions. We consider a finite horizon problem, i.e., $N < \infty$, so that $C_N^i(x^i|\theta_i, z_N^i)$ and $C_N^i(x^i|z_1^i, \ldots, z_N^i)$ denote the cost for the last period.

Proposition 5.7 *For any sequence of observed degradation signals, $z_1^i, \ldots, z_N^i$ we have:*

$$\bar{x}_N^i(z_1^i, \ldots, z_N^i) - \bar{x}_N^i(\theta_i, z_N^i) \overset{N\to\infty}{\to} 0,$$

$$C_N^i(x^i|z_1^i, \ldots, z_N^i) - C_N^i(x^i|\theta_i, z_N^i) \overset{N\to\infty}{\to} 0.$$

Note that this convergence result is for the expected cost and optimal order-up-to levels for the final period, period N, as discussed in the proof of the propostion.

6 Inventory Management in the Case of Spot Market Participation

Throughout this paper, we have considered the case in which the manufacturer purchases spare parts from a supplier at a constant unit cost, c. In this section, we briefly discuss the case in which the manufacturer replenishes spare parts from a spot market (SM). In this case, the manufacturer may either buy or sell parts on the spot market, depending on his current inventory status.

Let S_n denote the random spot market price at the start of period n, $n = 1, \ldots, N$. Similarly to how we defined $C_n^i(x^i|\theta_i, z_n^i)$ and $C_n^i(x^i|z_1^i, \ldots, z_n^i)$ above, for the SM case we define $C_{SM,n}^i(x^i|\theta_i, z_n^i)$ and $C_{SM,n}^i(x^i|z_1^i, \ldots, z_n^i)$ by omitting the condition $y \geq x_i$ and replacing $c(y - x^i)$ by $S_n(y - x^i)$ (since we may buy or sell at price S_n on the SM). We then have the following results:

Proposition 6.1 *When spare parts may be bought or sold on a spot market, a myopic critical fractile inventory policy is optimal in each period n, where the order-up-to level, $\bar{x}_{SM,n}^i$, satisfies:*

- *$\underline{\theta_i \text{ known}}$: $\bar{x}_{SM,n}^i(\theta_i, z_n^i)$ is the smallest value of y such that $F_{D|\theta_i, z_n^i}(y) \geq \frac{p - s_n + \alpha E[S_{n+1}]}{h+p}$.*
- *$\underline{\theta_i \text{ unknown}}$: $\bar{x}_{SM,n}^i(z_1^i, \ldots, z_n^i)$ is the smallest value of y such that $F_{D|z_1^i, \ldots, z_n^i}(y) \geq \frac{p - s_n + \alpha E[S_{n+1}]}{h+p}$.*
- *For any sequence of observed degradation signals, $z_1^i, \ldots, z_n^i, \ldots$, we have:*

$$\bar{x}_{SM,n}^i(z_1^i, \ldots, z_n^i) - \bar{x}_{SM,n}^i(\theta_i, z_n^i) \overset{n \to \infty}{\longrightarrow} 0,$$

$$C_{SM,n}^i(x^i|z_1^i, \ldots, z_n^i) - C_{SM,n}^i(x^i|\theta_i, z_n^i) \overset{n \to \infty}{\longrightarrow} 0.$$

7 Implementation Issues

In this paper, we have developed a framework for incorporating real-time condition monitoring information into inventory management for machine parts that are subject to deterioration, when the deterioration process can be captured via condition monitoring and modeled using a Wiener process. Many manufacturing organizations have been pursuing the development of condition monitoring technologies. Thus, models such as the one presented in this paper provide a useful framework for making use of information obtained through condition monitoring. However, one issue that arises when attempting to implement models such as this is that the optimal policies tend to be difficult to compute and may depend on large quantities of data. Therefore, we conclude with a discussion of a number of issues related to the implementation of these policies. The challenges can be divided into two categories: (1) updating the distribution of demand and (2) solving the DP formulation.

Regarding (1), when the degradation parameter is known, the computation of the pmf for demand, as presented in Proposition 4.2, is relatively straight-forward, requiring nothing more complex than the evaluation of the cumulative distribution function (cdf) of a normal distribution. When the degradation parameter is unknown, equation (7) indicates that the computation requires numerical integration of a one-dimensional integral. In addition, the pmf of demand must be recomputed each time a new signal is observed and will depend on some subset of observed signals. When the degradation parameter is unknown, one obvious way to simplify the computation of the pmf of demand is to apply the demand model for the *known* degradation parameter case, using some estimate of the unknown degradation parameter, e.g., the mean of the updated posterior distribution of θ. Such a procedure would make use of all relevant degradation signals (through the computation of the posterior distribution of θ) while greatly simplifying the computational requirements.

Regarding (2), solving even simple DPs can be a computational challenge, in large part due to the so-called curse of dimensionality (Bellman 1957). In our problem, the state space will inlclude some set of observed degradation signal values, a set which may be particularly large when the degradation parameter is unknown. Additionally, the observed degradation signal values have a continuous state space, $(-\infty, B)$. The derivation of the optimal order-up-to level for any period n, given any set of observed signals, requires the computation of the optimal order-up-to levels for all future periods, and, more importantly, for every possible value of the future degradation signal observations, $Z^i_{n'} \in (-\infty, B)$, $n' = n+1, \ldots, N-1, N$. Thus, exact computation of the optimal order-up-to levels is likely to be infeasible, particularly for the case of unknown θ.

Therefore, we consider the development of heuristic approaches to our problem. One key issue when developing such heuristics is to ensure that, like the optimal solution, the heuristics make use of the information obtained through condition monitoring, i.e., the history of degradation signal observations. One approach is to adapt the results of Lovejoy (1990), who studied a class of parameter adaptive demand models (which include traditional Bayesian inventory models) where the demand distribution depends on some unknown parameters and the beliefs regarding this parameter are updated in a statistical fashion as actual demand is realized over time. He suggested a simple heuristic: a myopic critical fractile order-up-to policy. In other words, the order-up-to level in each period n, say $\bar{x}_n$, should satisfy $F_n(\bar{x}_n) = \lambda$, where $F_n(\cdot)$ is the cdf of demand in period n, updated to reflect all information about demand observed up to period n, and λ, $0 \leq \lambda \leq 1$, is the critical fractile, a function of the cost parameters. Moreover, he developed upper bounds on the value loss relative to optimal cost and demonstrated that myopic policies can perform well in some cases.

Specifically, to develop the myopic policy for a multi-period inventory model, Lovejoy (1990) considers a model in which inventory may be returned with a full-refund. Such a model is referred to as the *disposal case* (DC) and is equivalent to our spot market model in Section 6 with $S_n = c$, i.e., in any period inventory may be purchased or returned at the unit price c. Thus, for machine i, when θ_i is known, the myopic order-up-to level would be the smallest value of x that satisfies $F_{D_n|\theta_i, z_n^i}(x) \geq \frac{p-c+\alpha c}{h+p}$. Similarly, when θ_i is unknown, the myopic order-up-to level would be the smallest value of x that satisfies $F_{D_n|z_1^i,\ldots,z_n^i}(x) \geq \frac{p-c+\alpha c}{h+p}$. It is important to note that this policy makes use of the entire relevant history of the observed degradation signals and, more importantly, is easy to compute and implement. On the other hand, such a policy ignores any potential for learning about demand in future periods. In addition, since the problem studied in this paper does not allow inventory to be returned (except for being salvaged at the end of the planning horizon), the myopic policy will result in overstocking relative to the optimal policy.

Given the myopic policy, we can develop an upper bound on the value loss relative to the optimal cost by following Lovejoy's analysis. We present the results for θ_i is known. The case of θ_i unknown is similar. First, define the following notation:

- $C_1^i(x^i|\theta_i, z_1^i)$, defined in (8), is the expected cost over the entire planning horizon for the model in Section 5.1, assuming the optimal control policy, specified in Proposition 5.1, is used.

- $\tilde{C}_1^i(x^i|\theta_i, z_1^i)$ is the expected cost over the entire planning horizon for the model presented in Section 5.1, assuming the myopic critical fractile policy is used in each period.

- $C_{DC,1}^i(x^i|\theta_i, z_1^i, c_s)$ is the expected cost over the entire planning horizon for the disposal case presented in Lovejoy (1990), assuming the myopic critical fractile policy is used in each period, where c_s is the unit stock disposal cost. This model is similar to our spot market model in Section 6, but with differing buying and selling prices, c and c_s, respectively.

Then, Proposition 4 in Lovejoy (1990) can be easily proved for our model (see the appendix) and we have the following upper bound on the value lost when using the myopic critical fractile policy:

$$\frac{\tilde{C}_1^i(x^i|\theta_i, z_1^i) - C_1^i(x^i|\theta_i, z_1^i)}{C_1^i(x^i|\theta_i, z_1^i)} \leq \frac{C_{DC,1}^i(x^i|\theta_i, z_1^i, c_s = \frac{h}{1-\alpha}) - C_{DC,1}^i(x^i|\theta_i, z_1^i, c_s = -c)}{C_{DC,1}^i(x^i|\theta_i, z_1^i, c_s = -c)},$$

where the two cost terms on the right hand side, which assume a myopic critical fractile policy is used, are straightforward to compute.

