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Global Analysis of an Expectations Augmented Evolutionary Dynamics

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Global Analysis of an Expectations Augmented Evolutionary Dynamics *

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Abstract

We consider a deterministic evolutionary model where players form expectations about future play. Players are not fully rational and have expectations that change over time in response to current payoffs and feedback from the past. We provide a complete characterization of the qualitative dynamics so induced for a two strategies population game, and relate our findings to standard evolutionary dynamics and equilibrium selection when agents have rational forward looking expectations.

Keywords: evolutionary games; dynamic systems; bounded rationality.

JEL classification: C73

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1 Introduction

In an evolutionary game, players typically have no reason to care about future rounds of play. But in some cases this assumption is not reasonable: in particular, when commitments of any sort take place (think, for example, of switching costs, or investment decisions), they have reason to be concerned about the future consequences of their current actions.

Starting from this premise, an increasing amount of literature is dealing with the issue of equilibrium selection under rational expectations in normal form games. For example, Matsui and Matsuyama (1995) (MM henceforth) consider an infinitely repeated two by two coordination game with random matching in which players can only change action at a random rate (friction). Therefore, players need to form expectations about the future evolution of play within the population. MM show that when the friction gets smaller, a unique equilibrium is selected, the risk dominant one (in the sense of Harsanyi and Selten). In a similar fashion, Burdzy et al. (2001) analyze a stochastic evolutionary model in which players play a two-by-two game with strategic complementarity, whose payoffs change over time. They consider the same sort of friction in players' ability to change strategies and find that the risk dominant equilibrium is played at any point in time when the friction is sufficiently small.

A departure from perfect foresight was taken first by Matsui and Rob (1992). They consider a game with stochastic overlapping generations of players whose actions are fixed for the entire life cycle. Players may have heterogeneous beliefs about the future evolution of play, and their individual behavior has to be rationalized by one of them. They find, among other things, that the Pareto efficient equilibrium can be the unique globally absorbing state. Lagunoff (2000) considers an infinitely repeated common interest game in which players play self-fulfilling equilibria. His model is close to Matsui and Rob (1992) in all the other features. It is shown that the Pareto dominant equilibrium is a globally absorbing state of the dynamics when there are relatively small inertia and discounting. More recently, Matsui and Oyama (2006) consider the same setup as in MM but move away from perfect foresight by assuming rationalizable expectations. They find that, when the level of friction is small enough, and players are playing a generic two-by-two game, the risk dominant equilibrium is the unique stable set of their dynamics.

In all these works, the main emphasis is on the limit properties of the absorbing states of the evolutionary dynamics in normal form games when a random parameter (typically describing the friction in the process of strategy

adjustment) becomes arbitrarily small.

Our main departure from the literature is given, in line with the trend from the new, more recent work, by the fact that we stick to bounded rationality. Moreover, we provide an explicit equation for the evolution of expectations. Specifically, players follow an adaptive expectation formation mechanism, whose dynamics depend only in part on how players tend to extrapolate the current outcomes into the future. We provide a global analysis of the induced (deterministic) dynamics as a function of the parameters governing the dynamics of expectations. In particular, we find that for any initial strategy distribution, the system can converge to any (if more than one) asymptotically stable fixed point, for a suitable choice of the initial value of the payoff expectations (Theorem 1). Moreover, starting from the same initial pair of strategy configuration and values of expectations, the dynamics may lead to different (if more than one) asymptotically stable fixed points depending on the values of the parameters that regulate the expectation formation process (Theorem 2, and Theorem 3).¹

Indeterminacy of the equilibrium selection is not novel. For example, both MM and Matsui and Oyama (2006) obtain it when the friction parameter is large enough. Yet, our formalization of the model allows for a (non trivial) global qualitative analysis of the evolutionary dynamics. Moreover, by including a specific equation for the evolution of the expectations, we can also address the issue of long run consistency of expectations.² Along this line of research, Hommes (1998) analyzes the consistency to rational expectations of backward looking expectations in a cobweb model.³ Similarly, Bischi et al. (2004) and Bischi and Kopel (2001) provide a global analysis of duopoly models in which players have backward looking expectations, and analyze the consistency of equilibria with rational expectations.

In this respect, our model generates convergence to rational expectations in a straightforward way. Since any of the Nash equilibria can be selected, expectations in the long run will determine which one of them is picked, but there cannot be any systematic error on the agents' side preventing the system from converging. As the system converges to a steady state, the

¹The distinction between history dependence and the role of expectations in selecting equilibria has been studied in the macroeconomics literature, with a focus on rational expectations (see for example, Matsuyama (1991), Krugman (1991) and Diamond and Fudenberg (1989)).

²The problem of convergence of bounded rationality to rational expectations is at the heart of learning in macroeconomics. See for example, Sargent (1993) and Evans and Honkapohja (2001).

³ Hommes and Sorger (1998) consider under which conditions beliefs that follow an AR(1) process are indistinguishable from rational expectations in a cobweb model.)

speed of the dynamics decreases, thus allowing for better expectations to be formed, thus reinforcing convergence to the steady state.

A similar approach to ours is provided by Conlisk (2001), who considers an evolutionary (zero-sum) game of competence selection in which players need to choose how much to invest in ability to overplay their opponents, and introduces a modification of the replicator dynamics as follows: the rate of growth of a strategy distribution depends on both the differential payoff of that strategy compared to the average and the gradient of such a differential. This last term is intended to take into account a simple extrapolation of the present that allows the dynamics to converge to an equilibrium rather than cycle around it.

Finally, our formalization is very similar to the one previously used in Antoci et al. (1992), who analyze the role of expectations in the transition process of an underdeveloped economy into a more advanced one. In such a context, different dynamics in the expectations lead to different levels of growth.

The remainder of the paper is organized as follows: in section 2 the model is introduced; in section 3 and section 4, some preliminary results are presented. Section 5 presents our main results, while section 6 compares them with the dynamics studied by MM by means of simulations. Lastly, section 7 contains the final comments. All the proofs are in the appendix.

2 The expectations augmented evolutionary dynamics

Evolutionary dynamics have been extensively studied. Samuelson (1997), and Weibull (1995) offer, among the others, a comprehensive review of the foundational literature. For a population game with two strategies, the standard specification of a dynamic evolutionary process is in terms of a vector field

$$\dot{x}(t) = F\{\Delta\pi[x(t)]\}, \quad (1)$$

where $\dot{x}(t) \equiv dx(t)/dt$ and $\Delta\pi[x(t)]$ is the payoff difference between strategies 1 and 2 at time t when the proportion of the population choosing strategy 1 is represented by $x(t)$. F is generally sign preserving and increasing in its argument.⁴ We know that, for a generic specification of the payoff

⁴Here we are following the terminology introduced by Friedman (1991). When F is sign preserving, we usually speak of weak compatible dynamics; if moreover it is increasing in its argument, we speak of order compatible dynamics.

function and the initial conditions $x(0)$, there exists an open interval $I_{x(0)}$ such that all the trajectories starting at $I_{x(0)}$ converge to the same asymptotically stable fixed point of (1). If the dynamics represented by (1) possess more than one asymptotically stable fixed point, then convergence to either one is just a matter of differences in the initial conditions $x(0)$.

In this paper we modify the dynamics given by equation (1) in order to take into account the expectations of the players. In particular, we consider a dynamic equation that describes how expectations evolve over time. Moreover, we allow for a variety of instantaneous payoffs $\pi_1[x(t)]$, and $\pi_2[x(t)]$ for strategies 1 and 2 that can account for positive or negative (network) externalities. Therefore $\Delta\pi[x(t)]$ will not in general be a monotone function of $x(t)$.

Our model, which we call expectations augmented evolutionary dynamics, is constructed as follows. Time is continuous and the population of players is infinite. At any instant $t \in [0, \infty)$ each player participates in a population game with two strategies. Players can choose only a pure strategy $i = 1, 2$. The present value (evaluated at time t) of the discounted flow of payoffs from playing strategy i from time t to time $t + T$ is

$$V_{i,t}^T \equiv \int_t^{t+T} \{\pi_i[x(s)]\} e^{-\alpha(s-t)} ds, \quad (2)$$

where the parameter $\alpha > 0$ represents the discount rate and T can be taken equal to $+\infty$. Similarly, the present value of the instantaneous payoff difference at instant t is

$$\Delta V_t^T \equiv \int_t^{t+T} \{\Delta\pi[x(s)]\} e^{-\alpha(s-t)} ds.$$

It is assumed that $x(t)$ is perfectly observable and common knowledge among the players at any t . Nevertheless, players need to form some expectations about the evolution of $x(t)$ from t to $t + T$. It is important to stress that T need not be finite. Consequently, this model can accommodate for scenarios where players are replaced, or can change action, only with some (i.i.d.) probability. In that case, α is the effective discount rate, which incorporates the replacement rate. Let $y(t)$ denote the expectation at t about ΔV_t^T .

