11-2002

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SMU ECONOMICS & STATISTICS
WORKING PAPER SERIES

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November 2002

Paper No. 22-2002

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November 2002

Abstract: In this paper we propose an analytically corrected plug-in method for constructing confidence intervals of the conditional quantiles of a response variable with data transformation. The method can be applied to (i) a general conditional regression quantile, (ii) a general monotonic transformation, and (iii) a transformation model with heteroscedastic errors. Our results extend those in Yang (2002a), in which the median of a response variable under the Box-Cox transformation with homoscedastic errors was considered. A Monte Carlo experiment is conducted to compare the performance of the corrected plug-in method, the plug-in method and the delta method. The corrected plug-in method provides superior results over the other two methods.

Key Words: Analytical calibration; Box-Cox transformation; Heteroscedasticity; Plug-in quantile limits; Regression quantile.

JEL Classification: C1, C5

1 Introduction

Data transformation is a useful technique in econometric and financial modelling (see, e.g., Zarembka, 1968; White, 1972; Leech, 1975; Granger and Newbold, 1976; Carroll and Ruppert, 1991; Collins, 1991; Higgins and Bera, 1992; Buchinsky, 1995; Hentschel 1995; Kim and Hill, 1995; Chen 2002; Yang and Abeysinghe, 2002a, 2002b). In a linear regression model or a quantile regression set up, transformations may be applied to either the response or the regressors, or both. Transformations may also be applied to nonlinear regression models, time series models and conditional volatility models. In many cases, the purpose of applying data transformations is to induce normality in the observations or to give a more flexible functional relationship.

In many practical applications, data transformations involve unknown parameter values, which have to be estimated from the data. In a transformed regression set up, Yang (2002a) showed that for inference concerning the median of a response, the extra variability introduced by estimating the transformation cannot be ignored.[1] He derived an analytical adjustment for the confidence interval of the median obtained by plugging in the estimated transformation parameter for the unknown true value under the Box-Cox power transformation (Box and Cox, 1964). His Monte Carlo results show that there is significant improvement in the accuracy of the empirical contents of the adjusted confidence intervals.

In a broader context the general quantiles of a response at given values of the exogenous variables may be of interest. For example, labor economists may want to estimate the quantile wages at certain levels of education and experience (Buchinsky, 1994, 1995). Industrial-organization economists may not only be interested in the average firm size obtained from an ordinary regression with or without transformation, but also in the quantile intervals (Mata and Machado, 1996; Machado and Mata, 2000). In this paper we shall extend Yang's (2002a) result to a general conditional quantile.

Besides developing the results for a general quantile, there is also a need for extending

[1]Other works on the effects of estimating the transformation include, among others, Bickel and Doksum (1981), Carroll and Ruppert (1981), Carroll (1982), Doksum and Wong (1983), Hinkley and Runger (1984), Taylor (1989), Duan (1993) and Yang (1999).

the method to more general transformations beyond the well-known Box-Cox transformation. First, the Box-Cox power transformation works only for nonnegative observations. As economic data are not always positive, a more general transformation family that allows for both positive and negative observations is desirable.[2] Second, even if the data observations are nonnegative, the Box-Cox transformation is unable to transform the data to *exact* normality unless the transformation parameter is zero. Modified families of power transformation such as the dual-power transformation proposed by Yang (2002b) may br more appropriate. We shall develop correction methods for a general monotonic data transformation, and specialize the approach to the Box-Cox transformation and a dual-power transformation that circumvents the truncation problem of the Box-Cox transformation. Furthermore, although the use of transformation methods mostly aims at achieving normality, other goals such as obtaining a simple model structure with homoscedastic errors are also desirable. It is, however, often difficult to achieve these goals simultaneously. In particular, homoscedasticity is usually not obtainable in a single transformation. Thus, a transformation model which allows for heteroscedastic errors will be useful (see Carroll and Ruppert, 1988).

In this paper we develop a method to calculate the confidence limits of a general conditional quantile that applies to (i) a general monotonic transformation, and (ii) a transformation model allowing for heteroscedastic errors. Our main task is to derive analytical adjustments to the plug-in type of confidence limits to account for the transformation estimation and also the estimation of the weighting parameters in the case of heteroscedastic errors. This method is simple to use and is built upon standard asymptotic theory for pivotal statistics with an asymptotic normal distribution. It leads to confidence limits that have excellent finite-sample performance, and compares favorably against the delta method which has been widely used in the literature.

We first extend the results of Yang (2002a) to the case of a general quantile and a general monotonic transformation. A greater challenge arises when heteroscedasticity is involved in the transformation model. We follow Carroll and Ruppert (1991) to model the heteroscedasticity as

[2]Alternative transformations that can handle negative observations include Manly (1976), John and Draper (1980), Bickel and Doksum (1981), Burbidge et al. (1988) and Yeo and Johnson (2000).

a function of the explanatory variables and/or some other variables, indexed by some unknown parameters called the weighting parameters. We show that the plug-in type of confidence limits (i.e., with plug-in for both the transformation and the weight function) for a general quantile has to be calibrated with respect to the estimation of the transformation and weighting parameters. A simple analytical correction is given. Our Monte Carlo results show that the effect of estimating the weights is much smaller than the effect of estimating the transformation. The calibrated confidence limits preserves its simplicity even when extra weighting parameters are estimated and plugged in. Once the maximum likelihood estimates of the model parameters are obtained, the analytically calibrated confidence limits can be calculated in the same way as for the case of homoscedastic errors after dividing the estimated weights into the response as well as the regressors.