8 References

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9 Appendix for Proofs and Derivations

Proof of Proposition 4.1: First, recall that we assume that the degradation signal process from the part used on machine i is the Wiener process Z^i satisfying $Z^i(t) = \theta_i t + \sigma W^i(t),\ \ Z^i(0) = 0$, with an absorbing barrier at $Z^i = B$, the failure threshold.

Clearly, to prove the proposition, it suffices to show that $\sum_{j=1}^{k} X_j^i = S_k^i = T(\theta_i, k)$. To this end, we take any path, from 0 to kB, of a Wiener process Z^i with absorbing barrier $Z^i = kB$. Note that the time duration of this path is a realization of random variable $T(\theta_i, kB)$. Consider a graph for this path with X-axis for the time and Y-axis for the value of Z^i.

We cut this path into k pieces using k horizontal lines, which in order are $Z^i = B$, $Z^i = 2B, \ldots, Z^i = kB$. Therefore, the jth piece is a path of Z^i first reaches $Z^i = jB$, starting from $(j-1)B$, for all $j = 1, \ldots, k$. Now for each j, we shift the jth piece starting from $(j-1)B$ down by $(j-1)B$. Thus, each shifted piece will start from 0. In other words, each piece itself is a path of degradation signal process Z^i with absorbing barrier B. Note that these pieces are mutually independent, which is due to fact that the Brownian motion W^i has independent increments. Therefore, the time duration of the jth piece is a realization of X_j^i, $j = 1, \ldots, k$. Note that the discussion above applies to any path of the Wiener process Z^i from 0 to kB. This concludes the proof. ∎

Proof of Proposition 4.2: Let $T(\theta_i, B_1, B_2)$ denote the time it takes for the Wiener process Z^i to first reach an absorbing barrier B_2, given that it starts from level B_1, for any $B_1 < B_2$. By shifting this process down by B_1, it is easy to see that $T(\theta_i, B_1, B_2)$ has the same distribution as $T(\theta_i, B_1 - B_2)$, the time it takes for Z^i, starting from 0, to first reach an absorbing barrier, $B_2 - B_1 > 0$. Therefore, using (1), we have the following cdf for $T(\theta_i, B_1, B_2)$, given θ_i:

$$P\{T(\theta_i, B_1, B_2) \le t\} = 1 - \Phi\left(\frac{(B_2 - B_1) - \theta_i t}{\sigma\sqrt{t}}\right) + \exp\left\{\frac{2\theta_i(B_2 - B_1)}{\sigma^2}\right\}\Phi\left(\frac{-(B_2 - B_1) - \theta_i t}{\sigma\sqrt{t}}\right). \tag{14}$$

Our goal is to compute, at some time t, the distribution of future demand for parts from machine i. To do this, we need to know the age of the part in use on machine i. We let $A_i(t)$ be the random variable representing the age of the part in use on machine i at time t and $a_i(t)$ be the observed value of the age of the part in use on machine i at time t. Finally, we let $z^i(a_i(t))$ denote the observed degradation signal value of the part in use at time t on machine i[3]. Since the part in use on machine i has not failed at time t, $z^i(a_i(t)) < B$.

Let D_{in} represent the demand for parts from machine i during the period n, $n = 1, \ldots, N$. Let $z_n^i = z^i\left(a_i((n-1)t_0)\right)$ represent the degradation signal value for the part operating on machine i, observed at the start of period n. We can find the pmf of D_{in} as follows:

$$\begin{aligned} P\left\{D_{in} = k|\theta_i, z_n^i\right\} &= P\left\{T(\theta_i, z_n^i, kB) \le t_0, T(\theta_i, z_n^i, (k+1)B) > t_0\right\} \\ &= P\left\{T(\theta_i, z_n^i, kB) \le t_0\right\} - P\left\{T(\theta_i, z_n^i, (k+1)B) \le t_0\right\}, \end{aligned} \tag{15}$$

for $k \in \mathbb{N}\bigcup\{0\}$, where $P\left\{T(\theta_i, z_n^i, 0B) \le t_0\right\} = 1$, for any $z_n^i \in \mathbb{R}$.

Finally, we can derive the pmf of the demand per period for the entire system of m machines. Let D_n represent the demand for parts from all machines during period n. We let $\vec{\theta} = [\theta_1, \ldots, \theta_m]$ denote the array of the drift parameters of the degradation signal process from each machine. We let $\vec{z}_n = [z_n^1, \ldots, z_n^m]$ denote the array of observed degradation signal values for the parts operating on all m machines at the start of period n. Since the demand for parts for the system as a whole is just the sum of the demands at the individual machines, we have the following expression for the pmf of D_n, given $\vec{\theta}$ and $\vec{z}_n$:

$$P\left\{D_n = k|\vec{\theta}, \vec{z}_n\right\} = \sum_{(l_1,\ldots,l_m):\ \sum_{i=1}^m l_i = k} \left(\prod_{i=1}^m P\left\{D_{in} = l_i|\theta_i, z_n^i\right\}\right), \quad k \in \mathbb{N}\bigcup\{0\}. \tag{16}$$

∎

Posterior Distribution of Θ_i: As noted in Section 4.2, it is easy to show that the updated (posterior) distribution of Θ_i, given $z_1^i, \ldots, z_n^i$, is Normal with mean, denoted by μ_{in}, and standard

[3] We will follow the convention that capital letters represent random variables / processes and lower case letters denote observed values of these random variables.

deviation, denoted σ_{in}, where:

$$\mu_{in} = \begin{cases} \left(\dfrac{\sigma^2}{t_0\sigma^2_{i(n-1)}+\sigma^2}\right)\mu_{i(n-1)} + \left(\dfrac{t_0\sigma^2_{i(n-1)}}{t_0\sigma^2_{i(n-1)}+\sigma^2}\right)\left(\dfrac{z^i_n - z^i_{n-1}}{t_0}\right) & \text{If SP,} \\ \left(\dfrac{\sigma^2}{a_i((n-1)t_0)\sigma^2_{i(n-1)}+\sigma^2}\right)\mu_{i(n-1)} + \left(\dfrac{a_i((n-1)t_0)\sigma^2_{i(n-1)}}{a_i((n-1)t_0)\sigma^2_{i(n-1)}+\sigma^2}\right)\left(\dfrac{z^i_n}{a_i((n-1)t_0)}\right) & \text{If not SP.} \end{cases}$$

$$\sigma^2_{in} = \begin{cases} \dfrac{\sigma^2_{i(n-1)}\sigma^2}{t_0\sigma^2_{i(n-1)}+\sigma^2} & \text{If SP,} \\ \dfrac{\sigma^2_{i(n-1)}\sigma^2}{a_i((n-1)t_0)\sigma^2_{i(n-1)}+\sigma^2} & \text{If not SP.} \end{cases} \tag{17}$$

Here, $a_i((n-1)t_0)$ represents the age of the part in use on machine i at time $(n-1)t_0$, i.e., at the start of period n. Case SP (same part) refers to the situation in which the degradation signals z^i_{n-1} and z^i_n come from a same part, i.e., the part under observation at the start of period n is the part in use on that machine at the start of period $n-1$.

Proof of Proposition 5.1: We first simplify (10). Note that $Z^i(A^i(nt_0))$ depends on whether $D_{in} = 0$ or $D_{in} > 0$. It is easy to see that $D_{in} = 0$ (i.e., $k = 0$) is equivalent to $A^i(nt_0) > t_0$, while $D_{in} > 0$ (i.e., $k > 0$) is equivalent $A^i(nt_0) \leq t_0$. Therefore, we can simplify the expectation term by first considering the case $D_{in} = 0$ and then considering the case $D_{in} > 0$.

For $D_{in} = 0$: It is easy to see that $D_{in} = 0$ is equivalent to the event that the process Z^i, starting from the observed value z^i_n at the start of period n, has not yet reached the failure threshold B. In other words,

$$1_{\{D_{in}=0|\theta_i, z^i_n\}} = 1_{\{T(\theta_i, z^i_n, B) > t_0\}}. \tag{18}$$

Also, given that $D_{in} = 0$ and $A^i((n-1)t_0) = a \in [0, (n-1)t_0]$, i.e., the age of the part in use at the start of period n was a, the signal value at the start of period $n+1$, $Z^i(A^i(nt_0))$, can be written as:

$$Z^i(A^i(nt_0)) = Z^i(a+t_0) = Z^i(a) + \theta_i t_0 + \sigma W^i(a+t_0) - \sigma W^i(a) = z^i_n + \theta_i t_0 + \sigma W(t_0),$$

where W is a standard Brownian motion independent of W^i, with restriction that the Wiener process, $\{\theta_i t + \sigma W(t) : t \geq 0\}$, has not reached an absorbing barrier $B - z_n^i$ by time t_0.