Players adopt the following adaptive expectation formation mechanism

$$\dot{y}(t) = \gamma [\omega[x(t)] - y(t)] + (1 + \beta) \frac{d\omega[x(t)]}{dt}, \quad (3)$$

where

$$\omega[x(t)] \equiv \int_t^{t+T} \{\Delta\pi[x(t)]\} e^{-\alpha(s-t)} ds = \frac{1}{\alpha} \Delta\pi[x(t)] (1 - e^{-\alpha T}) \quad (4)$$

is a variable used as an observable proxy of ΔV_t^T .

Equation (3) states that expectations change because of time changes in the proxy (momentum effect), and in response to discrepancies between actual and expected payoffs (error correction mechanism).

The parameter $\beta > -1$ measures how conservatively players tend to adjust their expectations as a consequence of the variability of their proxy. Ceteris paribus, if $\beta < 0$ ($\beta > 0$), a given change in the observed proxy will induce a relatively smaller (larger) change in the rate at which players adjust their expectations.

The error correction mechanism, shows that if players find out that their guess is too pessimistic on the basis of the available evidence, they revise their expectations upwards, and conversely in the opposite case. The parameter $\gamma > 0$ measures the strength of this informational feedback.

Note that if $x(t)$ becomes almost stationary, expectations $y(t)$ approach the value of the observable proxy, which in this case is a very good approximation of the true value.

Finally, equation (4) states that the observable proxy of ΔV_t^T is obtained by assuming that the distribution $x(t)$ does not change in the interval $[t, t + T]$.

To sum up, our expectations augmented evolutionary model can be expressed in terms of the following system of equations

$$\dot{x}(t) = g[x(t), y(t)] \quad (5)$$

$$\dot{y}(t) = \gamma [\omega[x(t)] - y(t)] + (1 + \beta) \frac{d\omega[x(t)]}{dt}, \quad (6)$$

where

$$\frac{d\omega[x(t)]}{dt} = \frac{1}{\alpha} (1 - e^{-\alpha T}) \frac{d\Delta\pi[x(t)]}{dx(t)} g[x(t), y(t)].$$

The following conditions represent a natural analogue of the order compatible dynamics assumptions for standard evolutionary models

- A1 $g(x, y)$ is C^1 for every $x \in (0, 1)$ and $y \in \mathbb{R}$.
- A2 $g(x, 0) = 0$, for every $x \in [0, 1]$;
- A3 $g(x, y)$ is (strictly) increasing in y , for every $x \in (0, 1)$ and $y \in \mathbb{R}$;
- A4 $g(0, y) > 0$ if $y > 0$ and $g(0, y) = 0$ if $y < 0$; $g(1, y) < 0$ if $y < 0$ and $g(1, y) = 0$ if $y > 0$.

A2 states that when the expected payoff difference is zero, i.e. $y = 0$, none of the players has an incentive to change action. Therefore any strategy distribution x is invariant for the population dynamics. A3 amounts to postulating that the proportion of a given strategy across the population is increasing (with a growth rate which depends positively on y) if and only if its payoff is expected to be higher than that of the rival strategy. Finally, A4 represents a boundary condition that allows the variable x to remain in the interval $[0, 1]$.⁵

Remark 1 Equation (6) may be alternatively written as

$$\frac{d}{dt} [y(t) - \omega[x(t)]] = -\gamma [y(t) - \omega[x(t)]] + \beta \frac{d\omega[x(t)]}{dt} \quad (7)$$

Thus, if $\beta = 0$, expectations dynamics are entirely characterized by the general solution

$$y(t) - \omega[x(t)] = [y(0) - \omega[x(0)]]e^{-\gamma t} \quad (8)$$

Notice that if $y(0) = \omega[x(0)]$, then $y(t) = \omega[x(t)]$ at every future time.

The aim of the following section is to study the dynamic properties of the system (5)-(6), and to derive some preliminary results.

3 Local stability

Definition 1 A fixed point of the system of equations (5)-(6) is a pair (x, y) such that $\dot{x} = \dot{y} = 0$.

The following result characterizes the set of fixed points of the expectations augmented dynamics.

Proposition 1 The set of fixed points of (5)-(6) is given by

1. Any $(x, 0) : x \in (0, 1)$ and $\omega[x(t)] = 0$ (i.e. $\Delta\pi[x(t)] = 0$)
2. $(0, \omega[0])$, if $\omega[0] \leq 0$ and $(1, \omega[1])$, if $\omega[1] \geq 0$.

⁵Possible specifications of \dot{x} satisfying this condition are $g(x, y) = x(1-x)y$ (defined for $0 < x < 1$) as well as $g(x, y) = \delta(1-x)y$ if $y \geq 0$ and $g(x, y) = \delta xy$ if $y < 0$, where δ can be interpreted as the replacement rate of players, as in MM. See section 6 for a numerical analysis of such a dynamics.

Definition 2 We call *mixed population fixed point* any fixed point of (5)-(6) of the first type, and *pure population fixed point* any fixed point of the second type.

Observe that $x = 0$ and $x = 1$ are attractive fixed points of (1) (and therefore are symmetric Nash equilibria of the underlying game) if and only if $(x, y) = (0, \omega[0])$ and $(x, y) = (1, \omega[1])$ are pure population fixed points of (5)-(6). Moreover, $\bar{x} \in (0, 1)$ is a fixed point of (1), if and only if $(\bar{x}, 0)$ is a mixed population fixed point of (5)-(6).

The following result concerns the local stability of fixed points.

Proposition 2 Suppose that $\omega[x(t)]$ always intersects the x -axis transversely, then we have that

1. Pure population fixed points are locally attractive.
2. A mixed population fixed point $(\hat{x}, 0)$ is a saddle point for the dynamics (5)-(6) if $\frac{d\omega[\hat{x}(t)]}{dx(t)} > 0$ (i.e. $\frac{d\Delta\pi[\hat{x}(t)]}{dx(t)} > 0$); it is an attractive fixed point if the opposite inequality holds.

Proposition 2 means that the stability properties of any fixed point of (5)-(6) are the same as those of the corresponding fixed point of (1). Therefore, the dynamics (5)-(6) preserve the stability properties of the dynamics (1).

Observe that to require a transversal intersection of $\omega[x(t)]$ with the x -axis is equivalent to limit our analysis to the case of hyperbolic fixed points. This assumption will be kept throughout the paper.

In the next two sections we provide the main contribution of our paper, which is given by the global analysis of the system (5)-(6).

4 Global analysis: preliminary results

In order to proceed with our analysis, some further preliminary results are needed. Consider the following sets

$$\Phi \equiv \{(x, y) : x \in (0, 1), y \in (0, +\infty)\},$$

and

$$\Lambda \equiv \{(x, y) : x \in (0, 1), y \in (-\infty, 0)\}.$$

We have that $\dot{x} > 0$ for any $(x, y) \in \Phi$ and $\dot{x} < 0$ for any $(x, y) \in \Lambda$. Therefore, in both Φ and Λ , $x(t)$ is strictly monotonic, and every trajectory

in these sets can be considered as the graph of a function of x , $y = Y(x)$, that must satisfy the differential equation

$$\frac{dy}{dx} = \gamma \frac{\omega[x(t)] - y}{g(x, y)} + (1 + \beta) \frac{d\omega[x(t)]}{dx}, \quad (9)$$

where $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$.

Lemma 1 *Equation (9) shows that*

1. *if $\beta = 0$, the curve $y = \omega[x(t)]$ is an invariant set.*
2. *if $\beta > 0$, then in Φ the trajectories cross the curve $y = \omega[x(t)]$ from below when $\frac{d\omega[x(t)]}{dx} > 0$, and from above when $\frac{d\omega[x(t)]}{dx} < 0$;*
3. *if $\beta < 0$, then in Φ the trajectories cross the curve $y = \omega[x(t)]$ from above when $\frac{d\omega[x(t)]}{dx} > 0$, and from below when $\frac{d\omega[x(t)]}{dx} < 0$;*
4. *A symmetric configuration holds in Λ .*

From proposition 2, we know that mixed population fixed points are saddles (resp. attractive) if the curve $y = \omega[x(t)]$ crosses the x -axis from below (resp. above) at the fixed point. Thus saddles alternate to attractive fixed points. Figure 1 shows a possible case. The horizontal line underneath the (x, y) axes represents the phase portrait in the case of the corresponding dynamics (1), and is drawn to allow an easy comparison (remember proposition 2).