The results obtained in this paper may be applied to econometric studies such as wage distribution and firm size. They can also be used as an alternative to the commonly used method of quantile regression.[3] An advantage of the transformed regression over the quantile regression is that standard inference methods are available when the transformation is known, and can be analytically adjusted when the transformation is estimated. Our Monte Carlo results show that the analytical calibration-based confidence intervals perform better than those based on the delta method suggested by Carroll and Ruppert (1991).

The plan of this paper is as follows. Section 2 provides the corrected plug-in confidence interval for a regression quantile with a general monotonic transformation and homoscedastic errors. In Section 3 we extend the results to a transformation model with heteroscedastic errors. Section 4 presents the Monte Carlo results, and Section 5 concludes the paper.

2 Transformation model with homoscedastic errors

Let $\mathbf{Y}$ be an $n \times 1$ vector of original observations, $h(\mathbf{Y}, \lambda)$ a vector of transformed observations, and $\mathbf{X}$ an $n \times k$ matrix the columns of which contain the values of the explanatory variables

[3] Quantile regression was originally developed by Koenker and Bassett (1978). Recent applications and developments include Chamberlain (1994), Hahn (1995), Buchinsky (1994, 1995), Mata and Machado (1996), and Machado and Mata (2000).

$X_1, X_2, \ldots, X_k$. The usual transformed linear regression model is given by

$$h(\mathbf{Y}, \lambda) = \mathbf{X}\beta + \sigma\mathbf{e}, \tag{1}$$

where β is a $k \times 1$ vector of parameters, σ is the error standard deviation, $\mathbf{e}$ is an $n \times 1$ vector of independent and identically distributed normal variates with zero mean and unit variance, and $h(\cdot, \lambda)$ is a general monotonically increasing function. Given the model assumptions, the maximum likelihood estimators (MLE) of β, σ^2 and λ are given by

$$\hat{\lambda} = \arg\min_{\ell} \dot{J}^{-1}(\ell)\|\mathbf{M}h(\mathbf{Y}, \ell)\|, \tag{2}$$

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'h(\mathbf{Y}, \hat{\lambda}), \tag{3}$$

$$\hat{\sigma}^2 = \frac{1}{n}\|\mathbf{M}h(\mathbf{Y}, \hat{\lambda})\|^2, \tag{4}$$

where $\|\cdot\|$ is the Euclidian norm, $\mathbf{M} = \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\dot{J}(\ell)$ is the geometric mean of $\{h_y(Y_i, \ell) = \partial h(Y_i, \ell)/\partial Y_i, i = 1, \cdots, n\}$. Note that $\hat{\beta}$ and $\hat{\sigma}^2$ are analytic functions of $\hat{\lambda}$, and we shall denote these quantities as $\hat{\beta}(\hat{\lambda})$ and $\hat{\sigma}^2(\hat{\lambda})$, respectively. When λ is known, the MLE of β and σ are $\hat{\beta}(\lambda)$ and $\hat{\sigma}(\lambda)$, respectively.

2.1 The plug-in quantile limits

Consider the problem of predicting the pth quantile of Y_0 at a given observation x_0. We denote the quantile as y_p. As the transformation is monotonic, we have $h(y_p, \lambda) = x_0'\beta + \sigma z_p$, where z_p is the pth quantile of the standard normal variate. A natural predictor of $h(y_p, \lambda)$ is $x_0'\hat{\beta}(\lambda) + \hat{\sigma}^*(\lambda)z_p$, where $\hat{\sigma}^{*2}(\lambda) = \hat{\sigma}^2(\lambda)[n/(n-k)]$ is an unbiased estimator of σ^2. As $\mathrm{Var}[x_0'\hat{\beta}(\lambda)] = \sigma^2\kappa_{n0}^{-2}$, where $\kappa_{n0}^{-2} = x_0'(\mathbf{X}'\mathbf{X})^{-1}x_0$, it is natural to consider the following pivotal quantity for inference about y_p,

$$T_p(\lambda) = \frac{x_0'\hat{\beta}(\lambda) + \hat{\sigma}^*(\lambda)z_p - h(y_p, \lambda)}{\kappa_{n0}^{-1}\hat{\sigma}^*(\lambda)}. \tag{5}$$

Note that $h(y_p, \lambda) = x_0'\beta + \sigma z_p$, and $T_p(\lambda)$ can be rewritten as

$$\frac{\kappa_{n0}[x_0'\hat{\beta}(\lambda) - x_0'\beta]/\sigma - \kappa_{n0}z_p}{\hat{\sigma}^*(\lambda)/\sigma} + \kappa_{n0}z_p, \tag{6}$$

from which we can see that $T_p(\lambda) \sim t_{n-k}(-\kappa_{n0}z_p) + \kappa_{n0}z_p$, where $t_\nu(\delta)$ denotes a noncentral t random variable with ν degrees of freedom and noncentrality parameter δ. Thus, we obtain the mean μ_T and variance σ_T^2 of $T_p(\lambda)$ as

$$\mu_T = \kappa_{n0} z_p \left[1 - \left(\frac{n-k}{2}\right)^{1/2} \frac{\Gamma((n-k-1)/2)}{\Gamma((n-k)/2)}\right], \tag{7}$$

$$\sigma_T^2 = \frac{n-k}{n-k-2}(1+\kappa_{n0}^2 z_p^2) - \frac{n-k}{2}\kappa_{n0}^2 z_p^2 \left[\frac{\Gamma((n-k-1)/2)}{\Gamma((n-k)/2)}\right]^2. \tag{8}$$

Note that when $p = 0.5$, $\mu_T = 0$ and $\sigma_T = (n-k)/(n-k-2)$. For a fixed value of p, it can be shown that as $n \to \infty$, $\mu_T \to 0$ and σ_T converges to a number larger than 1. For extreme values of p, however, the convergence may be very slow. These points are important for further discussions in the next section when adjustment to the pivotal quantity has to be made when λ is estimated.

