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A natural extension of the bucket brigade model of manufacturing is capable of chaotic
behavior in which the product intercompletion times are, in effect, random, even though the
model is completely deterministic. This is, we believe, the first proven instance of chaos
in discrete manufacturing. Chaotic behavior represents a new challenge to the traditional
tools of engineering management to reduce variability in production lines. Fortunately, if
configured correctly, a bucket brigade assembly line can avoid such pathologies.

1 Chaos

Some simple deterministic systems can generate surprisingly complicated behavior that has
been termed "chaotic". A system that is chaotic has long-term behavior that can be hard to
describe, hard to predict, and hard even to simulate. Indeed, long-term behavior of chaotic
systems seems to be deeply connected to randomness. (See Alligood et al. [1], Devaney [12],
or Martelli [17] for discussions.)

The possibility of chaos in manufacturing systems is important because it represents a
new type of variability, and variability is generally detrimental to efficiency. Chapter 9 of
Hopp and Spearman [13] explains how variability can degrade performance of manufacturing
systems.

Researchers such as Beaumariage and Kempf [8] have suggested the possibility of chaos
in manufacturing systems, but generally speaking, only circumstantial evidence has been
offered, such as seemingly complex behavior observed in simulations. Recently Schmitz
et al. [21] surveyed claims of chaos in manufacturing and concluded "Realistic, non-artificial

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discrete-event models of discrete production systems that show chaotic behavior, as defined in the theory of chaos in dynamic systems, were not found in this study".

To our knowledge, only one model of a production system has heretofore been formally shown to exhibit chaotic dynamics and that is the switched arrival system studied in Chase et al. [10]. This model treats manufacturing as a process and the product as a fluid. In this model there are no discrete or intermediate products and therefore it did not satisfy the criteria of Schmitz et al. [21]. Armbruster [2] agrees, saying "One major drawback of the switched arrival system is the fact that the chaotic dynamics are strictly internal to the production—the total throughput through the set of machines is always constant and does not reflect the chaotic dynamics". In short, it is not the strong example one would prefer to display if chaos can indeed be found in manufacturing systems.

In the model of Chase et al. [10] a single switching server distributes work over n parallel machines. The amount of work in the buffer in front of each machine is assumed to be a continuous variable and the processing rate of each machine is assumed to be constant. The server continues to fill the current buffer until some other buffer empties. The rate at which the server fills a buffer is equal to the sum of the processing rates of all machines. When the system is sampled at the instants when any buffer empties, the dynamics of the system can be represented by a function that, for $n = 3$, maps the unit interval into itself. Chase et al. [10] showed that this function, which describes the amount of work in the buffers, can be chaotic. Others have extended the model in various ways, as may be found in Ushio et al. [22], Katzorke and Pikovsky [14], Rem and Armbruster [20], Armbruster [2], Peters et al. [19], and citations therein. Of particular note, is Rem and Armbruster [20], which added complications such as setup times and maintenance to the switched arrival system to make the chaotic behavior have external effects.

We offer a model of bucket brigade assembly lines that meets the criteria of Schmitz et al. [21]. The model is realistic: Indeed, bucket brigades are currently used in a variety of manufacturing environments, examples of which are documented at Bartholdi and Eisenstein [4]. Furthermore, the model explicitly represents each product as a discrete entity with start and completion times. As we show, the product intercompletion times, and by implication product start and finish times, can be chaotic. The details of our analysis illustrate the strange challenges that chaotic manufacturing systems present to traditional tools of management and engineering.

2 A Generalization of the Normative Bucket Brigade Model

"Bucket brigades" are a way to coördinate the efforts of workers along an assembly line. The idea is to allow the workers to move where needed, thereby avoiding the imbalance typical of static partitions of work. The movement of the workers is coördinated by a simple decentralized rule: Each worker carries work forward, from work station to work station, until he either completes an item or it is taken by a downstream colleague; then he walks back to get more work, either from an upstream colleague or from a buffer at the start of the line. For previous models and applications see Bartholdi and Eisenstein [5], Bartholdi and Eisenstein [4], Bartholdi et al. [7], Villalobos et al. [24], and Villalobos et al. [23].