In this figure, and in the ones that follow, \bullet denotes attractive fixed points and \circ repulsive fixed points; saddle points are at the intersection of the curves defining their inset and outset. The central part of this section concerns the variations of the outset (unstable branch) and of the inset (stable branch) of each saddle with respect to changes of the parameters. The analysis highlights some interesting properties of the shape of the basins of attraction of fixed points, which are separated by the insets of saddle points. Such properties will be useful in characterizing our results.

We focus on the case in which there exist at least two mixed population fixed points, as the results can easily be extended to the remaining cases.

Definition 3 *Two mixed population fixed points are consecutive if between them (along the x axis) there are no other fixed points. Let $(x_n, 0)$ and $(x_{n+1}, 0)$ be two consecutive fixed points with $0 < x_n < x_{n+1} < 1$.*

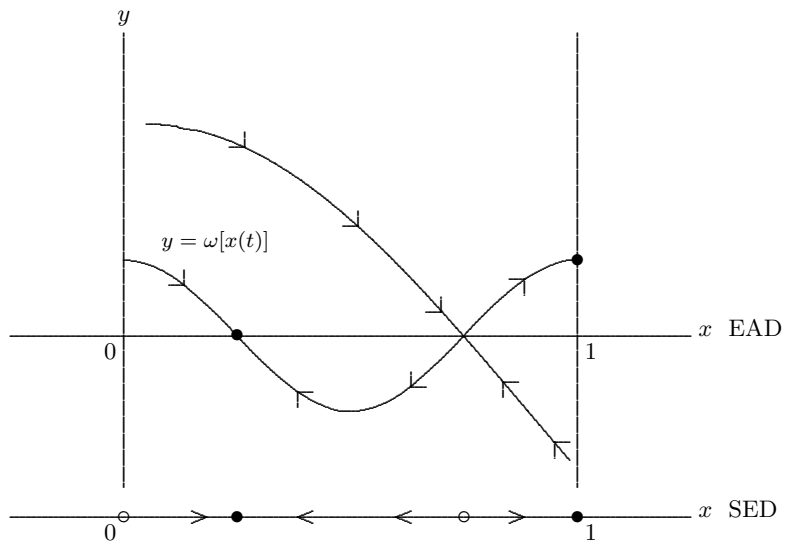


Figure 1: A possible phase portrait for the expectations augmented dynamics (EAD) in comparison with standard evolutionary dynamics (SED).

Without loss of generality, we let $\omega[x(t)] > 0$ in the interval (x_n, x_{n+1}) .⁶

Lemma 2 *If $\beta \geq 0$, the region of the phase space between the x -axis and the curve $y = (1 + \beta)\omega[x(t)]$ is a positively invariant set for the trajectories of the system (5)-(6)*

From the previous lemma, the following result follows.

Lemma 3 *if $\beta \geq 0$, then for $x > x_n$ the outset with $y > 0$ of $(x_n, 0)$ lies in the region delimited by the curve $y = (1 + \beta)\omega[x(t)]$ and the x -axis; therefore along it the fixed point $(x_{n+1}, 0)$ is reached.*

Lemma 3 also applies to the case in which $(x_n, 0)$ is attracting whereas $(x_{n+1}, 0)$ is a saddle. In this context, the result concerns the outset with $y < 0$ of $(x_{n+1}, 0)$ which, for $x \in (x_n, x_{n+1})$, connects these two fixed points. Therefore, for $\beta \geq 0$, the outset of any saddle point links each fixed point to its consecutive.

5 Global analysis: main results

We are now in the position to characterize globally the qualitative properties of the expectations augmented dynamics.

Theorem 1 *Suppose that $g(x, y)$ is an infinite of an order no smaller than y , for $y \rightarrow \infty$ and fixed $x \in (0, 1)$, and that $\beta \geq 0$. Then for any given $x(0) \in (0, 1)$, and any given attractive fixed point (x^*, y^*) of the dynamics (5)-(6), there exists an open set of initial values of the expectations $y(0)$ such that the trajectory starting at $(x(0), y(0))$ converges to (x^*, y^*) .*

Theorem 1 not only states that whatever the initial distribution of strategies in the population, the trajectories of the dynamics (5)-(6) converge to any attractive fixed point by suitably choosing the initial values of the expectations. It also provides us with a parametrization of the attractive fixed points by open sets, for given $x(0)$. As $y(0)$ varies, the asymptotic behavior of the expectations augmented dynamics jumps from one fixed point to another. Figures 2 and 3 give a graphical representation of theorem 1.

⁶In this case $(x_n, 0)$ is a saddle and $(x_{n+1}, 0)$ is attractive. The context in which $\omega[x(t)] < 0$ in (x_n, x_{n+1}) can be analyzed in the same way by considering the distribution of strategies across the population in terms of the variable $(1 - x)$ instead of x .

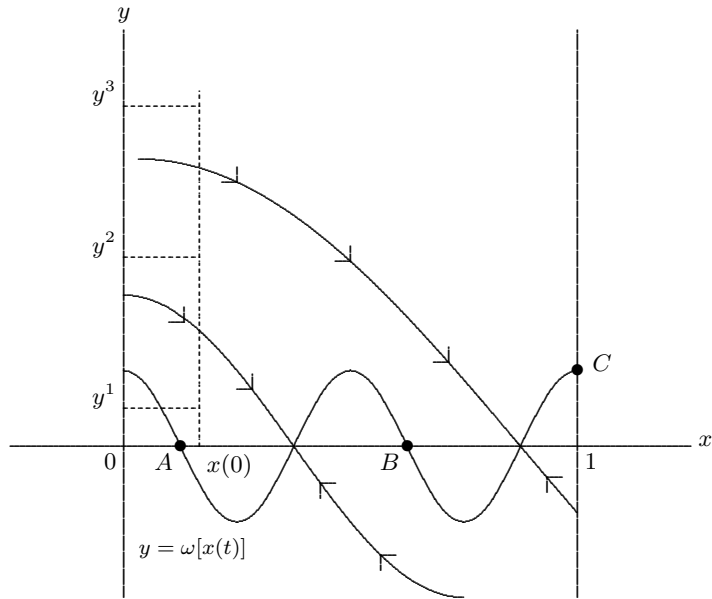


Figure 2: A graphical representation of theorem 1.

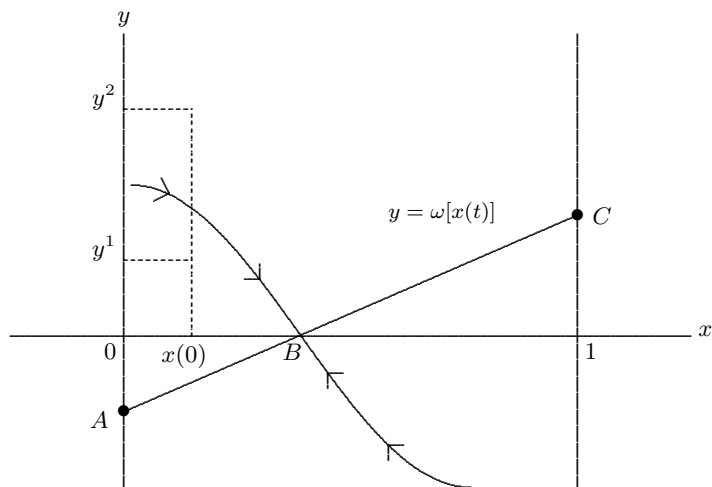


Figure 3: A possible phase portrait for a coordination game.

Figure 2 shows that when the system starts, say, at $x_0 \in (0, 1)$, it can converge to either A , B , or C , depending on the initial value of the expectations (respectively, y^1 , y^2 , or y^3).

Figure 3 shows instead the case of a coordination game, with two strict Nash equilibria (which are attractive fixed points) and one mixed Nash equilibrium (which is a saddle point). In the region below the stable branch of the saddle point (B), the dynamics (5)-(6) converge to the fixed point A . In the region above the stable branch of the saddle point, they converge to the fixed point C .