We now denote the αth quantile of $t_\nu(\delta)$ by $t_\nu^\alpha(\delta)$. With probability $1-\alpha$, the following inequality holds:

$$t_{n-k}^{\alpha/2}(-\kappa_{n0}z_p) + \kappa_{n0}z_p \le T_p(\lambda) \le t_{n-k}^{1-\alpha/2}(-\kappa_{n0}z_p) + \kappa_{n0}z_p, \tag{9}$$

from which we obtain a $100(1-\alpha)\%$ confidence interval for $h(y_p, \lambda)$ as

$$\left\{x_0'\hat{\beta}(\lambda) - t_{n-k}^{1-\alpha/2}(-\kappa_{n0}z_p)\frac{\hat{\sigma}^*(\lambda)}{\kappa_{n0}}, \quad x_0'\hat{\beta}(\lambda) - t_{n-k}^{\alpha/2}(-\kappa_{n0}z_p)\frac{\hat{\sigma}^*(\lambda)}{\kappa_{n0}}\right\}. \tag{10}$$

Applying inverse transformation to the lower limit $L(\lambda)$ and the upper limit $U(\lambda)$ in (10) gives the following $100(1-\alpha)\%$ confidence interval (CI) for y_p:

$$\left\{h^{-1}[L(\lambda), \lambda],\ h^{-1}[U(\lambda), \lambda]\right\}. \tag{11}$$

This interval is correct when exact normality is achieved by the transformation, and is asymptotically correct when approximate normality is achieved [4]. Thus, when λ is known, the interval can be recommended for practical applications and will be attractive to practitioners due to its simplicity. We shall call the confidence interval of y_p the quantile limits (QL).

In practical situations, λ is often unknown and has to be estimated using the same set of data. In this case, a popular approach is to 'plug' an estimate $\hat{\lambda}$ into (10) for the unknown λ

[4] For the case of the median, the interval is asymptotically correct if $\mathrm{E}[h(Y_0, \lambda)] = \mathrm{Median}[h(Y_0, \lambda)]$ for some h and λ. See Yang (2002a). The fact that expression (11) provides an exact confidence interval under normality may be a reason for the superior performance of the (corrected) plug-in method (as shown in the Monte Carlo results reported in Section 4) over the delta method in finite samples.

(Collins, 1991; Hahn and Meeker, 1991), so that the CI for y_p becomes

$$\left\{h^{-1}[L(\hat{\lambda}),\hat{\lambda}],\ h^{-1}[U(\hat{\lambda}),\hat{\lambda}]\right\}. \tag{12}$$

The interval (12), referred to as the plug-in quantile limits (PQL), is sometimes used without accounting for the extra variability introduced by $\hat{\lambda}$. In the next section we shall discuss the procedure to correct for the PQL when λ is unknown and is replaced by the MLE. Another popular approach to construct the QL is the delta method suggested by Carroll and Ruppert (1991), which provides asymptotically correct QL. In section 4, we compare the finite-sample performance of the corrected plug-in and the delta methods.

2.2 Asymptotics of the plug-in method

We define $T_p(\hat{\lambda})$ as $T_p(\lambda)$ with λ replaced by $\hat{\lambda}$, i.e.,

$$T_p(\hat{\lambda}) = \frac{\kappa_{n0}[x_0'\hat{\beta}(\hat{\lambda}) + \hat{\sigma}^*(\hat{\lambda})z_p - h(y_p,\hat{\lambda})]}{\hat{\sigma}^*(\hat{\lambda})}. \tag{13}$$

We use subscripts to denote the derivative or partial derivative of a function. For example, h_λ and $h_{\lambda\lambda}$ are, respectively, the first- and second-order partial derivatives of h with respect to λ. Let $\psi = (\beta', \sigma, \lambda)'$ and $\tau^2(\psi)$ be the asymptotic variance of $\sqrt{n}(\ \hat{\lambda} - \lambda)$. We state our first result as follows.

Theorem 1. *Assume that for some λ, the following are true: i) $\hat{\lambda} \xrightarrow{p} \lambda$ and $\sqrt{n}(\hat{\lambda} - \lambda)/\tau(\psi) \xrightarrow{D} N(0,1)$; ii) $\mathbf{X}'h_\lambda(\mathbf{Y},\lambda)/n$, $\mathbf{X}'h_{\lambda\lambda}(\mathbf{Y},\lambda)/n$, and $h'(\mathbf{Y},\lambda)\mathbf{M}h_\lambda(\mathbf{Y},\lambda)/n$ converge in probability; and iii) $\mathbf{X}'\mathbf{X}/n$ converges to a positive definite matrix. Then we have,*

$$T_p(\hat{\lambda}) = T_p(\lambda) + \frac{\sqrt{n}(\hat{\lambda} - \lambda)}{\tau(\psi)}c(\psi) + o_p(1). \tag{14}$$

Furthermore, if $T_p(\lambda)$ and $\hat{\lambda}$ are asymptotically independent,[5] *we have*

$$\mathrm{E}[T_p(\hat{\lambda})] = \mathrm{E}[T_p(\lambda)] + o(1), \tag{15}$$

$$\mathrm{Var}[T_p(\hat{\lambda})] = \mathrm{Var}[T_p(\lambda)] + c^2(\psi) + o(1), \tag{16}$$