Next, from Cox and Miller (1965), for the Wiener process Z^i with an absorbing barrier $B > 0$, the density function of $Z^i(t) \leq B$ is:

$$p(x,t;B,\theta_i) = \frac{1}{\sigma\sqrt{2\pi t}}\left[\exp\left\{-\frac{(x-\theta_i t)^2}{2\sigma^2 t}\right\} - \exp\left\{\frac{2\theta_i B}{\sigma^2} - \frac{(x-2B-\theta_i t)^2}{2\sigma^2 t}\right\}\right], \tag{19}$$

where x denotes one realization of $Z^i(t)$. Note that there are three parameters in this density function: t, B, θ_i, where t represents the time of the process Z^i, B represents the absorbing barrier for Z^i, and θ_i is the drift parameter of Z^i.

Using (19), we have the following result:

$$\begin{aligned} E\left[C_{n+1}^i\Big(y|\theta_i, Z^i(A^i(nt_0))\Big)1_{\{D_{in}=0|\theta_i,z_n^i\}}\right] &= E\left[C_{n+1}^i\Big(y|\theta_i, Z^i(A^i(nt_0))\Big)1_{\{T(\theta_i,z_n^i,B)>t_0\}}\right] \\ &= \int_{-\infty}^{B-z_n^i} C_{n+1}^i(y|\theta_i, z_n^i + x)p(x,t_0;B-z_n^i,\theta_i)dx \\ &= \int_{-\infty}^{B} C_{n+1}^i(y|\theta_i, x)p(x-z_n^i,t_0;B-z_n^i,\theta_i)dx. \end{aligned} \tag{20}$$

<u>For $D_{in} > 0$:</u> It is easy to see that $D_{in} = k$ for $k \geq 1$, is equivalent to the event that the Wiener process Z^i, starting from z_n^i at the start of period n, has reached kB, but has not yet reached $(k+1)B$, at the end of period n. In other words,

$$1_{\{D_{in}=k|\theta_i,z_n^i\}} = 1_{\{T(\theta_i,z_n^i,kB)<t_0, T(\theta_i,z_n^i,(k+1)B)>t_0\}} = 1_{\{T(\theta_i,z_n^i,kB)<t_0, T(\theta_i,z_n^i,kB)+A^i(nt_0)>t_0\}}, \tag{21}$$

which implies

$$\begin{aligned} &\sum_{k=1}^{\infty} E\left[C_{n+1}^i\Big(y-k|\theta_i, Z^i(A^i(nt_0))\Big)1_{\{D_{in}=k|\theta_i,z_n^i\}}\right] \\ &= \sum_{k=1}^{\infty} E\left[C_{n+1}^i\Big(y-k|\theta_i, Z^i(A^i(nt_0))\Big)1_{\{T(\theta_i,z_n^i,kB)<t_0, T(\theta_i,z_n^i,kB)+A^i(nt_0)>t_0\}}\right] \\ &= \sum_{k=1}^{\infty} \int_0^{t_0}\int_{-\infty}^{B} C_{n+1}^i(y-k|\theta_i,x)p(x,a;B,\theta_i)f_{T(\theta_i,z_n^i,kB)}(t_0-a)dxda, \end{aligned} \tag{22}$$

where a is a realization of $A^i(nt_0)$, x is a realization of $Z^i(a)$, $f_{T(\theta_i,z_n^i,kB)}(t_0-a)$ is the conditional density function of $A^i(nt_0)$, conditioning on $Z^i_n = Z^i(A^i((n-1)t_0)) = z^i_n$. In other words, $p(x, a; B, \theta_i) f_{T(\theta_i, z^i_n, kB)}(t_0 - a)$ is the joint density function for $A^i(nt_0)$ and Z^i_{n+1} for the case $D_{in} = k$. To understand this, we note that $D_{in} = k \geq 1$ implies $A^i(nt_0) < t_0$, i.e., the age of the part in use on machine i at the start of period $n+1$ is less than t_0. It also implies that it takes $t_0 - A^i(nt_0)$ time units for the process Z^i to first reaches kB starting from z^i_n. Integrating with respect to a and then with respect to x, we obtain (22).

Combining the results for $D_{in} = 0$ and $D_{in} > 0$, given in (20) and (22), we have:

$$\begin{aligned} &\sum_{k=0}^{\infty} E\left[C^i_{n+1}(y-k|\theta_i, Z^i_{n+1}) 1_{\{D_{in}=k|\theta_i, z^i_n\}}\right] \\ &= \int_{-\infty}^{B} C^i_{n+1}(y|\theta_i, x) p(x - z^i_n, t_0; B - z^i_n, \theta_i) dx \\ &\quad + \sum_{k=1}^{\infty} \int_0^{t_0} \int_{-\infty}^{B} C^i_{n+1}(y^i - k|\theta_i, x) p(x, a; B, \theta_i) f_{T(\theta_i, z^i_n, kB)}(t_0 - a) dx da. \end{aligned} \tag{23}$$

Given the dynamic programming formulation as specified in (8), we can now prove the form of the optimal inventory policy. We will prove the result by induction. First, note that $c(y-x^i)+L^i_n(y|\theta_i, z^i_n)$ is a convex function of y. Therefore, the cost function for the last period

$$C^i_N(x^i|\theta_i, z^i_N) = \min_{y \geq x^i} \left\{ c(y - x^i) + L_N(y|\theta_i, z^i_N) \right\},$$

is convex with respect to x^i and an order-up-to policy is optimal for the last period. This is a typical result for similar inventory models.

Next we need to show that the convexity results hold for period n. Note that the expression for $\sum_{k=0}^{\infty} E\left[C^i_{n+1}(y-k|\theta_i, Z^i_{n+1}) 1_{\{D_{in}=k|\theta_i, z^i_n\}}\right]$, given by (23), has variable y contained only in the cost functions $C^i_{n+1}(y-k|\theta_i, x)$, $k \in \mathbb{N} \bigcup \{0\}$. Moreover, these cost functions are inside the integration. Hence, we claim that $\sum_{k=0}^{\infty} E\left[C^i_{n+1}(y-k|\theta_i, Z^i_{n+1}) 1_{\{D_{in}=k|\theta_i, z^i_n\}}\right]$ preserves the convexity of y, which implies that the convexity results hold for $C^i_n(x^i|\theta_i, z^i_n)$ and an order-up-to policy is optimal for period n. This concludes the proof. ∎

Proof of Proposition 5.2: Note that, throughout this paper, we have considered integer demand, and thus integer order-up-to levels are optimal. Thus, it is easy to see that the derivative of the convex function $G_n^i(x^i, y|\theta_i, z_n^i)$ at $y = \bar{x}_N^i(\theta_i, z_N^i)$, the optimal order-up-to level for the last period, may or may not be 0. Without loss of generality, we assume that this derivative is positive and thus at $y = \bar{x}_N^i(\theta_i, z_N^i) - 1$, the derivative of $G_n^i(x^i, y|\theta_i, z_n^i)$ is negative.

We now continuize the cdf of $D_{iN}|\theta_i, z_N^i$ on support $[\bar{x}_N^i(\theta_i, z_N^i) - 1, \bar{x}_N^i(\theta_i, z_N^i)]$. From (15), we know that $P\{D_{iN} \leq k|\theta_i, z_N^i\} = 1 - P\{T(\theta_i, z_N^i, (k+1)B) \leq t_0\}$. Note that $1 - P\{T(\theta_i, z_N^i, x) \leq t_0\}$ is an increasing function of x for any θ_i and z_N^i. Thus, for any $y \in [\bar{x}_N^i(\theta_i, z_N^i) - 1, \bar{x}_N^i(\theta_i, z_N^i)]$, we assume that $P\{D_{iN} \leq y|\theta_i, z_N^i\} = 1 - \{T(\theta_i, z_N^i, (y+1)B) \leq t_0\}$ and thus the cdf of $D_{iN}|\theta_i, z_N^i$ for support $[\bar{x}_N^i(\theta_i, z_N^i) - 1, \bar{x}_N^i(\theta_i, z_N^i)]$ is continuous and differentiable.