Most previous models of bucket brigades have assumed that the time required for a worker to walk back upstream and get more work is insignificant compared to the time to work forward; and so workers have been modeled as having a common walk-back velocity $w_i = \infty$. One exception is Bartholdi and Eisenstein [6], which describes a case study wherein the time for worker i to walk back to get more work from worker $i - 1$ is a constant that does *not* depend on the progress of the item of either worker. Another exception is Bratcu and Dolgui [9], in which each worker shares the same constant walk back velocity.

Here we assume more generally that each worker i is characterized by two arbitrary but constant velocities: v_i in the forward direction and w_i in the backward direction. Our motivation for this generalized model is a low density order picking system such as we have observed, for example, at McMaster-Carr. In such systems workers may walk considerable distance between picks, and thus the time required to walk forward picking is comparable to the time required to walk back to get more work. Furthermore, a faster worker may overtake a slower worker when walking forward, and when walking back may (according to the rules that follow) pass a worker who is working in the forward direction.

Let there be n workers in the bucket brigade, indexed from 1 to n. Workers $1, \ldots, i - 1$ are the *predecessors* of worker i and workers $i + 1, \ldots, n$ are his *successors* and each worker must be able to distinguish his predecessors from his successors. Each worker follows the (generalized) Bucket Brigade Rules given by:

Forward Rule: Work forward with your item until

- your item is taken by a successor; or
- you complete your item;

then follow the Backward Rule.

Backward Rule: Walk back to get more work;

- if you encounter a predecessor working forward then take over his item;

- otherwise, begin a new item at the start of the line;

then follow the Forward Rule.

Under the generalized Bucket Brigade Rules several types of instantaneous events are possible. First, there are the events familiar from previous models of bucket brigades: *Starts*, in which a worker begins new work at position 0, the start of the assembly line; *Completions*, in which a worker finishes work at position 1, the end of the assembly line; and *Hand-offs*, in which a successor who is walking back takes over the work from a predecessor who is working forward.

Unlike the Normative Model, any worker may start or complete an item, and a worker i can hand off his item to *any* successor $j > i$. There are also two new behaviors: *Overtaking*, in which one worker catches up to and passes another as both walk back or as both work forward; and *Passing*, in which a worker going back to get more work walks past a successor who is working forward.

Under this generalized model of bucket brigades, even familiar events such as hand-offs appear in more complicated patterns. For example, because of finite velocities of walk-back, hand-offs are no longer contemporaneous. Furthermore, there can be multiple completions before the next hand-off or before the next start. It is even possible that there are never hand-offs, as when workers with velocities $v_1 = 2, w_1 = 1, v_2 = 1, w_2 = 2$ start together at the origin. In this case, the intended bucket brigade degenerates into the uncoördinated efforts of individual workers.

And because we now allow overtaking, it is no longer possible for one worker to block the movement of another. Therefore the production rate is as large as possible, regardless of how the workers are sequenced.

In the long run each worker must travel as far forward as he does backward and so worker i has an *effective* production rate of

$$\psi_i = (1/v_i + 1/w_i)^{-1}.\tag{1}$$

Therefore the long-run average production rate of a given set of workers is $\sum_{i=1}^{n} \psi_i$, and this is independent of the starting positions of the workers.

Expression (1) arises as follows. The term $1/v_i$ represents the *encumbered* transit time of worker i; that is, the time for worker i to accomplish one unit of work-content by himself. Similarly, $1/w_i$ represents the *unencumbered* transit time, which is the time for worker i to walk back past one unit of work-content. Therefore $1/v_i + 1/w_i$ is the total time required for worker i to assemble one item.