[5] The assumption of asymptotic independence between $T_p(\lambda)$ and $\hat{\lambda}$ is not restrictive. In the special case of the Box-Cox transformation, it follows directly from Theorem 3.1 of Yang (1999). For the purpose of generality, asymptotic independence is stated as an assumption for the (single parameter) transformation.

where

$$c(\psi) = \lim_{n\to\infty} \frac{x_0'\mathrm{E}[\hat\beta_\lambda(\lambda)] + \frac{1}{2\sigma}z_p\mathrm{E}[\hat\sigma^*_\lambda(\lambda)] - h_\lambda(y_p,\lambda)}{\sqrt{n}\sigma/[\kappa_{n0}\tau(\psi)]}. \tag{17}$$

The proof of Theorem 1 is given in Appendix A. The constant $c(\psi)$ quantifies the effect of estimating λ on the pivotal quantity $T_p(\lambda)$, and hence on the PQL. As $c^2(\psi)$ is a nonnegative number, $T_p(\hat\lambda)$ has a limiting mean the same as that of $T_p(\lambda)$, but has a limiting variance larger than that of $T_p(\lambda)$. This indicates that the QL without accounting for the estimation of λ is too tight. The value of $c^2(\psi)$ generally depends on the value of ψ, as well as the values of z_p and x_0. When $p = 0.5$, $z_p = 0$ and the results reduce to those of Yang (2002a).

2.3 The analytically calibrated quantile limits

We now derive an analytical calibration of $T_p(\hat\lambda)$ so that the calibrated quantity has the same limiting distribution as $T_p(\lambda)$. This leads to an analytical adjustment of the PQL. From the results of Theorem 1, it is clear that the following adjusted pivotal quantity

$$T_p^*(\hat\lambda) = \frac{T_p(\hat\lambda) - C_m(\psi))}{C_s(\psi)}, \tag{18}$$

where $C_m(\psi) = \mu_T(1 - C_s(\psi))$ and $C_s(\psi) = \sqrt{1 + c^2(\psi)/\sigma_T^2}$, has the same asymptotic mean and variance as those of $T_p(\lambda)$. This leads immediately to the following corrected (calibrated) plug-in quantile limits (CPQL):

$$\left\{h^{-1}[L^*(\hat\lambda),\hat\lambda],\ h^{-1}[U^*(\hat\lambda),\hat\lambda]\right\}, \tag{19}$$

with the two adjusted end points before transformation given by

$$L^*(\hat\lambda) = x_0'\hat\beta(\hat\lambda) + \widehat{C_m^*(\psi)} - t_{n-k}^{1-\alpha/2}(-\kappa_{n0}z_p)\widehat{C_s(\psi)}\frac{\hat\sigma^*(\hat\lambda)}{\kappa_{n0}}, \tag{20}$$

$$U^*(\hat\lambda) = x_0'\hat\beta(\hat\lambda) + \widehat{C_m^*(\psi)} - t_{n-k}^{\alpha/2}(-\kappa_{n0}z_p)\widehat{C_s(\psi)}\frac{\hat\sigma^*(\hat\lambda)}{\kappa_{n0}}, \tag{21}$$

and $\widehat{C_m^*(\psi)} = \hat\sigma^*(\hat\lambda)[1 - \widehat{C_s(\psi)}][z_p + \mu_T/\kappa_{n0}]$. What remains now is a method to estimate the correction factor $c(\psi)$ so that the estimates $\widehat{C_s^*(\psi)}$ and $\widehat{C_m^*(\psi)}$ can be obtained. We examine below the special case of the Box-Cox transformation, followed by the case of a general monotonic transformation.

3 The Box-Cox power transformation

For the Box-Cox power transformation (Box and Cox, 1964), we have

$$h(Y, \lambda) = \begin{cases} (Y^\lambda - 1)/\lambda, & \text{if } \lambda \neq 0, \\ \log Y, & \text{if } \lambda = 0. \end{cases} \tag{22}$$

In this case an explicit expression for $c(\psi)$ can be derived, so that an estimate of it, $c(\hat{\psi})$, can be easily obtained. We denote the elementwise multiplication operator (Hadamard product) of two vectors by $\odot$. Also, as a convention, functions applied to a vector, b say, are carried out elementwise, e.g., $\log b = \{\log b_i\}_{n\times 1}$. The following theorem provides the required correction for the Box-Cox transformation.

Theorem 2. *Let $h(., \lambda)$ be the Box-Cox power transformation. Assume i) the first six moments of e_i are the same as those of a standard normal random variable, ii) $T_p(\lambda)$ and $\hat{\lambda}$ are independent, and iii) $\max|\theta_i|$ is small.*[6] *Then we have, for large n, when $\lambda \neq 0$,*

$$c(\psi) \approx \frac{x_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[(1_n + \lambda\eta)\odot\phi + \frac{1}{2}\lambda\sigma\theta] - (1+\lambda\eta_0)\phi_0 + \lambda^2 a(z_p)}{\lambda\sigma k_{n0}^{-1}\left(\| \mathbf{M}(\theta^{-1}\odot\phi + \frac{1}{2}\theta) \|^2 + 2 \| \phi - \bar{\phi} \|^2 + \frac{3}{2} \| \theta \|^2\right)^{1/2}}, \tag{23}$$

and when $\lambda = 0$,

$$c(\psi) \approx \frac{x_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\eta^2 + \sigma^2 1_n) - \eta_0^2 + 2a(z_p)}{k_{n0}^{-1}(\| \mathbf{M}\eta^2 \|^2 + 8\sigma^2 \| \eta - \bar{\eta} \|^2 + 6n\sigma^4)^{1/2}}, \tag{24}$$