Let $\tilde{x}_n^i(\theta_i, z_n^i) \in \mathbb{R}$ denote the value of $y \in [\bar{x}_N^i(\theta_i, z_N^i) - 1, \bar{x}_N^i(\theta_i, z_N^i)]$ at which the derivative of $G_n^i(x^i, y|\theta_i, z_n^i)$ is 0. By definition, we have

$$
\begin{aligned}
& c + \frac{d}{dy} L_N^i(y|\theta_i, z_N^i)\Big|_{y=\tilde{x}_N^i(\theta_i, z_N^i)} = 0 \\
\Leftrightarrow\ & 1 - P\Big\{T(\theta_i, z_N^i, (\tilde{x}_N^i(\theta_i, z_N^i) + 1)B) \leq t_0\Big\} = \frac{p-c}{p+h} > 0 \qquad \text{(using (15))},
\end{aligned}
$$

Note that for any given z_N^i (θ_i) and x, as θ_i (z_N^i) increases, $1 - P\{T(\theta_i, z_N^i, x) \leq t_0\}$ decreases. Therefore, it is easy to see that if either θ_i increases or z_N^i increases, $\tilde{x}_N^i(\theta_i, z_N^i)$ should increase. Since $\tilde{x}_N^i(\theta_i, z_N^i)$ is between two integers, $\bar{x}_N^i(\theta_i, z_N^i) - 1$ and $\bar{x}_N^i(\theta_i, z_N^i)$, it is easy to see that $\bar{x}_N^i(\theta_i, z_N^i)$ should not decrease as θ_i increases or z_N^i increases. ∎

Proof of Proposition 5.3: Let $C_n(x|\vec{\theta}, \vec{z}_n)$ denote the minimum discounted expected cost for periods $n, n+1, \ldots, N$ for the entire system, given that the on-hand inventory level held for the system is x at the start of period n, $n = 1, 2, \ldots, N$. We then have the following dynamic programming formulation for $C_n(x|\vec{\theta}, \vec{z}_n)$, which is similar to (8):

$$
\begin{cases}
C_n(x|\vec{\theta}, \vec{z}_n) = \min\limits_{y \geq x} \left\{ c(y-x) + L_n(y|\vec{\theta}, \vec{z}_n) + \alpha \sum\limits_{k=0}^{\infty} E\left[C_{n+1}(y-k|\vec{\theta}, \vec{Z}_{n+1}) 1_{\{D_n = k|\vec{\theta}, \vec{z}_n\}} \right] \right\} \\
\qquad\qquad\quad = \min\limits_{y \geq x} G_n(x, y|\vec{\theta}, \vec{z}_n), \\
C_{N+1}(x|\vec{\theta}, \vec{z}_{N+1}) = 0, \quad x \in Z, \vec{z}_{N+1} \in \mathbb{R}^{N+1},
\end{cases}
\tag{24}
$$

where

$$L_n(y|\vec{\theta},\vec{z}_n) = \begin{cases} h\sum_{k=0}^{y}(y-k)P\left\{D_n = k|\vec{\theta},\vec{z}_n\right\} + p\sum_{k=y}^{\infty}(k-y)P\left\{D_n = k|\vec{\theta},\vec{z}_n\right\} & y \geq 0, \\ p\sum_{k=0}^{\infty}(k-y)P\left\{D_n = k|\vec{\theta},\vec{z}_n\right\} & y < 0, \end{cases} \tag{25}$$

in which $P\left\{D_n = k|\vec{\theta},\vec{z}_n\right\}$, $k \geq 0$, is given by (16). In addition, we can write:

$$\begin{aligned}
&\sum_{k=0}^{\infty} E\left[C_{n+1}(y-k|\vec{\theta},\vec{Z}_{n+1})1_{\left\{D_n=k|\vec{\theta},\vec{z}_n\right\}}\right] \\
&= \sum_{k=0}^{\infty}\sum_{\sum_{i=1}^{m} l_i=k} E\left[C_{n+1}(y-k|\vec{\theta},\vec{Z}_{n+1})1_{\{D_{in}=l_i|\theta_i,z_n^i,i=1,\ldots,m\}}\right] \\
&= \sum_{k=0}^{\infty}\sum_{\sum_{i=1}^{m} l_i=k} E\left[C_{n+1}(y-k|\vec{\theta},\vec{Z}_{n+1})1_{\{T(\theta_i,z_n^i,l_iB)<t_0,T(\theta_i,z_n^i,l_iB)+A^i(nt_0)>t_0,i=1,\ldots,m\}}\right] \text{ (use (21))} \\
&= \sum_{k=0}^{\infty}\sum_{\sum_{i=1}^{m} l_i=k} \int_0^{t_0}\int_{-\infty}^{B}\cdots\int_0^{t_0}\int_{-\infty}^{B} C_{n+1}(y-k|\vec{\theta},\vec{z}_{n+1})\prod_{i=1}^{m} g(z_{n+1}^i,a_i|\theta_i,z_n^i,l_i) \\
&\qquad\qquad dz_{n+1}^1 da_1\cdots dz_{n+1}^m da_m,
\end{aligned} \tag{26}$$

where $g(z_{n+1}^i,a_i|\theta_i,z_n^i,l_i)$ represents the joint density function of Z_{n+1}^i and $A^i(nt_0)$ given $l_i \geq 0$ and is similar to the joint density function used in (20) and (22) when $l_i = 0$ and $l_i > 0$, respectively. We derive this joint density function following the approach used in Section **??**.

For $l_i = 0$, $g(z_{n+1}^i,a_i|\theta_i,z_n^i,0)$ satisfies:

$$\int_0^{t_0}\int_{-\infty}^{B} g(z_{n+1}^i,a_i|\theta_i,z_n^i,0)dz_{n+1}^i da_i = \int_{-\infty}^{B} p(z_{n+1}^i - z_n^i,t_0;B-z_n^i,\theta_i)dz_{n+1}^i, \tag{27}$$

For $l_i > 0$, $g(z_{n+1}^i,a_i|\theta_i,z_n^i,l_i)$ satisfies:

$$g(z_{n+1}^i,a_i|\theta_i,z_n^i,l_i) = p(z_{n+1}^i,a_i;B-z_n^i,\theta_i)f_{T(\theta_i,z_n^i,l_iB)}(t_0-a_i). \tag{28}$$

To understand (26), we note that the pairs of random variables, $(Z_{n+1}^i, A^i(nt_0))$, $i = 1,\ldots,m$, are mutually independent. Integrating over $Z_{n+1}^1, A^1(nt_0),\ldots,Z_{n+1}^m, A^m(nt_0)$ using the corresponding joint density functions, we obtain (26).

Given this dynamic programming formulation, the same approach used to prove Proposition 5.1 can also be used for this case. Note that taking the summation over the minimum cost to go for each machine preserves convexity. ∎

Proof of Proposition 5.4: We first simplify (11) by replacing Z^i_{n+1} by $Z^i(A^i(nt_0))$ and considering two cases, $D_{in} = 0$ and $D_{in} > 0$, to obtain:

$$\begin{aligned}
&\sum_{k=0}^{\infty} E\left[C^i_{n+1}(y-k|z^i_1,\ldots,z^i_n,Z^i_{n+1})1_{\{D_{in}=k|z^i_1,\ldots,z^i_n\}}\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{B} C^i_{n+1}(y|z^i_1,\ldots,z^i_n,z^i_n+x)p(x-z^i_n,t_0;B-z^i_n,\theta)f_{\Theta_i|z^i_1,\ldots,z^i_n}(\theta)dxd\theta \\
&+\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\int_0^{t_0}\int_{-\infty}^{B} C^i_{n+1}(y^i-k|z^i_1,\ldots,z^i_n,x)p(x,a;B,\theta)f_{T(\theta,z^i_n,kB)}(t_0-a)f_{\Theta_i|z^i_1,\ldots,z^i_n}(\theta)dxdad\theta.
\end{aligned} \tag{29}$$

Next, the same approach used to prove Proposition 5.1 can also be used for this case with unknown degradation drift parameters. Note that taking the expectation with respect to Θ_i preserves convexity. ∎

Proof of Proposition 5.5: We first present the dynamic programming formulation for this problem.

$$\left\{\begin{aligned}
C_n(x|\vec{z}_1,\ldots,\vec{z}_n) &= \min_{y\geq x}\Big\{c(y-x)+L_n(y|\vec{z}_1,\ldots,\vec{z}_n) \\
&\qquad +\alpha\sum_{k=0}^{\infty} E\left[C_{n+1}(y-k|\vec{z}_1,\ldots,\vec{z}_n,\vec{Z}_{n+1})1_{\{D_n=k|\vec{z}_1,\ldots,\vec{z}_n\}}\right]\Big\} \\
&= \min_{y\geq x} G_n(x,y|\vec{z}_1,\ldots,\vec{z}_n), \\
C_{N+1}(x|\vec{z}_1,\ldots,\vec{z}_{N+1}) &= 0,\quad x\in Z, \vec{z}_j\in\mathbb{R}^m, j=1,\ldots,N+1,
\end{aligned}\right. \tag{30}$$

where for $k \geq 0$

$$\begin{aligned}
&E\left[C_{n+1}(y-k|\vec{z}_1,\ldots,\vec{z}_n,\vec{Z}_{n+1})1_{\{D_n=k|\vec{z}_1,\ldots,\vec{z}_n\}}\right] \\
&= \sum_{\sum_{i=1}^m l_i=k} E\left[C_{n+1}(y|\vec{z}_1,\ldots,\vec{z}_n,Z^i_{n+1})1_{\{D_{in}=l_i,i=1,\ldots,m|\vec{z}_1,\ldots,\vec{z}_n\}}\right]
\end{aligned}$$

$$= \sum_{\sum_{i=1}^m l_i = k} \int_{-\infty}^{\infty} \int_0^{t_0} \int_{-\infty}^{B} \cdots \int_{-\infty}^{\infty} \int_0^{t_0} \int_{-\infty}^{B} C_{n+1}(y-k|\vec{z}_1, \ldots, \vec{z}_n, \vec{z}_{n+1}) \prod_{i=1}^m g(z_{n+1}^i, a_i|\theta_i, z_n^i, l_i)$$
$$\prod_{i=1}^m f_{\Theta_i|z_1^i,\ldots,z_n^i}(\theta_i) dz_{n+1}^1 da_1 d\theta_1 \cdots dz_{n+1}^m da_m d\theta_m, \tag{31}$$

where $g(z_{n+1}^i, a_i|\theta_i, z_n^i, l_i)$ is given by (27) for $l_i = 0$, and is given by (28) for $l_i > 0$.