The generalized model will converge to a stable allocation of work if workers are sequenced as follows:

Convergence Condition: The workers on the bucket brigade assembly line should be indexed so that

$$\frac{1}{v_1} - \frac{1}{w_1} > \frac{1}{v_2} - \frac{1}{w_2} > \ldots > \frac{1}{v_n} - \frac{1}{w_n}.$$

The Convergence Condition may be interpreted as follows: The term $1/v_i - 1/w_i$ represents the difference in the encumbered and the unencumbered transit times of worker i and so gives the extent to which he is slowed by work. In other words, the Convergence Condition stipulates that workers should be sequenced from most-slowed to least-slowed.

This condition is somewhat surprising, because it may require a worker who is *slower* in both directions to be of higher index (that is, work downstream). For example, the two workers described by $v_1 = 10, w_1 = 40, v_2 = 9, w_2 = 20$ satisfies the Convergence Condition.

Under bucket brigades, workers share work-content by handing off items to successors. The locations at which hand-offs occur determine how the work is shared. The bucket brigade assembly line is *balanced* if each worker invests the same clock time and repeats the same interval of work content for each item produced, and, moreover, those intervals are non-overlapping. Let the balance point at which worker i hands off work, given as a fraction of work-content completed, be x_i^* and let $x^* = (x_1^*, x_2^*, \ldots, x_{n-1}^*)$.

Theorem 1. *For any bucket brigade the point*

$$x_i^* = \frac{\sum_{j=1}^{i} \psi_j}{\sum_{j=1}^{n} \psi_j} \quad \text{for } i\text{=}1,\ldots,\text{n-}1.$$

is a fixed point with respect to the map that relates successive points of hand-off and is, moreover, the unique point of balance.

Proof. See Appendix, Section A. □

It is worth remarking that there may be other fixed points, but the point of balance is unique. This uniqueness depends on our requirement that the work be *partitioned* among

the workers. (Consider again the bucket brigade line with workers $v_1 = 2$, $w_1 = 1$ and $v_2 = 1$, $w_2 = 2$. If these workers start at $1/2$ in opposite directions of travel then $1/2$ is a fixed point of balance; but if they start together at 1, each worker repeats the same (entire) interval of work content and 1 is a fixed point—but the line is not balanced because the work is not partitioned.)

Theorem 1 tells us that once the assembly line is balanced then, in the absence of perturbations, it will remain balanced. But for the balance point to be useful in practice, it must be an *attracting* fixed point, so that the assembly line will spontaneously seek balance. That is, successive hand-offs will move ever closer to the point of balance.

Theorem 2. *If workers are sequenced on the assembly line from most-slowed to least-slowed (the Convergence Condition) then x^* is an attractor.*

Proof. See Appendix, Section B. □

The proof of Theorem 2 shows that when the bucket brigade is "not too far" from balance, then it must converge to balance. In other words, the point of balance is a *local attractor*, and will assert itself to restore balance after perturbations, as long as they are not too disruptive. We believe, but have not proved, a stronger result: that if the Convergence Condition holds then the point of balance is a *global attractor*, which means that the bucket brigade will balance itself from any initial state.

3 Chaos in bucket brigades

Our main result is to prove that an instance of the generalized model of bucket brigades is capable of chaotic behavior when the Convergence Condition fails to hold.

Theorem 3. *There exists a 2-worker bucket brigade in which the sequence of hand-off positions is chaotic.*

We establish this theorem by showing that the bucket brigade emulates a system that is well-known to be chaotic.

Consider the bucket brigade composed of workers with the following velocities: $v_1 = 1$, $w_1 = 1/3$; $v_2 = 1$, $w_2 = 1$. This bucket brigade fails to satisfy the Convergence Condition and it is straightforward to verify that the dynamics function relating the positions of successive hand-offs is given by the following, where x^k denotes the location of the k-th hand-off.
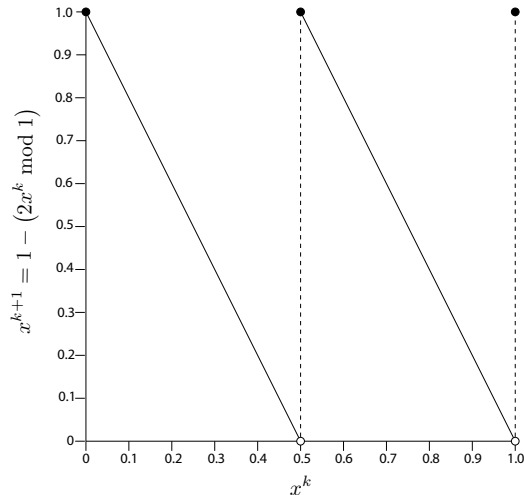
Figure 1: The dynamics map of a chaotic bucket brigade. If work is handed off at position x^k, then the next hand off will occur at position x^{k+1}.

