Corrigendum to "A Gaussian Approach for Continuous Time Models of the Short Term Interest Rate"

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Summary   An error is corrected in Yu and Phillips (2001) (*Econometrics Journal*, 4, 210-224) where a time transformation was used to induce Gaussian disturbances in the discrete time equivalent model. It is shown that the error process in this model is not a martingale and the Dambis, Dubins-Schwarz (DDS) theorem is not directly applicable. However, a detrended error process is a martingale, the DDS theorem is applicable, and the corresponding stopping time correctly induces Gaussianity. We show that the two stopping time sequences differ by $O(a^2)$, where a is the pre-specified normalized timing constant.

Keywords: *Nonlinear Diffusion, Normalizing Transformation, Level Effect, DDS Theorem.*

This note corrects an error in Yu and Phillips (2001) (hereafter YP) where a time transformation was used to induce Gaussian disturbances in the discrete time version of a continuous time model. The error occurs in equations (3.7)-(3.10) of YP where the Dambis, Dubins-Schwarz (hereafter DDS) theorem was applied to the quadratic variation of the error term in Equation (3.6), $[M]_h$, in order to induce a sequence of stopping time points $\{t_j\}$ for which the disturbance term in (3.10) follows a normal distribution, facilitating Gaussian estimation.

To apply the DDS theorem, the original error process, $M(h)$ needs to be a continuous martingale with finite quadratic variation. In YP, it was assumed that $M(h)$ was a continuous martingale. This note shows that the assumption is generally not warranted and so the DDS theorem does not induce a Brownian motion. However, a simple decomposition splits the error process into a trend component and a continuous martingale process. The DDS theorem can then be applied to the detrended error process, generating a Brownian motion residual. With the presence of the time varying trend component, the discrete time model is heteroskedastic and the regressor is endogenous. The endogeneity is addressed using an instrumental variable procedure for parameter estimation. In addition, we show that the new stopping time sequence differs from that in YP by a term of $O(a^2)$, where a is the pre-specified normalized timing constant. In the case where a is often chosen to be the average variance whose value is small, the difference between the two stopping time sequences is likely small.

The discrete time model of the following (nonlinear) continuous time model

$$dr(t) = (\alpha + \beta r(t))dt + \sigma r^\gamma(t)dB(t), \tag{0.1}$$

has the form

$$r(t+h) = \frac{\alpha}{\beta}(e^{\beta h} - 1) + e^{\beta h}r(t) + \int_0^h \sigma e^{\beta(h-\tau)}r^\gamma(t+\tau)dB(\tau), \qquad (0.2)$$

where B is standard Brownian motion. Let $M(h) = \sigma \int_0^h e^{\beta(h-\tau)}r^\gamma(t+\tau)dB(\tau)$. YP assumed that $M(h)$ is a continuous martingale with 'quadratic variation'

$$[M]_h^* = \sigma^2 \int_0^h e^{2\beta(h-\tau)}r^{2\gamma}(t+\tau)d\tau. \qquad (0.3)$$

Under this assumption, YP used the DDS theorem - see Revuz and Yor (1999) - to induce a Brownian motion to represent the process $M(h)$. That is, for any fixed 'timing' constant $a > 0$, YP set

$$h_{j+1} = \inf\{s | [M_j]_s^* \geq a\} = \inf\{s | \sigma^2 \int_0^s e^{2\beta(s-\tau)}r^{2\gamma}(t_j+\tau)d\tau \geq a\}, \qquad (0.4)$$

and constructed a sequence of time points $\{t_j\}$ using the iterations $t_{j+1} = t_j + h_{j+1}$, leading to the following version of (0.2) evaluated at $\{t_j\}$

$$r(t_{j+1}) = \frac{\alpha}{\beta}(e^{\beta h_{j+1}} - 1) + e^{\beta h_{j+1}}r(t_j) + M(h_{j+1}). \qquad (0.5)$$

If the DDS theorem were applicable, then $M(h_{j+1}) = B(a) \sim N(0, a)$.

Unfortunately, in general, $M(h)$ is NOT a continuous martingale. There is a trend factor in $M(h)$ and the quadratic variation calculation (0.3) in YP fails to take account of this factor. $M(h)$ is not a continuous martingale even when $\gamma = 0$. In this simple case, we have

$$M(h) = \sigma e^{\beta h} \int_0^h e^{-\beta s} dB(s),$$

which is an Ornstein–Uhlenbeck (OU) process satisfying $dM(h) = \beta M(h) dh + \sigma dB(h)$, whose quadratic variation process is

$$[M]_h = h\sigma^2 \neq \sigma^2 e^{2\beta h} \int_0^h e^{-2\beta s} ds.$$

To adjust for the drift in the residual of (0.2), let

$$M(h) = \sigma \int_0^h e^{\beta(h-s)}r^\gamma(s) dB(s) = e^{\beta h}\sigma \int_0^h e^{-\beta s}r^\gamma(s) dB(s) = e^{\beta h}H(h),$$

where $H(h) := \sigma \int_0^h e^{-\beta s}r^\gamma(s) dB(s)$ is a continuous martingale. Then $M(t)$ follows the process

$$dM(t) = \beta M(t) dt + e^{\beta t} dH(t) = \beta M(t) dt + \sigma r^\gamma(t) dB(t),$$

with

$$d[H]_t = \sigma^2 e^{-2\beta t}r^{2\gamma}(t) dt, \quad \text{and } d[M]_t = \sigma^2 r^{2\gamma}(t) dt.$$