Given this formulation, the same approach used to prove Proposition 5.3 can also be used for this case with unknown degradation drift parameters. Note that taking the expectation with respect to Θ_i preserves convexity. ∎

Proof of Proposition 5.6: We first define some additional notation. Let Θ_{i1} denote a Normal random variable with mean μ_{i1} and variance σ_{i1}^2, the prior mean and variance of Θ_i. Similarly, we then let Θ_{in} denote the Normal random variable with mean μ_{in} and variance σ_{in}^2, the posterior mean and variance of Θ_i, as given in Section 4.2. Note that μ_{in} and σ_{in}^2 are functions of $Z_1^i, \ldots, Z_n^i$.

Using inductive equations for μ_{in} and σ_{in}^2 given in (17), we start by deriving explicit expressions for μ_{in} and σ_{in}^2 and then prove the convergence results for two extreme cases:(i) $Z_1^i, \ldots, Z_n^i$ all come from a same stochastic process, i.e., the part under observation at the start of period n is the first part used on machine i since the monitoring was started, and (ii) $Z_1^i, \ldots, Z_n^i$ all come from different stochastic processes, i.e., the parts in use at the start of each period are all different.

For Case (i), we have

$$\mu_{in} = \left(\frac{\sigma^2}{(n-1)t_0\sigma_{i1}^2 + \sigma^2}\right)\mu_{i1} + \left(\frac{(n-1)t_0}{(n-1)t_0\sigma_{i1}^2 + \sigma^2}\right)\left(\theta_i + \frac{W^i((n-1)t_0)}{(n-1)t_0}\right), \tag{32}$$
$$\sigma_{in}^2 = \frac{\sigma_{i1}^2\sigma^2}{(n-1)t_0\sigma_{i1}^2 + \sigma^2}, \tag{33}$$

where μ_{i1} and σ_{i1}^2 are the prior mean and variance of Θ_i preset at time 0. Note that in (32), a same Brownian motion, W^i, is used to express $Z_1^i, \ldots, Z_n^i$, due to the assumption that $Z_1^i, \ldots, Z_n^i$ come from a same signal process. We also note that μ_{in} only depends on $Z_n^i - Z_1^i$ and the prior mean and variance of θ_i, μ_{i1} and σ_{i1}^2, where $Z_1^i = 0$ in our model.

It is obvious that $\sigma_{in}^2 \overset{n\to\infty}{\to} 0$. We next show that $\mu_{in} \overset{n\to\infty}{\to} \theta_i$ a.s.. It is easy to see that in (32) the first term converges to 0 and the first item of the second term converges to 1. Thus, it suffices to show that $\dfrac{W^i((n-1)t_0)}{(n-1)t_0}$ converges to 0 a.s.. To this end, we note that

$$\frac{W^i((n-1)t_0)}{(n-1)t_0} = \frac{\sum_{j=1}^{n-1}\left(\frac{W^i(jt_0)-W^i((j-1)t_0)}{t_0}\right)}{n-1},$$

where the Normal random variables, $\frac{W^i(jt_0)-W^i((j-1)t_0)}{t_0}$, $j = 1,\ldots,n-1$, are iid with mean 0. Therefore, according to Strong Law of Large Numbers (SLLN), we obtain that the average of these random variables convergence almost surely to 0, the mean of any of these random variables, i.e., $\frac{W^i((n-1)t_0)}{(n-1)t_0}$ converges to 0, a.s., as n goes to infinity.

For Case (ii), i.e., for the case in which $Z_1^i,\ldots,Z_n^i$ all come from different stochastic processes, we have

$$\begin{aligned}
\mu_{in} &= \left(\frac{\sigma^2}{\sigma^2_{i(n-1)}A^i((n-1)t_0)+\sigma^2}\right)\mu_{i(n-1)} + \left(\frac{\sigma^2_{in-1}A^i((n-1)t_0)}{\sigma^2_{i(n-1)}A^i((n-1)t_0)+\sigma^2}\right)\left(\frac{Z_n^i}{A^i((n-1)t_0)}\right) \\
&= \left(\frac{\sigma^2}{\sigma^2_{i1}\sum_{j=1}^{n}A^i(jt_0)+\sigma^2}\right)\mu_{i1} + \left(\frac{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)}{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)+\sigma^2}\right)\left(\frac{\sum_{j=2}^{n}Z_j^i}{\sum_{j=1}^{n-1}A^i(jt_0)}\right) \\
&= \left(\frac{\sigma^2}{\sigma^2_{i1}\sum_{j=1}^{n}A^i(jt_0)+\sigma^2}\right)\mu_{i1} + \left(\frac{\sigma^2_{i1}\sum_{j=1}^{n-1}\left(\theta_i A^i(jt_0)+W_j^i(A^i(jt_0))\right)}{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)+\sigma^2}\right) \\
&= \left(\frac{\sigma^2}{\sigma^2_{i1}\sum_{j=1}^{n}A^i(jt_0)+\sigma^2}\right)\mu_{i1} + \left(\frac{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)}{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)+\sigma^2}\right)\theta_i + \frac{\sigma^2_{i1}\sum_{j=1}^{n-1}W_j^i(A^i(jt_0))}{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)+\sigma^2},
\end{aligned} \tag{34}$$

$$\sigma^2_{in} = \frac{\sigma^2_{i,n-1}\sigma^2}{\sigma^2_{i(n-1)}A^i((n-1)t_0)+\sigma^2} = \frac{\sigma^2_{i1}\sigma^2}{\sigma^2_{i1}\sum_{j=1}^{n-1}A^i(jt_0)+\sigma^2}, \tag{35}$$

where $A^i(jt_0) < t_0$ represents the age of the part operating at time jt_0 on machine i, and W_j^i denotes the Brownian motion in the degradation signal process of the operating part on machine i at time jt_0. Since $Z_1^i,\ldots,Z_n^i$ all come from different signal processes, the subscript j of W_j^i is used to distinguish these different degradation signal processes. We also note that μ_{in} depends on $Z_2^i,\ldots,Z_n^i$ and the age of the parts under observation at the start of each period, together with the prior mean and variance of θ_i. σ^2_{in} depends on the age of the parts under observation at the start of each period and the prior variance of θ_i.

To prove the convergence result, we first show that $\sum_{j=1}^{n-1}A^i(jt_0) \overset{n\to\infty}{\to} \infty$ a.s., which implies that

$\sigma_{in}^2 \overset{n\to\infty}{\to} 0$, a.s. For this purpose, we first note that the demand process for parts for machine i, $\{D_i(t) : t \geq 0\}$, is in fact a renewal process due to our assumption that the life times of the parts used for the same machine are iid. Then following a same reasoning for *Proposition 3.4.6* in S. M. Ross (1983, page 71), we have the following result:

$$\begin{aligned}
\lim_{n\to\infty} E[A^i(nt_0)] &= \lim_{t\to\infty} E[A^i(t)] = \lim_{t\to\infty} t\bar{F}_i(t) + \int_0^t (t-y)\bar{F}_i(t-y)dm(y) \\
&= \frac{\int_0^\infty t\bar{F}_i(t)dt}{E[X_k^i]} = \frac{\int_0^\infty t\left(\int_t^\infty f_i(s)ds\right)dt}{E[X_k^i]} = \frac{\int_0^\infty \frac{s^2}{2} f_i(s)ds}{E[X_k^i]} \\
&= \frac{E[(X_k^i)^2]}{2E[X_k^i]} = \frac{E\left[(T(\theta_i,0,B))^2\right]}{2E\left[T(\theta_i,0,B)\right]} (= \beta > 0),\ k = 1,2,\ldots, \qquad (36)
\end{aligned}$$

where X_k^i represents the life time of the kth part used on machine i. Since $E\left[(T(\theta_i,0,B))^2\right]$ and $E\left[T(\theta_i,0,B)\right]$ are both positive and finite, it is easy to see $\beta > 0$. It follows immediately that $\lim_{n\to\infty} E\left[\sum_{j=1}^{n-1} A^i(jt_0)\right] = \infty$. This implies that $\sum_{j=1}^{n-1} A^i(jt_0) \overset{n\to\infty}{\to} \infty$, *a.s.*, using the fact $A^i(jt_0) \geq 0$, for any $j \geq 1$.