$$x^{k+1} = 1 - \left(2x^k \bmod 1\right). \tag{2}$$

As illustrated in Figure 1, this is an *expanding* map; that is, it has slope of absolute value strictly greater than 1, where defined (it has discontinuities at $1/2$ and 1). The point $1/3$ is the unique point of balance, but it is a *repelling* fixed point, which means that the system spontaneously avoids balance. The point $2/3$ is another repelling fixed point.

This dynamics function is a reflection of the *Bernoulli map* (also known as the shift map, the doubling map, or the baker's map):

$$x^{k+1} = 2x^k \bmod 1.$$

Martelli [17] calls the Bernoulli map "one of the most quoted examples of chaotic behavior" and proves it is chaotic in the sense that there exists x^0 such that the orbit $O(x^0) = \{x^0, x^1, \ldots\}$ is both dense and unstable in $[0, 1]$. Devaney [11] agrees that the map is chaotic but uses a slightly different definition of chaos, which he summarizes as "unpredictability, indecomposability, and an element of regularity". More formally, $f : J \to J$ is chaotic if f satisfies:

- Sensitive dependence on initial conditions: There exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

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- Topological transitivity: For any pair of open sets $U, V \in J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

- Density of periodic points

(It is a common weakness of the literature on production and manufacturing to assert a weak form of sensitive dependence on initial conditions as sufficient to establish chaos.)

The reflected Bernoulli map (2) is chaotic under either definition (indeed, under any definition of chaos that we know). This is easy to see if one considers the values of the x^k to be represented by their binary expansions. Then each iteration of either map simply shifts digits leftward one position and then drops any integer part. The reflected Bernoulli map then complements each bit. A consequence is that the two-fold composition of the Bernoulli map is identical to the two-fold composition of the reflected Bernoulli map (except at $0, 1/4, 1/2, 3/4$, and 1).

It is worthwhile to examine in some detail the types of behavior of which our chaotic bucket brigade is capable. For example, one of the characteristics of chaotic behavior is that long term behavior depends sensitively on initial conditions. Both the Bernoulli map and its reflection are *expansive*, which means that the orbits of *all* nearby starting points eventually separate (Devaney [11]). The difference between an initial hand off at x^0 and $x^0 + \epsilon$ grows as $2^n \epsilon$.

Sensitivity to initial conditions becomes even more troubling when coupled with this observation: If, in a simulation, x^0 is given to an accuracy of n binary digits, then after n hand-offs all information will have evaporated. Thus, when the Convergence Condition fails to hold, it is impossible to predict the future state of our bucket brigade due to unavoidable inaccuracy in the measurement of initial conditions.

There are more reasons to distrust simulations of chaotic systems. If x^0 is the binary representation of a *rational* number, then the orbit $O(x^0)$ is periodic because the binary expansion repeats. Therefore, the starting points that lead to periodic orbits are dense in the unit interval. Furthermore, there are a countably infinite number of periodic orbits having arbitrarily large period, which may look chaotic but are not. Finally, if x^0 has a finite binary expansion — as it must in any finite precision machine — then the orbit converges to the period-1 cycle, $x = 1$. Yet it can be argued that all such behavior is spurious because it is restricted to a set of initial conditions of measure 0.

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If, on the other hand, x^0 is the binary representation of an *irrational* number, then the orbit $O(x^0)$ is non-periodic. Therefore there are uncountably many starting points (the irrational numbers) for which the resultant orbit of hand-offs never repeats. For example, in a bucket brigade based on the reflected Bernoulli map, if the first hand-off occurs at say $\pi/4$ then subsequent hand-offs will be determined by the digits of π, which are famously without apparent pattern.