Hence, instead of (0.3), the actual quadratic variation of M is

$$[M]_h = \sigma^2 \int_0^h r^{2\gamma}(t+s)ds.$$

The equation of interest is

$$r\left(t\right)=\left[r\left(0\right)+\frac{\alpha}{\beta}\right]e^{\beta t}-\frac{\alpha}{\beta}+e^{\beta t}H\left(t\right),$$

so that

$$\begin{aligned}
r\left(t+h\right) &= \left[r\left(0\right)+\frac{\alpha}{\beta}\right]e^{\beta(t+h)}-\frac{\alpha}{\beta}+e^{\beta(t+h)}H\left(t+h\right)\\
&= e^{\beta h}r\left(t\right)+\frac{\alpha}{\beta}\left(e^{\beta h}-1\right)+e^{\beta(t+h)}H\left(t+h\right)-e^{\beta(t+h)}H\left(t\right)\\
&= e^{\beta h}r\left(t\right)+\frac{\alpha}{\beta}\left(e^{\beta h}-1\right)+e^{\beta(t+h)}\left(H\left(t+h\right)-H\left(t\right)\right)\\
&= e^{\beta h}r\left(t\right)+\frac{\alpha}{\beta}\left(e^{\beta h}-1\right)+e^{\beta(t+h)}\sigma\int_{t}^{t+h}e^{-\beta s}r^{\gamma}\left(s\right)dB\left(s\right)\\
&= e^{\beta h}r\left(t\right)+\frac{\alpha}{\beta}\left(e^{\beta h}-1\right)+e^{\beta h}\sigma\int_{0}^{h}e^{-\beta p}r^{\gamma}\left(t+p\right)dB\left(t+p\right).
\end{aligned}$$

Now

$$Q_{t}\left(h\right)=\sigma\int_{0}^{h}e^{-\beta p}r^{\gamma}\left(t+p\right)dB\left(t+p\right)$$

is a continuous martingale with $dQ_{t}\left(h\right)=e^{-\beta h}\sigma r^{\gamma}\left(t+h\right)dB\left(t+h\right)$ and

$$d\left[Q_{t}\right]_{h}=e^{-2\beta h}\sigma^{2}r^{2\gamma}\left(t+h\right)dh.$$

Applying the DDS Theorem to Q_{t} with timing constant a so that

$$\widetilde{h}_{j+1}=\inf\left\{s:\left[Q_{t_{j}}\right]_{s}\geq a\right\}=\inf\left\{s:\sigma^{2}\int_{0}^{s}e^{-2\beta p}r^{2\gamma}\left(t_{j}+p\right)dp\geq a\right\},\qquad(0.6)$$

we have

$$r\left(t_{j+1}\right)=e^{\beta\widetilde{h}_{j+1}}r\left(t_{j}\right)+\frac{\alpha}{\beta}\left(e^{\beta\widetilde{h}_{j+1}}-1\right)+e^{\beta\widetilde{h}_{j+1}}Q_{t_{j}}\left(\widetilde{h}_{j+1}\right),$$

which has Gaussian $N(0,a)$ innovations and where $t_{j+1}=t_{j}+\widetilde{h}_{j+1}$. However, the step size and stopping times $\widetilde{h}_{j+1}$ are endogenous. As a result, the ordinary least squares or weighted least squares procedures are inconsistent. To consistently estimate α and β, we note that $(1,r(t_{j}))$ is a valid instrument. The estimating equations are

$$\sum_{j}\left(e^{-\beta\widetilde{h}_{j+1}}r(t_{j+1})-\frac{\alpha}{\beta}(1-e^{-\beta\widetilde{h}_{j+1}})-r(t_{j})\right)r(t_{j})=0,\qquad(0.7)$$

and

$$\sum_{j}\left(e^{-\beta\widetilde{h}_{j+1}}r(t_{j+1})-\frac{\alpha}{\beta}(1-e^{-\beta\widetilde{h}_{j+1}})-r(t_{j})\right)=0.\qquad(0.8)$$

Solving these two equations for α and β yields the instrumental variable (IV) estimators of (α,β) which we denote as $(\widehat{\alpha},\widehat{\beta})$. The analytic expression for $\widehat{\alpha}$ is

$$\widehat{\alpha}=\widehat{\beta}\frac{\sum_{j}\left[e^{-\widehat{\beta}\widehat{h}_{j+1}}r(t_{j+1}(\widehat{\beta}))-r(t_{j}(\widehat{\beta}))\right]}{\sum_{j}(1-e^{-\widehat{\beta}\widehat{h}_{j+1}})},$$

and $\widehat{\beta}$ is obtained by numerically solving the following equation:

$$
\sum_j \left[\left(e^{-\beta \widehat{\widetilde{h}}_{j+1}} r(\widehat{t}_{j+1}) - r(\widehat{t}_j) \right) r(\widehat{t}_j) \right] \sum_j (1 - e^{-\beta \widehat{\widetilde{h}}_{j+1}})
$$
$$
- \sum_j \left(e^{-\beta \widehat{\widetilde{h}}_{j+1}} r(\widehat{t}_{j+1}) - r(\widehat{t}_j) \right) \sum_j \left[(1 - e^{-\beta \widehat{\widetilde{h}}_{j+1}}) r(\widehat{t}_j) \right] = 0,
$$

where $\widehat{\widetilde{h}} = \widetilde{h}(\widehat{\beta})$ and $\widehat{t}_j = t_j(\widehat{\beta})$.

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