We next show that $\mu_{in} \overset{n\to\infty}{\to} \theta_i$ a.s. by considering each term separately in (34). For the first term, since $\sum_{j=1}^{n-1} A^i(jt_0) \overset{n\to\infty}{\to} \infty$ a.s. and μ_{i1} is a constant number, the first term converges to 0 almost surely. Similarly, the second term can be shown converge to θ_i almost surely. Finally, for the last term, we claim

$$\frac{\sigma_{i1}^2 \sum_{j=1}^{n-1} W_j^i(A^i(jt_0))}{\sigma_{i1}^2 \sum_{j=1}^{n-1} A^i(jt_0) + \sigma^2} = \frac{\frac{\sum_{j=1}^{n-1} W_j^i(A^i(jt_0))}{\sum_{j=1}^{n-1} A^i(jt_0)}}{1 + \frac{\sigma^2}{\sigma_{i1}^2\left(\sum_{j=1}^{n-1} A^i(jt_0)\right)}} \overset{n\to\infty}{\to} 0, \quad a.s. \qquad (37)$$

To prove this result, we first note that the Normal random variables, $W_j^i(A^i(jt_0))$, $j = 1,\ldots,n-1$, are mutually independent, given $A^i(jt_0) = t_j^i$, for any $t_j^i \in \mathbb{R}^+$. Without loss of generality, suppose each t_j^i is an integer. Using the property of independent increments for each W_j^i, we can separate $\sum_{j=1}^{n-1} W_j^i(t_j^i)$, a Normal random variable with mean 0 and variance $\sum_{j=1}^{n-1} t_j^i$, into $\sum_{j=1}^{n-1} t_j^i$ iid Normal random variables each with mean 0 and variance 1. Applying the SLLN, we obtain

$$\frac{\sum_{j=1}^{n-1} W_j^i(t_j^i)}{\sum_{j=1}^{n-1} t_j^i} \overset{n\to\infty}{\to} 0 \ a.s. \text{ for any } t_j^i \in \mathbb{R}^+ \ \Rightarrow \ \frac{\sum_{j=1}^{n-1} W_j^i(A^i(jt_0))}{\sum_{j=1}^{n-1} A^i(jt_0)} \overset{n\to\infty}{\to} 0 \ a.s.. \qquad (38)$$

Since $\sum_{j=1}^{n-1} A^i(jt_0) \overset{n\to\infty}{\to} \infty$ a.s. and σ_{i1}^2 and σ^2 are both constants, it is easy to see that (37) holds. Thus, in all, we have shown that $\mu_{in} \overset{n\to\infty}{\to} \theta_i$ a.s.

So far, we have proved the convergence for μ_{in} and σ_{in}^2 for the two extreme cases. We next use these results to show that the convergence results hold in general.

Note that as n goes to infinity, among the random variables $Z_1^i, \ldots, Z_n^i, \ldots$, there are either infinitely many coming from a same stochastic process, i.e., there is a part operating on machine i that never fails, or there are infinitely many all coming from different stochastic processes. If the first case happens, we can obtain the convergence result as follows: We first use the random variables that are not coming from the same stochastic process and the prior mean and variance, $(\mu_{i1}, \sigma_{i1}^2)$, to obtain a posterior mean and a posterior variance for Θ_i. We then treat this posterior mean and posterior variance as a new prior mean and a new prior variance, respectively. Finally, we utilize the proof we have given for the extreme case (i) with these new prior mean and variance. If the second case happens and the part in use at the start of each period is different, then we can apply the results for extreme case (ii) directly to obtain convergence. Otherwise, if a part is in use for multiple periods, say part j being used on machine i, we let $Z_{j_1}^i, \ldots, Z_{j_{n_j}}^i$ denote all of its observed signal values. Note that $\mu_{ij_{n_j}}$ and $\sigma_{ij_{n_j}}^2$ can be updated directly using $Z_{j_{n_j}}^i - Z_{j_1}^i$, μ_{ij_1}, and $\sigma_{ij_1}^2$, where μ_{ij_1} and $\sigma_{ij_1}^2$ are updated using $Z_{j_1}^i$, $\mu_{i(j_1-1)}$ and $\sigma_{i(j_1-1)}^2$, i.e., the posterior mean and variance determined a period before part j's first signal, $Z_{j_1}^i$, is observed. Thus $\mu_{ij_{n_j}}$ and $\sigma_{ij_{n_j}}^2$ can be updated directly using $Z_{j_{n_j}}^i$, the age of part j when its last signal is observed, $\mu_{i(j_1-1)}$, and $\sigma_{i(j_1-1)}^2$. This is as if we update the posterior mean and variance using only the last signals observed from each different part being monitored and thus the signals we use are all from different parts. Therefore, we can apply the proof we have given for the extreme case (ii) and obtain the convergence. Thus, in all, we have proved the convergence of μ_{in} and σ_{in}^2 in general.

Finally, since $\Theta_{in} \sim N(\mu_{in}, \sigma_{in}^2)$, it follows immediately that $\Theta_{in} \overset{n\to\infty}{\to} \theta_i$, in distribution, finishing the proof. ∎

Proof of Proposition 5.7: First, we note that the almost surely convergence result in Proposition 5.6, $\mu_{in} \overset{n\to\infty}{\to} \theta_i, a.s.$, and $\sigma_{in}^2 \overset{n\to\infty}{\to} 0, a.s.$, has the following implication: Except for a set of Ω that is of $P-$measure 0, denoted by Ω_0, for any $w \in \Omega\backslash\Omega_0$, we have that $\mu_{in} \overset{n\to\infty}{\to} \theta_i$ and $\sigma_{in}^2 \overset{n\to\infty}{\to} 0$, where μ_{in} and σ_{in}^2, given $w \in \Omega$, are numbers rather than random variables. For the remainder of this section, we only consider the domain $\Omega\backslash\Omega_0$. To keep the notation simple, we denote $\Omega\backslash\Omega_0$ by Ω.

Next, notice that

$$C_N^i(x^i|z_1^i,\ldots,z_N^i) = \min_{y\geq x^i} c(y-x^i) + L_N(y|z_1^i,\ldots,z_N^i),$$
$$C_N^i(x^i|\theta_i,z_N^i) = \min_{y\geq x^i} c(y-x^i) + L_N(y|\theta_i,z_N^i).$$

$\bar{x}_N^i(z_1^i,\ldots,z_N^i)$ is the solution to $\frac{d}{dy}\left(c(y-x^i)+L_N(y|z_1^i,\ldots,z_N^i)\right)=0$. And $\bar{x}_N^i(\theta_i,z_N^i)$ is the solution to $\frac{d}{dy}\left(c(y-x^i)+L_N(y|\theta_i,z_N^i)\right)=0$. It is easy to see that it suffices to show that $L_N(y|z_1^i,\ldots,z_N^i)-L_N(y|\theta_i,z_N^i)\overset{N\to\infty}{\to}0$, for any $y\in\mathbb{R}$. Comparing the expression for $L_N(y|z_1^i,\ldots,z_N^i)$ to the expression for $L_N(y|\theta_i,z_N^i)$, we find that it suffices[4] to show

$$P\{D_{iN}=k|z_1^i,\ldots,z_N^i\} - P\{D_{iN}=k|\theta_i,z_N^i\} \overset{N\to\infty}{\to} 0,\ \text{uniformly in } k\in\mathbb{N}\bigcup\{0\}, \tag{39}$$

which, using (4) and (5), reduces to:

$$\int_{-\infty}^{\infty} P\{T(\theta,z_N^i,kB)\leq t_0\} f_{\Theta_i|z_1^i,\ldots,z_N^i}(\theta)d\theta - P\{T(\theta_i,z_N^i,kB)\leq t_0\} \overset{N\to\infty}{\to} 0,\ \text{uniformly in } k\in\mathbb{N}\bigcup\{0\}. \tag{40}$$

In order to prove (40), we next show a stronger result: for any $\epsilon\in(0,1)$, there exists $M\in\mathbb{N}$ such that for any $N\geq M$, we have for all $k\in\mathbb{N}\bigcup\{0\}$,

$$\int_{-\infty}^{\infty} \Big|P\{T(\theta,z_N^i,kB)\leq t_0\} - P\{T(\theta_i,z_N^i,kB)\leq t_0\}\Big| f_{\Theta_i|z_1^i,\ldots,z_N^i}(\theta)d\theta < \epsilon. \tag{41}$$