Of particular import to assembly lines, there exist orbits that are dense. That is, there exists an irrational x^0 for which the resultant orbit approaches every number in $[0, 1]$ arbitrarily closely. This means that hand-offs can occur anywhere along the interval of work content.

The binary expansion of such a starting point x^0 can be constructed as follows. At the j-th step, append two copies of each of the 2^j sequences of j binary digits, so that the first digits of x^0 would be

$$0.0\ 0\ 1\ 1\ \ \ 00\ 00\ 01\ 01\ 10\ 10\ 11\ 11\ \ \ \ldots,$$

and thus any number will appear, in successively more accurate approximations, within the binary expansion of x^0. Consequently, within 2^{j+2} iterations, the reflected Bernoulli map will be within $1/2^j$ of that number.

In addition, orbits that do not map to either point of discontinuity, $x = 1/2$ or $x = 1$, have Lyapunov exponent greater than 0: Letting f represent the reflected Bernoulli map, the Lyapunov exponent is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left| f'\left(x^k\right) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln 2 = \ln 2 > 0.$$

Therefore there are an uncountably infinite number of orbits that are chaotic in the sense of Alligood et al. [1] (not asymptotically periodic and of Lyapunov exponent greater than 0).

Figure 2 shows the behavior of the bucket brigade based on the reflected Bernoulli map. The left graph shows the positions of the first hundred hand-offs when the workers are sequenced from most-slowed to least-slowed. The hand-offs quickly converge to the fixed point predicted by Theorem 1. The right graph plots thousands of hand-offs of the same workers sequenced in reverse order, so that hand offs occur chaotically. In fact, they seem to be rather uniformly distributed over the unit interval of work content. This is consistent with the fact that the natural distribution of the reflected Bernoulli map is the uniform
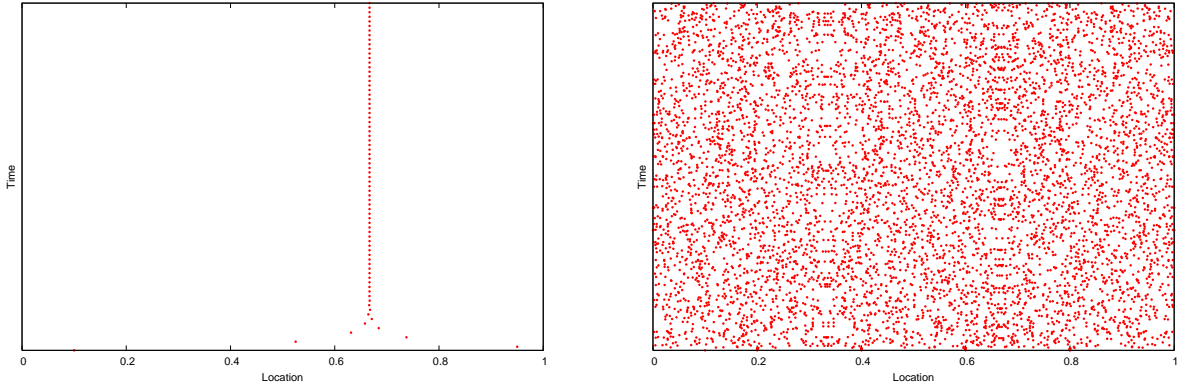
Figure 2: Locations of hand-offs under a stable bucket brigade (left) and a chaotic one (right). When the Convergence Condition holds (left), hand-off locations quickly converge to a single point and products are completed at regular intervals. When the Convergence Condition is violated (right), hand-offs appear to be distributed uniformly throughout the interval of work-content and completion times are erratic. (On the left, worker velocities are $v_1 = w_1 = 1$ and $v_2 = 1$, $w_2 = 1/3$. On the right the workers are swapped.)

distribution, which means that almost all starting positions generate orbits that visit every subinterval of work content with frequency proportional to the width of the subinterval (see, for example, Section 6.6 of Alligood et al. [1]).