where $\eta = \mathbf{X}'\beta$, $\phi = \log(1+\lambda\eta)$, $\theta = \lambda\sigma(1_n + \lambda\eta)^{-1}$, $\bar{\eta} = 1_n'\eta/n$, $\bar{\phi} = 1_n'\phi/n$, $1_n = \{1\}_{n\times 1}$, $\eta_0 = x_0'\beta$, $\theta_0 = \lambda\sigma/(1+\lambda\eta_0)$, $\phi_0 = \log(1+\lambda\eta_0)$, and

$$a(z_p) = \begin{cases} \sigma z_p \left(\frac{1}{n-k}\sum_{i=1}^n m_{ii}(\phi_i - \frac{3}{2}\theta_i^2) - \phi_0 - \frac{1}{2}\theta_0 z_p + \frac{1}{2}\theta_0^2 z_p^2\right)\lambda^{-1}, & \text{if } \lambda \neq 0, \\ \sigma z_p \left(\frac{1}{n-k}\sum_{i=1}^n m_{ii}\eta_i - \eta_0 - \frac{1}{2}\sigma z_p\right), & \text{if } \lambda = 0, \end{cases} \tag{25}$$

with m_{ii} being the ith *diagonal element of* $\mathbf{M}$.

The proof of Theorem 2 can be found in Appendix A. Note that when $p = 0.5$, $a(z_p) = 0$ and Theorem 2 above reduces to Theorem 2 in Yang (2002a). Using the Box-Cox power transformation, the implementation of the CPQL is straightforward. Once $\hat{\lambda}$ is obtained, it is easy to calculate $c^2(\hat{\psi})$ and the rest is similar to the calculation of the usual regression QL.

[6] *This is to ensure that the truncation effect in the Box-Cox transformation model is negligible. See Yang (1999) for a detailed discussion.*

Note that $\hat{\lambda}$ defined in (2) can be conveniently calculated by solving the following equation for λ:[7]

$$-n\frac{h'(\mathbf{Y},\lambda)\mathbf{M}h_\lambda(\mathbf{Y},\lambda)}{h'(\mathbf{Y},\lambda)\mathbf{M}h(\mathbf{Y},\lambda)} + \sum_{i=1}^{n} \log Y_i = 0. \tag{26}$$

Theorem 2 provides an estimate of $c^2(\psi)$ in a very convenient way. More importantly, it characterizes the factors governing the magnitude of the variance correction factor. In particular, $c^2(\psi)$ depends on (i) the value of x_0, (ii) the value of z_p through the term $a(z_p)$, (iii) the model structure through the $\mathbf{M}$ term, (iv) the mean spread through the term $\|\phi - \bar{\phi}\|^2$, and (v) the residual standard deviation. Among these factors, the distance from x_0 to the center of the design is most important. The calculations given in Table 1 show that $c(\psi)$ is a concave function of x_0, and that when x_0 is far from the design region, $c^2(\psi)$ may be very large no matter what value p takes. Therefore, there may be a large effect on the distribution of the pivotal quantity used for constructing the QL due to the estimation of the transformation. As a result, the QL without accounting for the transformation estimation can be extremely liberal. This is in contrast to Carroll and Ruppert (1981), who claimed that while there is a cost in estimating the transformation on the inference concerning the median, the cost is generally not severe.

3.1 General transformations

In general, negative observations may arise in a transformed regression model and a general transformation that allows for both positive and negative observations would be desirable. Candidates for such transformations were suggested by Manly (1976), John and Draper (1980), Bickel and Doksum (1981), Burbidge et al. (1988) and Yeo and Johnson (2000). Theorem 1 incorporates such transformations. However, even if observations are nonnegative, the Box-Cox power transformation is, technically speaking, unable to transform the observations to exact normality unless $\lambda = 0$. This is the well-known truncation problem associated with the Box-Cox power transformation. Yang (2002b) proposed a modified family of power transformation, called the *dual-power transformation*, that can be used to circumvent the truncation problem.

[7]A Fortran subroutine for doing so is available from the authors on request. For more general applications, a SAS/IML program is also available. The program calculates the quantity in equation (2) over a grid of values and searches for the minimum.

The transformation is given by (for $y > 0$)

$$h(y, \lambda) = \begin{cases} (y^{\lambda} - y^{-\lambda})/2\lambda, & \lambda > 0, \\ \log y, & \lambda = 0. \end{cases} \tag{27}$$

Similar results as in Theorem 2 may be difficult (if not impossible) to obtain for a general transformation. Here we adopt an alternative method to estimate the correction factor $c(\psi)$ that works for any monotonic transformation. From the way $c(\psi)$ is defined, it is natural to introduce the following estimate:

$$\widehat{c(\psi)} = \frac{x_0'\hat{\beta}_{\lambda}(\hat{\lambda}) + z_p\hat{\sigma}_{\lambda}^*(\hat{\lambda}) - h_{\lambda}(\hat{y}_p, \hat{\lambda})}{\hat{\sigma}^*(\hat{\lambda})/[\kappa_{n0}J^{\lambda\lambda}(\hat{\psi})^{1/2}]}, \tag{28}$$

where $J^{\lambda\lambda}(\psi)$ is the last diagonal element of $J^{-1}(\psi)$. It is easy to see that $\widehat{c(\psi)}$ is a consistent estimate of $c(\psi)$. Thus, the adjusted QL in equations (20) and (21) can be estimated based on the estimate $\widehat{c(\psi)}$. The MLE $\hat{\lambda}$ now solves

$$-n\frac{h'(\mathbf{Y}, \lambda)Qh_{\lambda}(\mathbf{Y}, \lambda)}{h'(\mathbf{Y}, \lambda)Qh(\mathbf{Y}, \lambda)} + \sum_{i=1}^{n}\frac{h_{y\lambda}(Y_i, \lambda)}{h_y(Y_i, \lambda)} = 0, \tag{29}$$

where $h_{y\lambda}(\cdot, \cdot)$ is the second order partial derivative of h with respect to Y and λ.