To complete the proof, for any fixed ϵ, we need to find M such that (41) is satisfied for any $N\geq M$. To achieve this goal, we first recall that $P\{T(\theta,z_N^i,kB)\leq t_0\}$ represents the probability that by time t_0 the Wiener process Z^i has not yet reached the absorbing barrier $Z^i=kB-z_N^i$. Note that by time t_0, if the Wiener process Z^i has reached a higher absorbing barrier, then it must have reached a lower absorbing barrier as well. In other words, it is clear that this probability function decreases in k. Also note that as k goes to infinity, $P\{T(\theta,z_N^i,kB)\leq t_0\}$ goes to 0. Mathematically,

[4]Note that for k large, (39) cannot guarantee the convergence of $L_N(y|z_1^i,\ldots,z_N^i)-L_N(y|\theta_i,z_N^i)$. However, since $E[D_{in}]<\infty$, we can find an integrable function, denoted by $H(k)$, such that $k|P\{D_{iN}=k|z_1^i,\ldots,z_N^i\}-P\{D_{iN}=k|\theta_i,z_N^i\}|\leq H(k)$. Thus, the convergence holds due to Dominated Convergence Theorem when k is large.

this implies that, for any $\theta > 0$, for fixed ϵ, there exists $K(\theta) \in \mathbb{N}$ such that for any $k \geq K(\theta)$, we have:

$$P\{T(\theta, z_N^i, kB) \leq t_0\} < \frac{\epsilon}{2}.$$

We next prove (41) by three steps. Step 1: We claim that

$$K = \max_{\theta \in [\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2}]} K(\theta) = K(\theta_i - \frac{\theta_i}{2}) \vee K(\theta_i + \frac{\theta_i}{2}) < \infty. \tag{42}$$

This claim can be proved as follows. (i) Note that $p(x, t_0; kB - z_N^i, \theta)$ is the pdf of $Z^i(t_0)$ with the absorbing barrier $Z^i = kB - z_N^i$. We know that

$$P\{T(\theta, z_N^i, kB) \leq t_0\} = 1 - \int_{-\infty}^{kB - z_N^i} p(x, t_0; kB - z_N^i, \theta) dx, \tag{43}$$

where $p(x, t_0; kB - z_N^i, \theta)$ is given by (19). (ii) Since $p(x, t_0; kB - z_N^i, \theta)$ is continuous in θ, we obtain that $P\{T(\theta, z_N^i, kB) \leq t_0\}$ is also continuous in θ. (iii) We can show that $P\{T(\theta, z_N^i, kB) \leq t_0\}$ is unimodal opening upwards as a function of θ. From (43), we know that it suffices to show that $p(x, t_0; kB - z_N^i, \theta)$ is unimodal opening downwards as a function of θ. We then calculate the first derivative of $p(x, t_0; kB - z_N^i, \theta)$ with respect to θ. Using (19), we have the following for any $t > 0$ and any $b > 0$:

$$\begin{aligned}
&\frac{d}{d\theta} p(x, t; b, \theta) \\
&= \frac{d}{d\theta} \left\{ \frac{1}{\sigma\sqrt{2\pi t}} \left[\exp\left\{ -\frac{(x - \theta t)^2}{2\sigma^2 t} \right\} - \exp\left\{ \frac{2\theta b}{\sigma^2} - \frac{(x - 2b - \theta t)^2}{2\sigma^2 t} \right\} \right] \right\} \\
&= \left(\frac{x - \theta t}{\sigma^2} \right) \left(\frac{1}{\sigma\sqrt{2\pi t}} \right) \left[\exp\left\{ -\frac{(x - \theta t)^2}{2\sigma^2 t} \right\} - \exp\left\{ \frac{2\theta b}{\sigma^2} - \frac{(x - 2b - \theta t)^2}{2\sigma^2 t} \right\} \right] \\
&= \left(\frac{x - \theta t}{\sigma^2} \right) p(x, t; b, \theta),
\end{aligned} \tag{44}$$

which is positive if $\theta < \frac{x}{t}$ and is negative if $\theta > \frac{x}{t}$. By taking $t = t_0, b = kB - z_N^i$, this implies that $p(x, t_0; kB - z_N^i, \theta)$ is a unimodal function of θ opening downwards. (iv) The unimodal opening upwards property of $P\{T(\theta, z_N^i, kB) \le t_0\}$ implies

$$\max_{\theta \in [\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2}]} P\{T(\theta, z_N^i, kB) \le t_0\} = \max\left\{P\{T(\theta_i - \frac{\theta_i}{2}, z_N^i, kB) \le t_0\}\ , \right.$$
$$\left. P\{T(\theta_i + \frac{\theta_i}{2}, z_N^i, kB) \le t_0\}\right\}. \tag{45}$$

Therefore, for any $\theta \in (\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2})$, for any $k \ge K = K(\theta_i - \frac{\theta_i}{2}) \vee K(\theta_i + \frac{\theta_i}{2})$, we have

$$\begin{aligned}
& P\{T(\theta, z_N^i, kB) \le t_0\} \\
\le\ & P\{T(\theta_i - \frac{\theta_i}{2}, z_N^i, kB) \le t_0\} \vee P\{T(\theta_i + \frac{\theta_i}{2}, z_N^i, kB) \le t_0\} \qquad (\text{ using } (45)\ ) \\
<\ & \frac{\epsilon}{2} \vee \frac{\epsilon}{2} \qquad\qquad (\text{ because } k \ge K(\theta_i - \tfrac{\theta_i}{2}) \text{ and } k \ge K(\theta_i + \tfrac{\theta_i}{2})\ ) \\
\Rightarrow\ & K \ge K(\theta) \qquad\qquad (\text{ using definition of } K(\theta)\ ),
\end{aligned}$$

finishing the proof for the claim in (42).

Step 2: We note that (42) implies that for any $k \ge K$

$$\left|P\{T(\theta, z_N^i, kB) \le t_0\} - P\{T(\theta_i, z_N^i, kB) \le t_0\}\right| < \frac{\epsilon}{2},$$

for any $\theta \in [\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2}]$. It follows immediately that

$$\int_{\theta_i - \frac{\theta_i}{2}}^{\theta_i + \frac{\theta_i}{2}} \left|P\{T(\theta, z_N^i, kB) \le t_0\} - P\{T(\theta_i, z_N^i, kB) \le t_0\}\right| f_{\Theta_i|z_1^i,\ldots,z_N^i}(\theta) d\theta < \frac{\epsilon}{2}.$$

To prove (41) for $k \ge K$, we show that there exists $M \in \mathbb{N}$ such that for any $N \ge M$,

$$\left(\int_{-\infty}^{\theta_i - \frac{\theta_i}{2}} + \int_{\theta_i + \frac{\theta_i}{2}}^{\infty}\right) \left|P\{T(\theta, z_N^i, kB) \le t_0\} - P\{T(\theta_i, z_N^i, kB) \le t_0\}\right| f_{\Theta_i|z_1^i,\ldots,z_N^i}(\theta) d\theta < \frac{\epsilon}{2}. \tag{46}$$

To find an M such that (46) is satisfied, we use the result that $\mu_{iN} \overset{N\to\infty}{\rightarrow} \theta_i$, and $\sigma_{iN}^2 \overset{N\to\infty}{\rightarrow} 0$, for any $z_1^i, z_2^i, \ldots$ and $\Theta_i | z_1^i, \ldots, z_N^i \overset{N\to\infty}{\rightarrow} \theta_i$ in distribution. Mathematically, this implies that for ϵ, there exists $M_1 \in \mathbb{N}$ such that for any $N \ge M_1$, we have

$$\left(\int_{-\infty}^{\theta_i - \delta} + \int_{\theta_i + \delta}^{\infty}\right) f_{\Theta_i|z_1^i,\ldots,z_N^i}(\theta) d\theta < \frac{\epsilon}{2}, \tag{47}$$

where

$$\delta = \frac{\epsilon}{4L} \wedge \frac{\theta_i}{2}, \tag{48}$$

$$L = \max_{\xi \in [\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2}]} \int_{-\infty}^{KB - z_N^i} \Big| \frac{x - \xi t_0}{\sigma^2} \Big| p(x, t_0; KB - z_N^i, \xi) dx < \infty. \tag{49}$$

The result that $L < \infty$ is due to the fact that $p(x, t_0; KB - z_N^i, \xi)$ is of the order e^{-x^2}.

Now, we let $M = M_1$, which does not depend on k. Since $\delta \leq \frac{\theta_i}{2}$ and $\Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| \leq 1$, we know that (47) implies (46). So far, we have shown that for fixed ϵ, for any $N \geq M = M_1$, (41) holds true for any $k \geq K$, where K is defined by (42).

Step 3: We demonstrate that for any $N \geq M$, (41) holds for any $k \leq K$ as well, i.e.,

$$\int_{-\infty}^{\infty} \Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| f_{\Theta_i | z_1^i, \ldots, z_N^i}(\theta) d\theta < \epsilon, \tag{50}$$

for any $k \leq K$, and then we will have completed the proof.