A description of the proof of theorem 1 may help to better appreciate its significance and its limitations. From lemma 2 we know that the inset for each saddle is given by two trajectories (as well as the fixed point itself): one along which $y > 0$ (so that it lies entirely in Φ), and one along which $y < 0$ (so that it lies entirely in Λ). Furthermore, the condition on $g(x, y)$ (which basically means that x needs to be sufficiently reactive to expectations y) implies that all the trajectories, included those that generate the stable branches of each saddle, do not have vertical asymptotes at any $\bar{x} \in (0, 1)$. As a result, each inset is the graph of a function of x defined in the open interval $(0, 1)$. We can conclude that, for any $x(0)$, the vertical line passing through that point has a non-empty intersection with the insets of all the saddles, and thus with all the basins of attraction of all the fixed points.

Unfortunately nothing can be said, analytically, about the robustness of this result to some restrictions about the initial values of the expectations, one natural candidate being $y(0) \in \frac{1-e^{-\alpha T}}{\alpha} [\min \Delta\pi(x), \max \Delta\pi(x)]$. In general, such a restriction about initial beliefs may not be too stringent in the case where players do not know the payoffs of the game.⁷ Moreover, as section 6 shows, the initial values of expectations for which our results hold can well be within that range. In other words, the initial expectations need not necessarily be too irrational for our results to emerge.

From theorem 3 it also follows that, for $\beta \geq 0$, along the system's trajectories the sign of \dot{x} can change at most once; thus we cannot observe persistent oscillations of the distribution of strategies. This excludes the possibility of limit-cycles, so that the system always converges to a fixed point. In other words, the dynamics converge to a point in which expectations are rational: in the long run, agents cannot make mistakes. This result is due to the fact that the speed of adjustment decreases as the dynamics approach steady state, thus facilitating the onset of new expectations, and

⁷It is also important to stress that all the other results in the paper do not depend either on $y(0)$ or on the restriction on $g(x, y)$.

making it harder for agents to make systematic mistakes.

On the other hand, this may not be the case if $\beta < 0$. More generally, when β and γ are sufficiently close to, respectively, -1 and 0 , any attractive fixed point is a focus, and therefore the trajectories oscillate infinitely many times around it.

The last thing we check is how the basins of attraction change with respect to the parameters. Consider two subsequent fixed points $(x_n, 0)$ (a saddle) and $(x_{n+1}, 0)$ (attractive), with $x_{n+1} > x_n$.

Theorem 2 *Assume that $x(0) \in (x_n, x_{n+1})$, $y(0) \geq 0$, and $\omega[x(0)] > 0$. Then there exists an interval $[\beta^*, +\infty)$ such that, if $\beta \in [\beta^*, +\infty)$, the trajectory starting at $[x(0), y(0)]$ approaches $(x_{n+1}, 0)$. The same (symmetric) result holds for $y(0) \leq 0$ when $(x_{n+1}, 0)$ is a saddle and $(x_n, 0)$ is attractive.*

The above result says that, when we have two consecutive fixed points $(x_{n+1}, 0)$ and $(x_n, 0)$, if $x_n < x(0) < x_{n+1}$ and

$$y(0) \geq 0, \omega[x(0)] > 0 \text{ respectively } y(0) \leq 0, \omega[x(0)] < 0, \quad (10)$$

the trajectory starting from $(x(0), y(0))$ converges to the attracting fixed point $(x_{n+1}, 0)$ (respectively, to $(x_n, 0)$) if β is large enough.⁸ Figure 4 shows a graphical representation of the theorem. As we can see, the set between the horizontal axis and the curve whose equation is $y = (1 + \beta^*)\omega[x(t)]$ represents the invariant set of theorem 2 for positive values of y_0 . Since this set expands with β^* , for any initial value of $[x(0), y(0)]$ it is possible to converge to the next (mixed population) fixed point. It is important to stress that this result holds even if there are many stable fixed points.

Condition (10) requires that both $y(0)$ and the proxy have the same sign. When this is true, theorem 2 states that for any initial level of the expectations, a sufficiently high level of β allows the system to converge to $(x_{n+1}, 0)$. Both dynamics (1) and (5)-(6) converge to the same strategy distribution. Note also that a large β is not sufficient to generate such a qualitative equivalence between the two dynamics. We also need that the signs of the initial value of the expectations and of the proxy be the same.

The next theorem takes into account the case in which β is large but the proxy and the initial value of the expectations have different signs. As we will see, the dynamics (1) and (5)-(6) converge to different strategy distributions.

To fix ideas, assume first that there exist only three mixed population fixed points: $(x_1, 0)$, $(x_2, 0)$ and $(x_3, 0)$, $0 < x_1 < x_2 < x_3 < 1$, where the

⁸Remember that the parameter β represents the reactivity of expectations to changes of the observable proxy $\omega[x(t)]$.

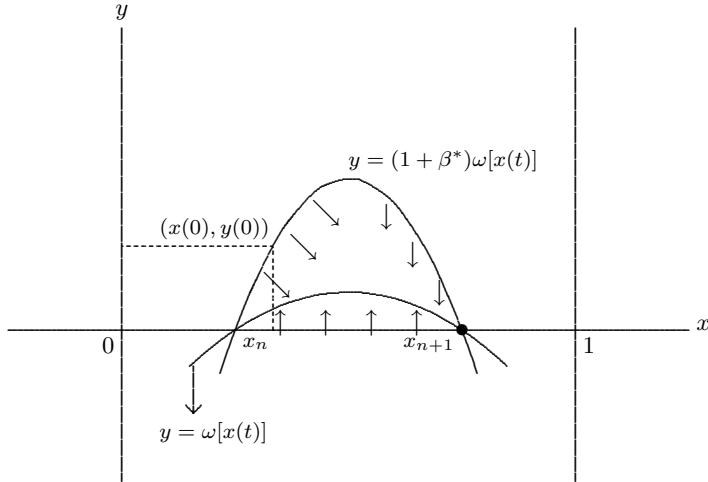


Figure 4: A graphical representation of theorem 2.

middle one is a saddle and the other two are attracting. Thus $\omega[x(t)] > 0$ for $x \in (x_2, x_3)$ and $\omega[x(t)] < 0$ for $x \in (x_1, x_2)$.

Theorem 3 *Assume that $x(0) \in (x_1, x_2)$ and $\omega[x(0)] < 0 < y(0)$. If $\omega[x(t)]$ is increasing in $[x(0), x_2]$ and if in $\{(x, y) : x \in [x(0), x_2], y > 0\}$ the following inequality is satisfied*

$$\left[\frac{\omega[x(t)] - y}{g(x, y)} \right]_y \geq 0, \text{ i.e. } g_y(x, y) \geq \frac{g(x, y)}{y - \omega[x(t)]} \quad (11)$$

*then there exists an interval $[\beta^{**}, +\infty)$ such that the trajectory starting from $[x(0), y(0)]$ converges to $(x_3, 0)$ if $\beta \in [\beta^{**}, +\infty)$.*

The symmetric case holds if $x(0) \in (x_2, x_3)$ and $y(0) < 0 < \omega[x(0)]$.

Condition (11) simply requires that the reactivity of \dot{x} with respect to the expectations be high enough. Theorem 3 says that when the initial value of the expectations has a different sign from that of the observable proxy and β is big enough, a trajectory starting with a strategy distribution between two subsequent fixed points, converges to the third fixed point of the system. Figure 5 provides a graphical interpretation of the theorem. Here, the trajectory starting at $(x(0), y(0))$, with $y(0) > 0$ and $x(0) \in (x_*, B)$, converges to the point $C < 1$ when β is sufficiently large. On the other

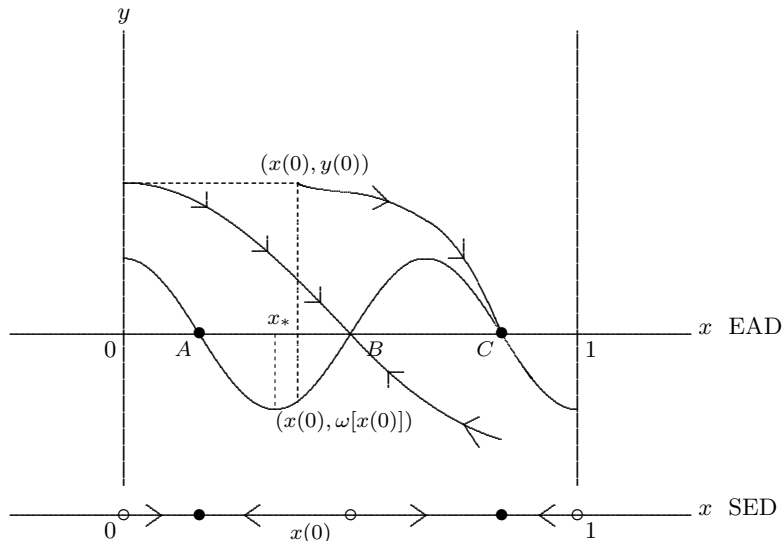


Figure 5: A graphical representation of theorem 3.

hand, under the dynamics (1), the trajectory starting at $x = x(0)$ converges to a point whose coordinate on the x axis is the same as A .