4 Conclusions

Our generalized model of bucket brigades, though fully deterministic, is capable of chaotic behavior if the Convergence Condition fails to hold. This is the first example of provably chaotic behavior in a realistic model of discrete manufacturing.

Furthermore, there is strong evidence that, for bucket brigades for which the Convergence Condition fails, chaos is pervasive. Chaotic behavior is associated with the existence of an invariant distribution (Alligood et al. [1]). While not known to be equivalent to chaos, it is taken as strongly suggestive. Theorem 1 of Li and Yorke [15] (and restated in Theorem 6.15 of Alligood et al. [1]) shows that any function has an invariant measure if it maps a unit interval into itself, is piecewise smooth, and is piecewise expanding. The dynamics function for 2-worker bucket brigades always satisfies the first two criteria and satisfies the third when the Convergence Condition fails (Lim [16]).

Chaos has implications external to the assembly line, the most notable being that *a chaotic assembly line will appear to start and to complete products at random — even though the assembly line is completely deterministic.* The external costs of variability in start times

or completion times include increased safety stock and difficulty coördinating with upstream and downstream processes.

Chaos also has implications internal to the assembly line. Because hand-offs can occur anywhere, the first worker can be interrupted in the midst of any subtask, however small. This can slow learning because workers would not experience a stable assignment of work. (See Muñoz and Villalobos [18] and Armbruster et al. [3] for discussions of bucket brigades under models of learning.) Indeed, in our chaotic bucket brigade, no portion of the work-content remains the specialty of either worker. Each worker has to learn every subinterval of work content.

This also renders uneconomical the reëngineering of work to make hand-offs more efficient. In contrast, such improvements are possible when hand-off positions are known in advance, even if only approximately, as for traditional assembly lines, or for bucket brigades in which the workers have been indexed to satisfy the Convergence Condition. But it is hard to know where to improve the process when work is passed without pattern.

The possibility of chaotic behavior presents a new challenge for the management of manufacturing systems. A central goal of manufacturing systems control is the reduction of variability, such as results from machine breakdowns, vagaries in the positioning of work and in task execution, human inconsistency, and so on (Hopp and Spearman [13]). But in a chaotic bucket brigade, product starts and completions can appear irreducibly random even if every traditional source of variability has somehow been eliminated so that the system is purely deterministic. Such apparent randomness is inherent in the system and is resistant to the traditional tools of industrial engineering and operations management.

Furthermore, one must be extremely careful in simulating a system that may be chaotic. The bucket brigade based on the reflected Bernoulli map provides a vivid example, for almost all starting points of the workers lead to chaotic behavior; yet simulation on a finite precision machine must always result in periodic behavior.

Finally, chaotic behavior might be useful in some contexts. For example, military sentries patrolling a perimeter might avoid regular, easily predictable movements if they adopted different speeds of travel in each direction so that their meeting points would appear without obvious pattern.

Acknowledgements

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A Theorem 1: Existence of Fixed Point as Unique Point of Balance

Proof. At the balance point there is no passing or overtaking and each worker i repeats a simple loop for each item produced, retrieving work from worker $i - 1$ at point x_{i-1}^* and relinquishing his work to worker $i + 1$ at point x_i^* (where for convenience we define $x_0^* = 0$ and $x_n^* = 1$).

Since each worker must repeat his portion of work-content in a common cycle time for each item produced, the balance point is the unique solution to the $n - 1$ equations

$$(x_i^* - x_{i-1}^*)/\psi_i = (x_{i+1}^* - x_i^*)/\psi_{i+1} \quad \text{for each } i = 1, \ldots, n - 1. \tag{3}$$

$\square$

B Theorem 2: Local Convergence of n-Worker Bucket Brigades

Proof. Iteration t follows the hand-off points of the t-th disassembled item from the end of the line to the start. We let x_i^t be the hand-off point where worker i receives the t-th disassembled item from worker $i + 1$.