For this purpose, we first rewrite the left hand side of (50) and obtain an upper bound for it as follows:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| f_{\Theta_i | z_1^i, \ldots, z_N^i}(\theta) d\theta \\
= & \left(\int_{\theta_i - \delta}^{\theta_i + \delta} + \int_{-\infty}^{\theta_i - \delta} + \int_{\theta_i + \delta}^{\infty} \right) \Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| f_{\Theta_i | z_1^i, \ldots, z_N^i}(\theta) d\theta \\
< & \int_{\theta_i - \delta}^{\theta_i + \delta} \Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| f_{\Theta_i | z_1^i, \ldots, z_N^i}(\theta) d\theta + \frac{\epsilon}{2}.
\end{aligned} \tag{51}$$

To understand (51), we first note that $\Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| \leq 1$. Then, using (47), (51) is attained.

Finally, we will show that for any $N \geq M$, the following result holds true:

$$\int_{\theta_i - \delta}^{\theta_i + \delta} \Big| P\{T(\theta, z_N^i, kB) \leq t_0\} - P\{T(\theta_i, z_N^i, kB) \leq t_0\} \Big| f_{\Theta_i | z_1^i, \ldots, z_N^i}(\theta) d\theta \leq \frac{\epsilon}{2}, \text{ for any } k \leq K. \tag{52}$$

To do this, we first apply (43) to the left hand side of (52) and obtain the following results:

$$\begin{aligned}
&\int_{\theta_i-\delta}^{\theta_i+\delta} \Big|P\{T(\theta, z_N^i, kB) \le t_0\} - P\{T(\theta_i, z_N^i, kB) \le t_0\}\Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta \\
&\le \int_{\theta_i-\delta}^{\theta_i+\delta} \int_{-\infty}^{kB-z_N^i} \Big|p(x, t_0; kB - z_N^i, \theta) - p(x, t_0; kB - z_N^i, \theta_i)\Big| dx f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta \\
&= \int_{-\infty}^{kB-z_N^i} \int_{\theta_i-\delta}^{\theta_i+\delta} \Big|p(x, t_0; kB - z_N^i, \theta) - p(x, t_0; kB - z_N^i, \theta_i)\Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx \text{ (Fubini Thm)} \\
&= \int_{-\infty}^{kB-z_N^i} \int_{\theta_i-\delta}^{\theta_i} \Big| \int_{\theta}^{\theta_i} \frac{d}{d\theta} p(x, t_0; kB - z_N^i, \xi) d\xi \Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx \\
&\quad + \int_{-\infty}^{kB-z_N^i} \int_{\theta_i}^{\theta_i+\delta} \Big| \int_{\theta_i}^{\theta} \frac{d}{d\theta} p(x, t_0; kB - z_N^i, \xi) d\xi \Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx, \quad \text{for any } k \le K. \qquad (53)
\end{aligned}$$

For the first term in (53), we apply (44) and (48) and obtain the following result for any $k \le K$:

$$\begin{aligned}
&\int_{-\infty}^{kB-z_N^i} \int_{\theta_i-\delta}^{\theta_i} \Big| \int_{\theta}^{\theta_i} \frac{d}{d\theta} p(x, t_0; kB - z_N^i, \xi) d\xi \Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx \\
&= \int_{-\infty}^{kB-z_N^i} \int_{\theta_i-\delta}^{\theta_i} \Big| \int_{\theta}^{\theta_i} \left(\frac{x-\xi t}{\sigma^2}\right) p(x, t_0; kB - z_N^i, \xi) d\xi \Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx, \quad \text{(use (44))} \\
&= \int_{\theta_i-\delta}^{\theta_i} \int_{\theta}^{\theta_i} \Big|\frac{x-\xi t}{\sigma^2}\Big| p(x, t_0; kB - z_N^i, \xi) d\xi f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx \\
&= \int_{\theta_i-\delta}^{\theta_i} \int_{\theta_i-\delta}^{\xi} \left(\int_{-\infty}^{kB-z_N^i} \Big|\frac{x-\xi t}{\sigma^2}\Big| p(x, t_0; kB - z_N^i, \xi) dx \right) f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta d\xi, \quad \text{(Fubini Theorem)} \\
&\le \delta \max_{\xi \in [\theta_i-\delta, \theta_i]} I(\xi, kB - z_N^i) \\
&\le \delta \max_{\xi \in [\theta_i-\delta, \theta_i]} I(\xi, KB - z_N^i) \quad (\text{ since } k \le K \text{ and } I(\xi, kB - z_N^i) \text{ is increasing in } k\ ) \\
&\le \delta \max_{\xi \in [\theta_i-\frac{\theta_i}{2}, \theta_i+\frac{\theta_i}{2}]} I(\xi, KB - z_N^i) \quad (\text{ since } \delta \le \tfrac{\theta_i}{2}\ ) \\
&= \delta L \le \frac{\epsilon}{4} \quad (\text{ since } \delta \le \tfrac{\epsilon}{4L}\ ), \qquad (54)
\end{aligned}$$

where

$$I(\xi, kB - z_N^i) = \int_{-\infty}^{kB-z_N^i} \Big|\frac{x-\xi t}{\sigma^2}\Big| p(x, t_0; kB - z_N^i, \xi) dx < \infty, \text{ for any } \xi \in [\theta_i - \frac{\theta_i}{2}, \theta_i + \frac{\theta_i}{2}].$$

Similarly, for the second term in (53), we have the following:

$$\int_{-\infty}^{kB-z_N^i} \int_{\theta_i}^{\theta_i+\delta} \Big| \int_{\theta_i}^{\theta} \frac{d}{d\theta} p(x, t_0; kB - z_N^i, \xi) d\xi \Big| f_{\Theta_i|z_1^i,\dots,z_N^i}(\theta) d\theta dx \le \frac{\epsilon}{4}, \quad \text{for any } k \le K. \qquad (55)$$

Since (54) and (55) together imply (50), we have proved (41), finishing the proof.

Note that this convergence result is for the expected cost and optimal order-up-to levels for the final period, period N. We are only able to prove convergence for the final period for several reasons. For machine i, $i = 1, \ldots, m$, at the start of period $n \leq N$, the expectation of the cost-to-go function is assessed and the expectation is taken over the random degradation signal value that will be observed at the start of the next period, Z^i_{n+1}. One use of this value is to update the distribution of the unknown degradation parameter, Θ_i, i.e., the updated distribution of the unknown degradation parameter depends on the observed value of this random degradation signal, z^i_{n+1}. Note that we have proven that, for any sequence of observed degradation signal values, $z^i_1, z^i_2, \ldots, z^i_n, z^i_{n+1}, \ldots$, the updated distribution for Θ_i converges to its actual value, θ_i. This convergence, however, is not uniform in all sequences of observed degradation signal values, but is uniform in a finite number of sequences of observed degradation signal values. Such a uniform convergence can only ensure a uniform convergence for the cost-to-go function values that are associated with a finite number of values of Z^i_{n+1}. Since this random degradation signal, Z^i_{n+1}, can take infinitely many values (i.e., $z^i_{n+1} \in (-\infty, B)$), the convergence for the expected cost-to-go functions may not hold.∎

Proof of Proposition 6.1: Following the proof of Proposition 5.1, we can easily show that a myopic fixed level inventory policy is optimal for the cases with known and unknown θ_i. The fixed level, i.e., $\bar{x}^i_{SM,n}(\theta_i, z^i_n)$, for the case in which θ_i is known, is obtained by differentiating the function $s_n - \alpha E[S_{n+1}])y + L_n(y|\theta_i, z^i_n)$. Since demand is discrete and the inventory level is an integer, the solution should be the smallest value of y such that $s_n - \alpha E[S_{n+1}] + L'_n(y|\theta_i, z^i_n) \geq 0$. Similarly, we have the result for the fixed level, $\bar{x}^i_{SM,n}(z^i_1, \ldots, z^i_n)$, when θ_i is unknown. Finally, using the same proof as for Proposition 5.2, we can show that $\bar{x}^i_{SM,n}(\theta_i, z^i_n)$ is a non-decreasing function of θ_i and z^i_n.

To prove the convergence result, notice that we show $L_N(y|z^i_1, \ldots, z^i_N) - L_N(y|\theta_i, z^i_N) \overset{N\to\infty}{\to} 0$ in the proof for Proposition 5.7. This result implies that $L_n(y|z^i_1, \ldots, z^i_n) - L_n(y|\theta_i, z^i_n) \overset{n\to\infty}{\to} 0$, for any $y \in \mathbb{R}$. From the definition of $\bar{x}^i_{SM,n}(\theta_i, z^i_n)$ and $\bar{x}^i_{SM,n}(z^i_1, \ldots, z^i_n)$, we can easily see that this result implies the convergence of $\bar{x}^i_{SM,n}(z^i_1, \ldots, z^i_n) - \bar{x}^i_{SM,n}(\theta_i, z^i_n)$ and the convergence of $C^i_{SM,n}(x^i|z^i_1, \ldots, z^i_N) - C^i_{SM,n}(x^i|\theta_i, z^i_n)$, for any sequence of observed degradation signals, $z^i_1, \ldots, z^i_n, \ldots$.
∎