Figure 6 represents theorem 3 in the case of a coordination game, where $x = 0$ and $x = 1$ are the only two attractive fixed points. In this case, since the proxy is always increasing in x , theorem 3 holds for any $x(0) \in [0, 1]$. For any $x(0) < B$, we have that the current payoff differential between strategies 1 and 2 is negative. Yet, strategy 1 grows faster with x . Once x gets bigger than B , the current payoff differential becomes positive.

A large β means that the dynamics of the expectations give a high importance to the fact that strategy 1 has a steeper gradient. As a result, the dynamics (5)-(6) converge to point C , where everybody adopts strategy 1.

Clearly, theorem 3 also applies if the system admits more than three mixed population fixed points. In this case the trajectory converges to one of the non-subsequent fixed points. Moreover, it also applies to the case considered in figure 6, where the unique mixed population fixed point is a saddle and the pure population fixed points at $x = 0$ and $x = 1$ are stable.

To sum up, the dynamics (1) and (5)-(6) generate the same prediction about the asymptotic strategy distribution only when the proxy and the initial value of the expectations have the same sign, with β sufficiently large (theorem 2). In fact, when β is large but the proxy and the initial value of

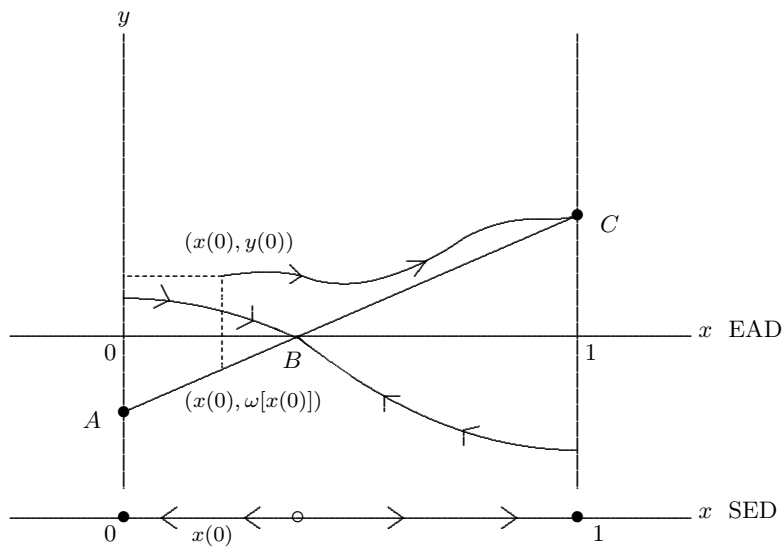


Figure 6: A graphical representation of theorem 3 in the case of a coordination game.

the expectations have opposite signs, the two dynamics generate different predictions (theorem 3).

These results, together with the basic indeterminacy from theorem 1, imply that for large β the expectations matter more than the initial strategy distribution.

6 Some applications

This section contains some applications of our analysis. We provide simulations for different parameter values of the expectations augmented dynamics, and compare the results with MM and standard evolutionary dynamics.

First, we consider a coordination game, then we look at a public contracting game from Antoci and Sacco (1995), that generates non-linear payoff differences.

Consider the normal form game reported below.

| | | | |
|----|---|-----|-----|
| | | P2 | |
| | | R | P |
| P1 | R | 4,4 | 8,1 |
| | P | 1,8 | 9,9 |

It is well known that there are two pure strategy strict Nash equilibria. One, (R,R) , is risk dominant in the sense of Harsanyi and Selten. The other, (P,P) , is Pareto efficient. Moreover, the basins of attraction of the pure strategy equilibria in the standard evolutionary dynamics are separated by $x = \frac{1}{4}$, where x denotes the share of population adopting strategy R . Thus, when the system starts at $x(0) < \frac{1}{4}$, it converges to $x = 0$. Conversely, it converges to $x = 1$ when it starts at $x(0) > \frac{1}{4}$.

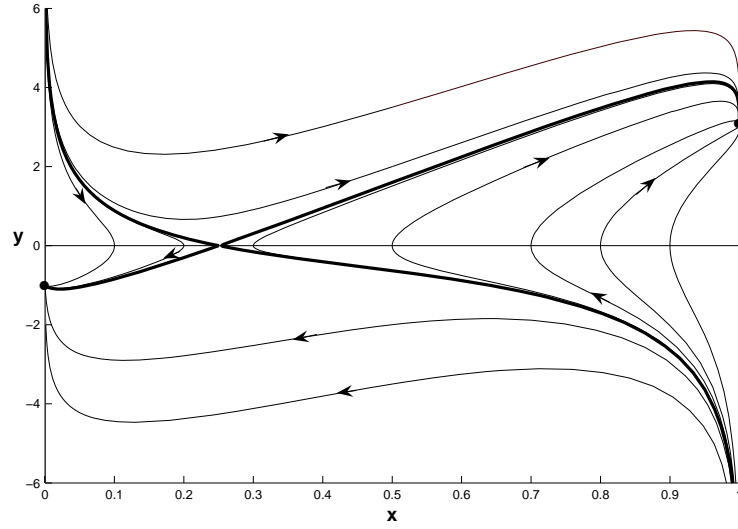
For any $x \in (0, 1)$, we consider the following specification of our expectations augmented evolutionary dynamics:

$$\begin{aligned} \dot{x} &= x(1-x)y & (12) \\ \dot{y} &= \gamma(4x-1-y) + 4(1+\beta)\dot{x} \end{aligned}$$

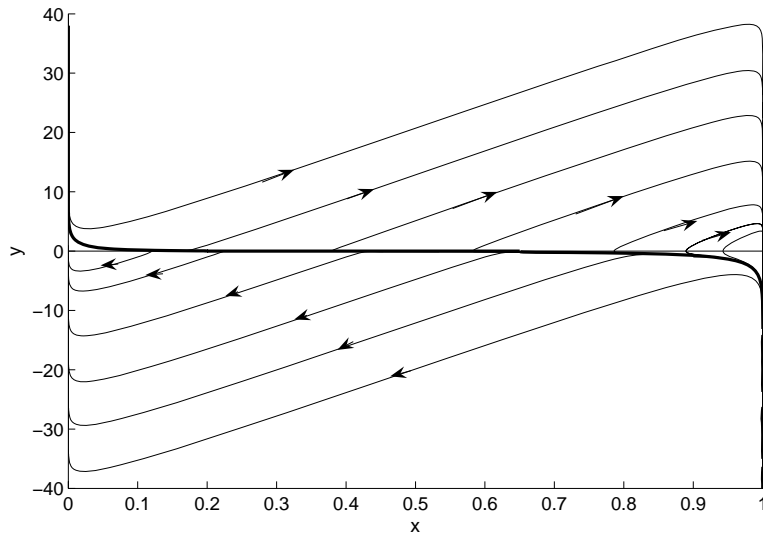
where, again, x denotes the share of the population adopting strategy R .^{9 10} We solve numerically for $\beta = 1, 10$, and $\gamma = 1, 10$. Figure 7 shows how the

⁹In all our applications, $\alpha = 1$ and $T = \infty$ are maintained assumptions.

¹⁰Observe that we are not simulating the dynamics on the boundary of x . We chose to proceed this way in order to get more readable phase portraits without affecting our results.



(a) $\beta = 1$.



(b) $\beta = 10$.

Figure 7: Phase portraits of the simulated dynamics (12), in the coordination game, when $\gamma = 1$ and $\beta = 1, 10$.

dynamics change as β increases, given $\gamma = 1$. As we can see, by moving from $\beta = 1$ to $\beta = 10$, the inset of the saddle point (which is drawn in bold, like in all the following figures) flattens towards the x axis. Thus, when $\beta = 10$, for $y_0 > 0$, and very close to zero, the system converges to the fixed point with $x = 1$, for almost any possible value of $x(0)$. Likewise, when $y_0 < 0$, and very close to zero, the dynamics converge to the fixed point $x = 0$. (See theorem 3.)