4 Transformation model with heteroscedastic errors

We now consider a variation of model (1) in which the error standard deviation is no longer constant but varies with a set of 'weights'. Thus, we assume

$$h(Y_i, \lambda) = x_i'\beta + \sigma\omega(v_i, \gamma)e_i, \qquad i = 1, \cdots, n, \tag{30}$$

where $\omega(v_i, \gamma) \equiv \omega_i(\gamma)$ is the weight function, v_i is a vector of observations on a set of variables, called the weighting variables, and γ is a vector of weighting parameters. The weighting variables may include (some of) the regressors and/or other variables and the weighting parameter-vector may include (some of) the regression coefficients and/or other parameters.[8] A special weight function which is very popular in the econometrics literature is the multiplicative heteroscedastic function suggested by Harvey (1976), for which

$$\omega^2(v_i, \gamma) = \exp(v_i'\gamma). \tag{31}$$

[8]See Carroll and Ruppert (1988, 1991) for the general specification of the weight function.

Another important special case is $\omega^2(v_i, \gamma) = X_i^\gamma$. We now define $\Omega(\gamma) = \text{diag}\ \{\omega_1^2(\gamma), \cdots, \omega_n^2(\gamma)\}$. If the weights are given, then equation (30) can be converted into the form of equation (1) by premultiplying $\Omega^{-\frac{1}{2}}(\gamma)$ on each side of equation (30), where $\Omega^{-\frac{1}{2}}(\gamma) = \text{diag}\ \{\omega_1^{-1}(\gamma), \cdots, \omega_n^{-1}(\gamma)\}$. Thus, the results given in Section 2 apply after replacing h by $\Omega^{-\frac{1}{2}}(\gamma)h$, $\mathbf{X}$ by $\Omega^{-\frac{1}{2}}(\gamma)\mathbf{X}$, x_0 by $\omega_0^{-1}(\gamma)x_0$, and $h(y_p, \lambda)$ by $\omega_0^{-1}(\gamma)h(y_p, \lambda)$.

When the weights are unknown, the maximum likelihood method can be used to estimate jointly the parameters. The log likelihood function is

$$\ell(\beta, \sigma^2, \gamma, \lambda) \propto -\frac{n}{2}\log\sigma^2 - \sum_{i=1}^{n}\log\omega_i(\gamma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[\frac{h(y_i,\lambda) - x_i'\beta}{\omega_i(\gamma)}\right]^2 + \sum_{i=1}^{n}\log h_y(y_i, \lambda). \quad (32)$$

For given γ and λ, the MLE of β and σ are

$$\hat{\beta}(\gamma, \lambda) = [\mathbf{X}'\Omega^{-1}(\gamma)\mathbf{X}]^{-1}\mathbf{X}'\Omega^{-1}(\gamma)h(\mathbf{Y}, \lambda), \quad (33)$$

$$\hat{\sigma}^2(\gamma, \lambda) = \frac{1}{n}[h(\mathbf{Y}, \lambda) - \mathbf{X}\hat{\beta}(\gamma, \lambda)]'\Omega^{-1}(\gamma)[h(\mathbf{Y}, \lambda) - \mathbf{X}\hat{\beta}(\gamma, \lambda)]. \quad (34)$$

Substituting $\hat{\beta}(\gamma, \lambda)$ and $\hat{\sigma}(\gamma, \lambda)$ into the log likelihood function gives the profile likelihood function of γ and λ, i.e.,

$$\ell_p(\gamma, \lambda) = \ell[\hat{\beta}(\gamma, \lambda), \hat{\sigma}^2(\gamma, \lambda), \gamma, \lambda] \propto -\frac{n}{2}\log\sum_{i=1}^{n}\left[\dot{\omega}(\gamma)s_i(\gamma, \lambda)/\dot{J}(\lambda)\right]^2, \quad (35)$$

where $s_i(\gamma, \lambda) = [h(y_i, \lambda) - x_i'\hat{\beta}(\gamma, \lambda)]/\omega_i(\gamma)$, and $\dot{\omega}(\gamma)$ and $\dot{J}(\lambda)$ are the geometric means of $\omega_i(\gamma)$ and $J_i(\lambda) = h_y(Y_i, \lambda)$, respectively. The unrestricted MLE can thus be written as

$$(\hat{\gamma}', \hat{\lambda})' = \arg\min_{(\gamma,\lambda)} \left[\dot{\omega}(\gamma)/\dot{J}(\lambda)\right]^2 \sum_{i=1}^{n} s_i^2(\gamma, \lambda), \quad (36)$$

$$\hat{\beta}(\hat{\gamma}, \hat{\lambda}) = [\mathbf{X}'\Omega^{-1}(\hat{\gamma})\mathbf{X}]^{-1}\mathbf{X}\Omega^{-1}(\hat{\gamma})h(\mathbf{Y}, \hat{\lambda}), \quad (37)$$

$$\hat{\sigma}^2(\hat{\gamma}, \hat{\lambda}) = \frac{1}{n}h'(\mathbf{Y}, \hat{\lambda})\mathbf{M}(\hat{\gamma})\Omega^{-1}(\hat{\gamma})h(\mathbf{Y}, \lambda), \quad (38)$$

where $\mathbf{M}(\gamma) = I_n - \mathbf{X}[\mathbf{X}'\Omega^{-1}(\gamma)\mathbf{X}]^{-1}\mathbf{X}'\Omega^{-1}(\gamma)$.