From the hand-off at x_i^t, worker $i+1$ moves forward to hand-off point x_{i+1}^{t+1} and then back to hand-off point x_i^{t+1}; and at the same time, worker i moves back to hand-off point x_{i-1}^t and then forward to hand-off point x_i^{t+1}. The following equates these movements of worker i and $i + 1$ from one iteration to the next:

$$\frac{x_i^t - x_{i-1}^t}{w_i} + \frac{x_i^{t+1} - x_{i-1}^t}{v_i} = \frac{x_{i+1}^{t+1} - x_i^{t+1}}{w_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^t}{v_{i+1}}, \text{ for each } i = 1, \ldots, n - 1, \tag{4}$$

where we define $x_0^t = 0$ and $x_n^t = 1$ for all t.

Rewriting yields:

$$x_i^{t+1} = \left(\frac{1/v_i + 1/w_i}{1/v_i + 1/w_{i+1}} \right) x_{i-1}^t - \left(\frac{1/v_{i+1} + 1/w_i}{1/v_i + 1/w_{i+1}} \right) x_i^t + \left(\frac{1/v_{i+1} + 1/w_{i+1}}{1/v_i + 1/w_{i+1}} \right) x_{i+1}^{t+1}. \tag{5}$$

Or we can write

$$x_i^{t+1} = (1 + \alpha_i)\gamma_i x_{i-1}^t - \alpha_i x_i^t + (1 + \alpha_i)(1 - \gamma_i)x_{i+1}^{t+1}, \tag{6}$$

where

$$\alpha_i = \frac{1/v_{i+1} + 1/w_i}{1/v_i + 1/w_{i+1}}, \tag{7}$$

and $0 < \alpha_i < 1$ corresponds to our Convergence Condition; and where

$$\gamma_i = \frac{1/v_i + 1/w_i}{1/v_i + 1/w_i + 1/v_{i+1} + 1/w_{i+1}}.$$

We show that the dynamics of the system can be described by an affine linear mapping,

$$y^{t+1} = Ay^t + b, \tag{8}$$

where $y^t = (x_1^t, x_2^t, \ldots, x_{n-2}^t, x_{n-1}^{t+1})^T$. (Our vector y^t indeed holds the last $n - 2$ disassembly hand-offs of item t, followed by the first disassembly hand-off of item $t + 1$—as we will see, this is to accommodate the vector b, which affects only the updating of x_{n-1}^{t+1}.)

We now factor the matrix $A = A_{n-1}A_1A_2...A_{n-2}$, where each matrix A_i updates x_i according to Equation (6), and we have

$$b = (0, 0, \ldots, 0, (1 + \alpha_{n-1})(1 - \gamma_{n-1}))^T.$$

In this way we first update x_{n-2}^t, then x_{n-3}^t, and so forth until x_1^t, and then finally x_{n-1}^{t+1} which utilizes the last component of b.

Each A_i is the identity matrix except for row i. Each $A_2, A_3, \ldots, A_{n-2}$ has three non-zero terms in row i that sum to one, with values $(1 + \alpha_i)\gamma_i$, $-\alpha_i$, and $(1 + \alpha_i)(1 - \gamma_i)$ in columns $i - 1$, i, and $i + 1$ respectively. For A_1 the first term $(1 + \alpha_1)\gamma_1 > 0$ is omitted from row 1, and thus the first row sum has absolute value less than one. And for A_{n-1} the last term $(1 + \alpha_{n-1})(1 - \gamma_{n-1}) > 0$ is omitted from row $n - 1$, so the last row sum has absolute value less than one.

For the full transition matrix A, all the eigenvalues have modulus less than one. In short, this follows since each $A_2, \ldots, A_{n-2}$ can be replaced with a stochastic matrix, while both A_1 and A_{n-1} can be replaced with a strictly sub-stochastic matrix. And since all states communicate, the system tends to the zero matrix. Thus the orbit $y^0, y^1, \ldots$ converges to the unique fixed point y^* of hand-off positions. (For dynamics of affine systems, see, for example, [17]). $\qquad\square$