Moreover, in both $\beta = 1$ and $\beta = 10$ cases, the initial values of the expectations, y_0 , for which it is possible to move from one fixed point to the other are not too big, and therefore reasonable.

Figure 8 shows that when $\gamma = 10$, the values of y_0 for which the indeterminacy results hold tend to increase (in magnitude) for every x_0 . In other words, only very high or low values of y_0 can push the system out of its ‘natural’ basin of attraction. This is because a large γ (for fixed β), implies that changes in expectations are mainly driven by the current share of population x (i.e. by the current payoff difference), like in the standard evolutionary dynamics.

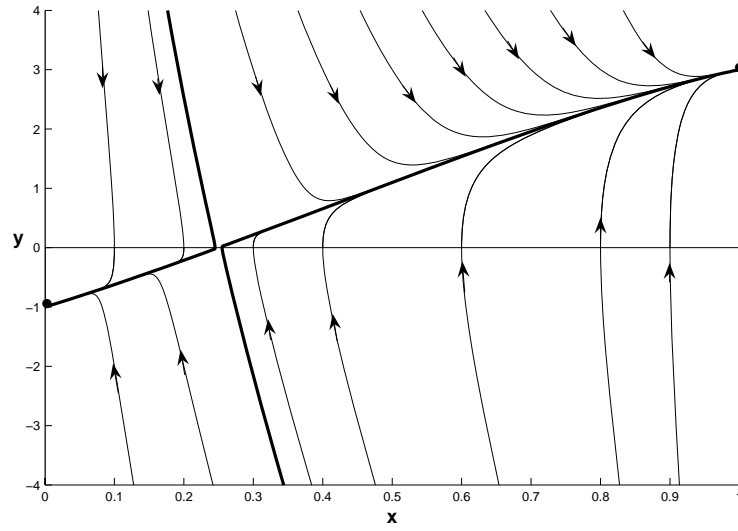
In particular, we observe that for $\beta = 1$, the inset of the saddle point (drawn in bold) is almost vertical. Therefore, given $x(0)$, the initial value of the expectations, $y(0)$, plays only a marginal role in the selection process of the equilibria. (See figure 8(a)). On the other hand, this feature is weakened by an increase in β , as shown in figure 8(b).

Next, we provide a comparison with the results obtained by MM. Adapting MM to our case, the expectations augmented evolutionary dynamics are given by

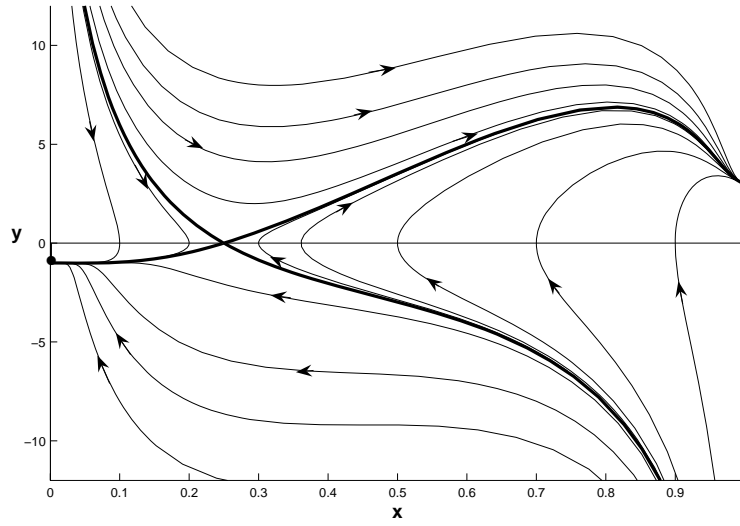
$$\begin{aligned} \dot{x} &= \begin{cases} \delta y x & \text{if } y < 0 \\ \delta y(1-x) & \text{if } y \geq 0 \end{cases} & (13) \\ \dot{y} &= \gamma(4x - 1 - y) + 4(1 + \beta)\dot{x} \end{aligned}$$

where x denotes the share of the population adopting strategy R , and δ is the parameter indicating the replacement rate of players.¹¹ We numerically solve the system of equations (13) for $\delta = 0.05, 3, \beta = 1, 10$, and $\gamma = 1$. The two values of δ are chosen so as to meet the conditions of proposition 1 in MM: for small δ , they have that (R, R) is the only absorbing state, whereas for large δ , they find that both (R, R) and (P, P) are absorbing. As we can see from figures 9(a) and 9(b), even when δ is small, the expectations augmented dynamics still converge to either equilibrium for reasonable values of

¹¹Clearly $g(x, y)$ is not C^1 at $y = 0$. While this helps in the implementation of the simulations, it does not affect the theoretical results of the paper.



(a) $\beta = 1$



(b) $\beta = 10$

Figure 8: Phase portraits of the simulated dynamics (12), in the coordination game, when $\gamma = 10$ and $\beta = 1, 10$.

y_0 . Figure 10 reports the case of $\delta = 3$, whose interpretation at this point is pretty straightforward. On the other hand, it is interesting to observe how δ , the friction parameter in MM, plays a role that is analogous to β (which measures the reactivity of players' expectations to payoff differences) and opposite to γ (which measures the strength of the informational feedback). As δ and β get smaller (i.e. as γ gets bigger), the initial values of the expectations, y_0 , that are called for to move from one fixed point to the other become larger.

Moving to our second example, Antoci and Sacco (1995) study the following evolutionary model of corruption: a continuum of firms are randomly matched in pairs to play a procurement auction game in which they decide first, between investing in a low and a high quality project and second, whether to bribe the official running the auction. Revenues from the procurement and bribes are fixed in amounts. Moreover, the officer can be caught and penalized with positive probability.

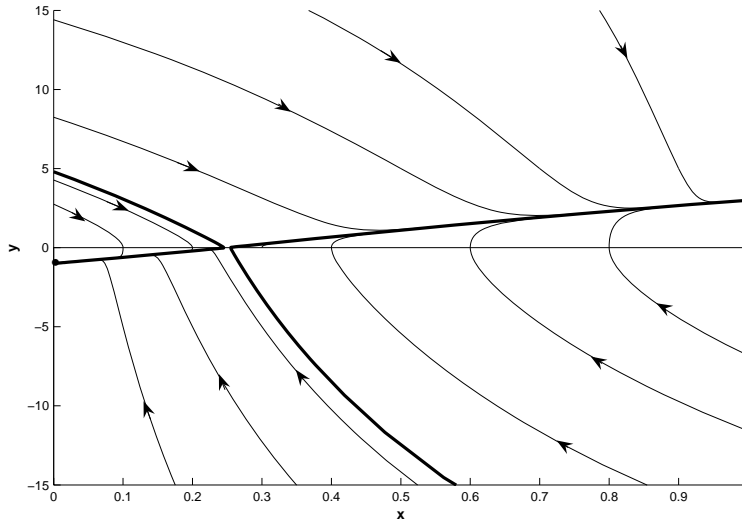
In this context, x denotes the share of firms that bid high quality and $1 - x$ is the share of firms that bid low quality and bribe the officer. For some parameters ranges, the model generates non linear payoffs difference.¹² We extend their model to account for expectations as defined in the current paper. As a result, the system of equations to be analyzed becomes

$$\begin{aligned} \dot{x} &= x(1-x)y \\ \dot{y} &= \gamma \left(\frac{271x - 180x^2 - 95}{180(1-x)} - y \right) + (\beta + 1) \frac{44 - 90x + 45x^2}{45(1-x)^2} \dot{x} \end{aligned} \quad (14)$$

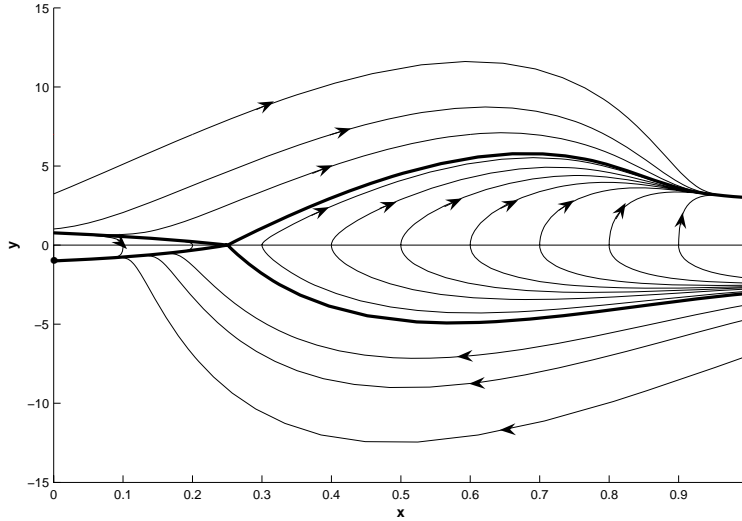
Figure 11 reports the results of the simulations for the following parameters configurations: $\beta = 1, 10$, and $\gamma = \delta = 1$. As we can see, there are two attractive fixed points: one is a pure population fixed point, at $x = 0$, and the other is a mixed population fixed point, at $x = 0.95$. The two are separated by a mixed population fixed point, at $x = 0.5$, which is a saddle. Moreover, the qualitative features of the inset for the saddle point highlighted in the previous example are preserved. Finally, observe that the initial values of the expectations, y_0 , needed to move from one fixed point to the other are still reasonable.