Similar to the case of homoscedastic errors, to predict the pth quantile y_p of the response Y_0 at values x_0 and v_0 of the regressors and weighting variables, respectively, we start with the pivotal quantity for the case of known λ and γ, i.e.,

$$T_p(\gamma, \lambda) = \frac{x_0'\hat{\beta}(\gamma, \lambda) + \hat{\sigma}^*(\gamma, \lambda)\omega(v_o, \gamma)z_p - h(y_p, \lambda)}{\kappa_{n0}^{-1}(\gamma)\hat{\sigma}^*(\gamma, \lambda)}, \quad (39)$$

where $\kappa_{n0}^{-2}(\gamma) = x_0'[\mathbf{X}\Omega^{-1}(\gamma)\mathbf{X}]^{-1}x_0$. It is easy to see that $T_p(\gamma, \lambda) \sim t_{n-k}[-\delta(\gamma)] + \delta(\gamma)$, where $\delta(\gamma) = \kappa_{n0}(\gamma)\omega_0(\gamma)z_p$. Thus, the confidence limits for y_p can be constructed in the same way as in equations (10) and (11).

When both γ and λ are unknown and are replaced by their MLE, the pivotal quantity becomes

$$T_p(\hat{\gamma}, \hat{\lambda}) = \frac{x_0'\hat{\beta}(\hat{\gamma}, \hat{\lambda}) + \hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})\omega(v_o, \hat{\gamma})z_p - h(y_p, \hat{\lambda})}{\kappa_{n0}^{-1}(\hat{\gamma})\hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})}. \tag{40}$$

The PQL can be constructed in the same way as in equation (12). The issue now is the adjustment of the PQL to account for the estimation of the weighting parameter as well as the transformation parameter. The following theorem provides a convenient way to perform the adjustment. We now denote $\psi = \{\beta', \sigma^2, \gamma', \lambda\}'$.

Theorem 3. *Under the specification of the model in equation (30), we assume further that the following are true: i) $\hat{\lambda} \xrightarrow{p} \lambda$ and $\hat{\gamma} \xrightarrow{p} \gamma$, ii) $\sqrt{n}(\hat{\lambda} - \lambda)/\tau(\psi) \xrightarrow{D} N(0, 1)$, iii) $\mathbf{X}'\Omega^{-1}(\gamma)h_\lambda(\mathbf{Y}, \lambda)/n$, $\mathbf{X}'\Omega^{-1}(\gamma)h_{\lambda\lambda}(\mathbf{Y}, \lambda)/n$, and $h'(\mathbf{Y}, \lambda)\mathbf{M}(\gamma)\Omega^{-1}(\gamma)h_\lambda(\mathbf{Y}, \lambda)/n$ converge in probability, and iv) $\mathbf{X}'\Omega^{-1}(\gamma)\mathbf{X}/n$ converges to a positive definite matrix. Then, we have,*

$$T_p(\hat{\gamma}, \hat{\lambda}) = T_p(\gamma, \lambda) + B_1\sqrt{n}(\hat{\gamma} - \gamma) + B_2\sqrt{n}(\hat{\lambda} - \lambda) + o_p(1), \tag{41}$$

where

$$B_1 = \lim_{n\to\infty} \frac{x_0'\mathrm{E}[\hat{\beta}_\gamma(\gamma, \lambda)] + z_p\omega_0(\gamma)\mathrm{E}[\hat{\sigma}_\gamma^*(\gamma, \lambda)] + \hat{\sigma}^*(\gamma, \lambda)\omega_{0\gamma}(\gamma)z_p}{\sqrt{n}\sigma\kappa_{n0}^{-1}(\gamma)}, \tag{42}$$

$$B_2 = \lim_{n\to\infty} \frac{x_0'\mathrm{E}[\hat{\beta}_\lambda(\gamma, \lambda)] + z_p\omega_0(\gamma)\mathrm{E}[\hat{\sigma}_\lambda^*(\gamma, \lambda)] - h_\lambda(y_p, \lambda)}{\sqrt{n}\sigma\kappa_{n0}^{-1}(\gamma)}. \tag{43}$$

Furthermore, if $T_p(\gamma, \lambda)$ is asymptotically independent of $\hat{\gamma}$ and $\hat{\lambda}$, we have

$$\mathrm{E}[T_p(\hat{\gamma}, \hat{\lambda})] = \mathrm{E}[T_p(\gamma, \lambda)] + o(1), \tag{44}$$

$$\mathrm{Var}[T_p(\hat{\gamma}, \hat{\lambda})] = \mathrm{Var}[T_p(\gamma, \lambda)] + c^2(\psi) + o(1) \tag{45}$$

where $c^2(\psi) = B'\Sigma B$, $B' = \{B_1', B_2\}$, and Σ is the asymptotic variance of $\sqrt{n}\{\hat{\gamma}', \hat{\lambda}\}'$.