Our results hold under the assumption of $\beta \geq 0$, because in that case the trajectories that are on the inset of a saddle cannot cross the $y = 0$ axis.

¹²One case for which this is true, and that is adopted in this paper, is given by $\epsilon = \frac{1}{3}$ (cost of the bribe), $l = 3$ (value from a low quality bid), $h = 1$ (value from a high quality bid), and $Pk = 5$ (officer's expected cost of being detected). Note also that for computational reasons the dynamics here are slightly simplified. Again, the main results are not affected.

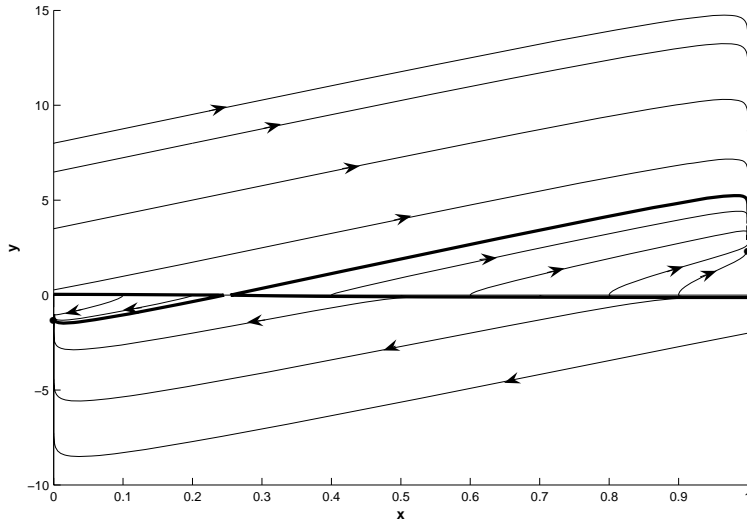


(a) $\beta = 1$

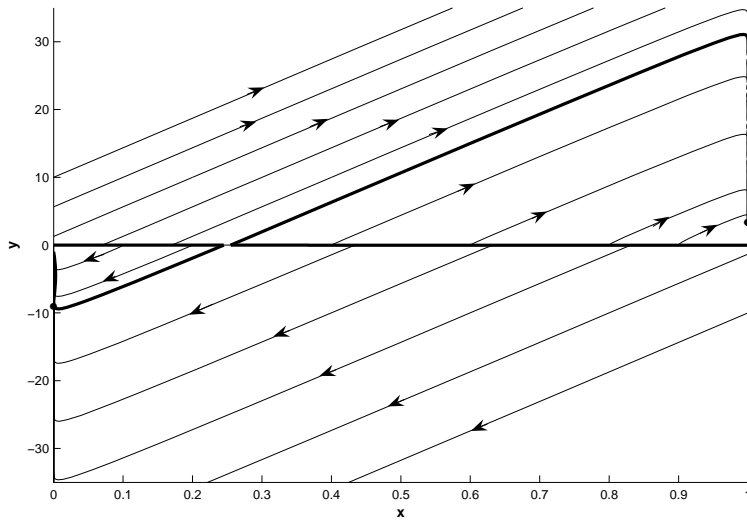


(b) $\beta = 10$

Figure 9: Phase portrait of the simulated dynamics (13), in the coordination game, when $\gamma = 1$, $\delta = 0.05$ and $\beta = 1, 10$.

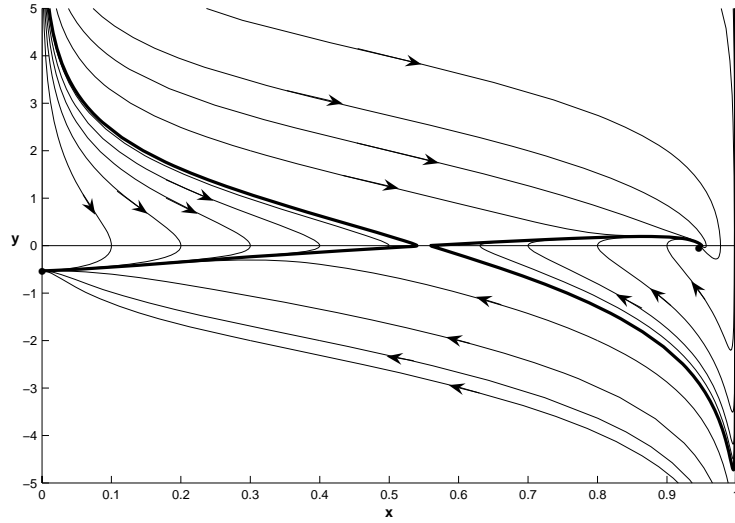


(a) $\beta = 1$

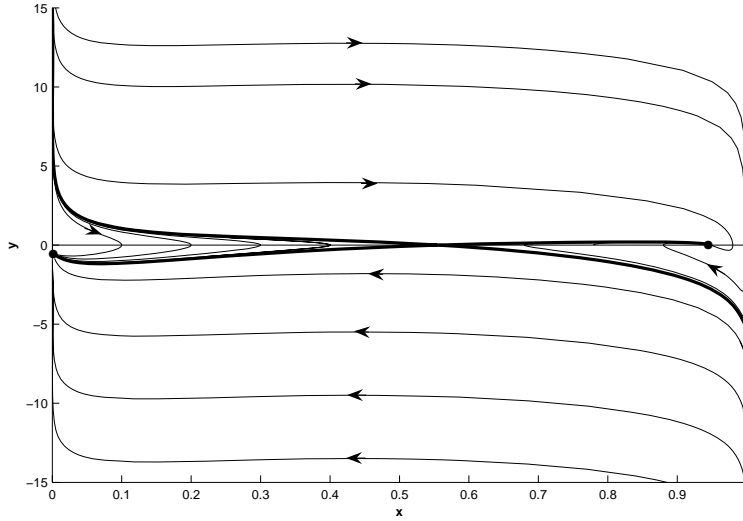


(b) $\beta = 10$

Figure 10: Phase portrait of the simulated dynamics (13), in the coordination game, when $\gamma = 1$, $\delta = 3$ and $\beta = 1, 10$.



(a) $\beta = 1$.



(b) $\beta = 10$.

Figure 11: Phase portrait of the simulated dynamics (14), in the public contracting game, when $\gamma = 1$ and $\beta = 1, 10$.

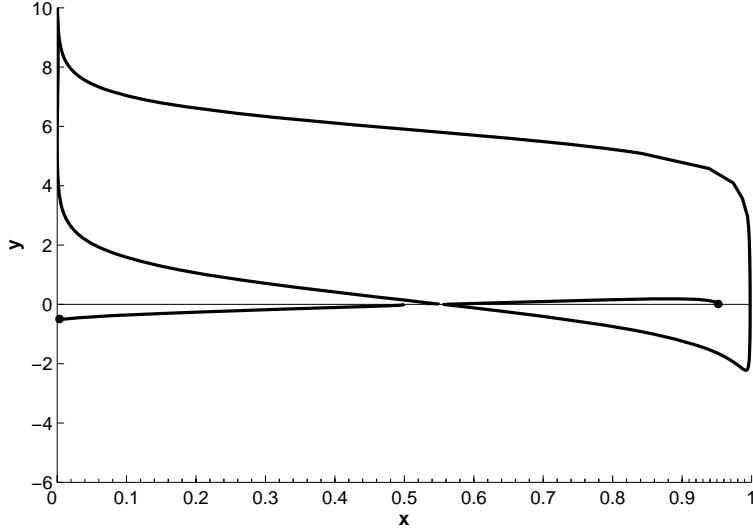


Figure 12: Phase portrait of the simulated dynamics (14), in the public contracting game, when $\beta = -0.95$ and $\gamma = 0.5$.

When $\beta < 0$ the dynamics become even less predictable and much harder to study analytically. For β sufficiently close to -1 and γ sufficiently close to 0 , any attractive fixed point is a focus, and therefore the trajectories oscillate infinitely many times around it. Otherwise, other behaviors can emerge. Figure 12 shows this possibility, when $\beta = -0.95$ and $\gamma = 0.5$. The inset is represented by the trajectories that start from $y = 4$ and from $y = 10$. The outset is represented by the trajectories connecting the fixed point $x = 0$ to the fixed point $x = 0.95$.

Observe first that the trajectory starting at $y = 10$ goes from the region with $y > 0$ to the region with $y < 0$. Again, this is a consequence of $\beta < 0$. Moreover, the basin of attraction of the fixed point $x = 0$ is now given by the region between the trajectory starting at $y = 4$ and $y = 10$. This means that, for fixed x_0 , it is possible that the system converges to $x = 0$ for values of y_0 that are sufficiently small or large, and to $x = 0.95$ for values of y_0 that are intermediate.

7 Conclusion

This paper provides a global analysis of an expectations augmented evolutionary dynamics and compares them to standard evolutionary dynamics. It is found that more radical kinds of indeterminacies arise in addition to the already well known ones. Moreover, boundedly rational agents converge to rational expectations in the long run.

A necessary condition for our results is that β , the measure of the reactivity of the expectations to changes in the observable proxy, be large enough. Moreover, the behavior of our expectations augmented dynamics becomes very similar to standard evolutionary dynamics when γ , which measures the strength of the informational feedback, increases.

Last we showed, by means of numerical simulations, the features of our model in linear and non-linear cases, and compared them to the rational expectation evolutionary dynamics explored by Matsui and Matsuyama (1995).

A Proofs of the results

In this section we provide the proofs of our results.

Proof of Lemma 1. The first point has already been shown in remark 1. Observe that at $y = \omega[x(t)]$ equation (9) shows that $\frac{dy}{dx}$ has the same sign as $\frac{d\omega[x(t)]}{dx}$, for any $\beta > -1$. This implies that, for $y > 0$, when $\beta > 0$ ($\beta < 0$), the trajectories have a greater (lower) slope, evaluated at $y = \omega[x(t)]$, than $y = \omega[x(t)]$ itself. This demonstrates points 2 and 3. The last point can be proved in a similar way. ■

Proof of Lemma 2. The slope of the trajectories evaluated along the curve $y = (1 + \beta)\omega[x(t)]$ can be written as

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{y=(1+\beta)\omega[x(t)]} &= \gamma \frac{\omega[x(t)] - (1 + \beta)\omega[x(t)]}{g(x, y)} + (1 + \beta) \frac{d\omega[x(t)]}{dx} \\ &= -\beta\gamma \frac{\omega[x(t)]}{g(x, y)} + (1 + \beta) \frac{d\omega[x(t)]}{dx}, \end{aligned} \quad (15)$$

whereas the slope of $y = (1 + \beta)\omega[x(t)]$ is $(1 + \beta) \frac{d\omega[x(t)]}{dx}$. Since along $y = (1 + \beta)\omega[x(t)]$ we have that $\text{sign}\{(1 + \beta)\omega[x(t)]\} = \text{sign}\{g(x, y)\}$, the lemma is proved. ■

Proof of Lemma 3. Let $\bar{Y}^u(x)$ denote the outset with $y > 0$ of $(x_n, 0)$. The case with $\beta = 0$ follows simply by noticing that for $x_n < x < x_{n+1}$ one has $\bar{Y}^u(x) \equiv \omega[x(t)]$. Consider now the case with $\beta > 0$ and suppose that, for x close to x_n , $\bar{Y}^u(x)$ lies above the curve $y = (1 + \beta)\omega[x(t)]$. We shall show that this generates a contradiction. Every trajectory $Y(x)$ of the system (5)-(6) satisfies the following integral equation

$$\begin{aligned} Y(x^*) &= Y(x_*) + \gamma \int_{x_*}^{x^*} \frac{\omega[x(t)] - Y(x)}{g(x, Y(x))} dx \\ &\quad + (1 + \beta) [\omega[x^*(t)] - \omega[x_*(t)]], \end{aligned}$$

where $x^* > x_*$.

For $x_* \rightarrow x_n$ and $x^* > x_n$, if $\bar{Y}^u(x) > (1 + \beta)\omega[x(t)]$ in (x_n, x^*) , then it

must be the case that

$$\begin{aligned}
\bar{Y}^u(x^*) - (1 + \beta)\omega[x^*(t)] &= \\
&= \bar{Y}^u(x_n) + \gamma \int_{x_n}^{x^*} \frac{\omega[x(t)] - \bar{Y}^u(x)}{g(x, \bar{Y}^u(x))} dx + \\
&+ (1 + \beta) [\omega[x^*(t)] - \omega[x_n(t)]] - (1 + \beta)\omega[x(t)] = \\
&= \gamma \int_{x_n}^{x^*} \frac{\omega[x(t)] - \bar{Y}^u(x)}{g(x, \bar{Y}^u(x))} dx > 0.
\end{aligned}$$

This generates the desired contradiction since $\frac{\omega[x(t)] - \bar{Y}^u(x)}{g(x, \bar{Y}^u(x))} < 0$ when $\bar{Y}^u(x) > (1 + \beta)\omega[x(t)] > \omega[x(t)]$. ■

Proof of Theorem 2. For $0 \leq y(0) \leq \omega[x(0)]$, the interval is $[0, +\infty)$; in fact, the curve $y = \omega[x(t)]$ is invariant for $\beta = 0$. If $y(0) > \omega[x(0)]$, we choose β^* to be the solution of the equation $y(0) = (1 + \beta^*)\omega[x(0)]$. The proof for the case in which $y(0) \geq 0$, $(x_{n+1}, 0)$ is attractive, and $(x_n, 0)$ is a saddle, is completed by just applying lemma 2. The proof of the symmetric case follows the same steps of the previous one. ■

Proof of Theorem 3. First observe that the partial derivative of $\frac{dy}{dx}$ with respect to β in (9) is given by

$$\left(\frac{dy}{dx} \right)_{\beta} = \frac{d\omega[x(t)]}{dx}. \tag{16}$$

This means that the slope $\left(\frac{dy}{dx} \right)$ of trajectories in $\Phi \cup \Lambda$ increases with β if $\frac{d\omega[x(t)]}{dx} > 0$ and decreases with β if $\frac{d\omega[x(t)]}{dx} < 0$.

Consider now the case: $x(0) \in (x_1, x_2)$ and $\omega[x(0)] < 0 < y(0)$. Take an $x^* \in (x(0), x_2)$; the trajectory $Y(x)$ passing through $[x(0), y(0)]$ must satisfy the integral equation

$$\begin{aligned}
Y(x^*) &= Y[x(0)] + \gamma \int_{x(0)}^{x^*} \frac{\omega[x(t)] - Y(x)}{g(x, Y(x))} dx \\
&+ (1 + \beta) \{ \omega[x^*] - \omega[x(0)] \}
\end{aligned} \tag{17}$$

Notice that

1. by (16), if we take two values of β , $\beta^1 < \beta^2$, then in the interval $[x(0), x_2]$, where $\omega[x(t)]$ is increasing in x , the trajectory passing through $[x(0), y(0)]$ when $\beta = \beta^2$ lies above the one passing through $[x(0), y(0)]$ when $\beta = \beta^1$.

2. By (16), in the interval $[x(0), x_2]$ the inset $\bar{Y}^s(x)$ (with $y > 0$) of $(x_2, 0)$ for $\beta = \beta^2$ lies below the inset $\bar{Y}^s(x)$ of $(x_2, 0)$ for $\beta = \beta^1$.
3. Since, by assumption, $\omega[x(t)]$ is increasing in $[x(0), x_2]$, it follows that $\omega[x^*(t)] - \omega[x(0)] > 0$.
4. By (11), the (negative) value of $\int_{x(0)}^{x^*} \frac{\omega[x(t)] - Y(x)}{g(x, Y(x))} dx$ increases if β increases.

Therefore, from (17) we have that, as β increases, $Y(x^*)$ becomes arbitrarily big and there must exist a β^{**} such that, for every $\beta \in [\beta^{**}, +\infty)$, $Y(x) > \bar{Y}^s(x)$ for every $x \in [x(0), x_2]$. This implies that the trajectory starting from $[x(0), y(0)]$ converges to $(x_3, 0)$. By the same token we can prove the symmetric case of the theorem. ■

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