Theorem 3 is similar to Theorem 1, the proof of which is given in Appendix A. Similar to the results in Section 2, a consistent estimator of $c(\psi)$ is given by

$$\widehat{c(\psi)} = (\hat{B}'\hat{\Sigma}\hat{B})^{1/2}, \tag{46}$$

with $\hat{\Sigma}/n = J^{22}(\hat{\psi})$, the $(\gamma', \lambda)'$ diagonal block of $J^{-1}(\psi)$ evaluated at $\hat{\psi}$, and

$$\hat{B}_1 = \frac{x_0'\hat{\beta}_\gamma(\hat{\gamma}, \hat{\lambda}) + z_p\omega_0(\hat{\gamma})\hat{\sigma}^*_\gamma(\hat{\gamma}, \hat{\lambda}) + \hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})\omega_{0\gamma}(\hat{\gamma})z_p}{\sqrt{n}\hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})\kappa_{n0}^{-1}(\hat{\gamma})}, \tag{47}$$

$$\hat{B}_2 = \frac{x_0'\hat{\beta}_\lambda(\hat{\gamma}, \hat{\lambda}) + z_p\omega_0(\hat{\gamma})\hat{\sigma}^*_\lambda(\hat{\gamma}, \hat{\lambda}) - h_\lambda(\hat{y}_p, \hat{\lambda})}{\sqrt{n}\hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})\kappa_{n0}^{-1}(\hat{\gamma})}. \tag{48}$$

With Theorem 3, it is clear that the pivotal quantity $T_p(\hat{\gamma}, \hat{\lambda})$ should be corrected as

$$T_P^*(\hat{\gamma}, \hat{\lambda}) = \frac{T_p(\hat{\gamma}, \hat{\lambda}) - C_m(\psi)}{C_s(\psi)}, \tag{49}$$

where $C_m(\psi) = \mu_T(\gamma)(1 - C_s(\psi))$ and $C_s(\psi) = \sqrt{1 + c^2(\psi)/\sigma_T^2(\gamma)}$, with $\mu_T(\gamma)$ and $\sigma_T^2(\gamma)$ being the mean and variance of $T_p(\gamma, \lambda)$, which have similar expressions as μ_T and σ_T^2 with a different noncentrality parameter. The implementation of the CPQL is again very simple. All the partial derivatives have simple analytical expressions except $\hat{\beta}_\gamma(\gamma, \lambda)$ which could be complicated in a model containing more than one explanatory variable. In this case, numerical differentiation may be used. The final analytical expression of the CPQL takes the form

$$\left\{h^{-1}[L^*(\hat{\gamma}, \hat{\lambda}), \hat{\lambda}],\ h^{-1}[U^*(\hat{\gamma}, \hat{\lambda}), \hat{\lambda}]\right\}, \tag{50}$$

with the two adjusted end points before transformation being

$$L^*(\hat{\gamma}, \hat{\lambda}) = x_0'\hat{\beta}(\hat{\gamma}, \hat{\lambda}) + \widehat{C_m^*(\psi)} - t_{n-k}^{1-\alpha/2}[-\delta(\hat{\gamma})]\widehat{C_s(\psi)}\frac{\hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})}{\kappa_{n0}(\hat{\gamma})}, \tag{51}$$

$$U^*(\hat{\gamma}, \hat{\lambda}) = x_0'\hat{\beta}(\hat{\gamma}, \hat{\lambda}) + \widehat{C_m^*(\psi)} - t_{n-k}^{\alpha/2}[-\delta(\hat{\gamma})]\widehat{C_s(\psi)}\frac{\hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})}{\kappa_{n0}(\hat{\gamma})}, \tag{52}$$

and $\widehat{C_m^*(\psi)} = \hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})[1 - \widehat{C_s(\psi)}][\omega_0(\hat{\gamma})z_p + \mu_T(\hat{\gamma})/\kappa_{n0}(\hat{\gamma})]$.

5 Some Monte Carlo results

The results given in the last section provide asymptotically correct confidence limits for the regression quantiles using the CPQL. However, it is not clear how the CPQL performs when

n is not large. Also, it would be interesting to compare the performance of the CPQL method against other methods used in the literature, such as the delta method. In this section we report some Monte Carlo results on the performance of the CPQL, the PQL and the delta method in small samples.

We consider cases of homoscedastic and heteroscedastic errors. For homoscedastic errors we consider the following model as the data generation process (DGP):

$$\text{Model 1:} \quad h(Y_i, \lambda) = \beta_0 + \beta_1 X_i + \sigma e_i, \qquad i = 1, \cdots, n. \tag{53}$$

For heteroscedastic errors we use the following DGP:

$$\text{Model 2:} \quad h(Y_i, \lambda) = \beta_0 + \beta_1 X_i + \sigma \exp(\gamma X_i) e_i, \qquad i = 1, \cdots, n. \tag{54}$$

In each of the above models, X_i are fixed values of the regressor that are uniformly spaced in $[0, X_m]$. For Model 1, we consider both the Box-Cox transformation and the dual-power transformation. For Model 2, to simplify the numerical computations, only the Box-Cox transformation is considered.

The Monte Carlo experiment can be described as follows. For a given set of parameter values, we generate n standard normal random numbers (e_i). Using these random numbers we calculate the values of Y_i based on the assumed DGP and estimate the model parameters. We then compute the 95% quantile limits, and repeat this process a number times (10,000 for Model 1 and 5,000 for Model 2). The proportion of the quantile limits that cover the true quantile provides a Monte Carlo estimate of the coverage probability of the quantile limits. All simulations are performed using a FORTRAN 90 compiler which calls the IMSL subroutines for numerical optimization. To study the effect of the adjustment, we report the results corresponding to the PQL. Furthermore, the coverage probabilities of the commonly used delta-method QL are also estimated.

Table 1 provides a summary of the values of $c(\psi)$ when the DGP is Model 1 and the Box-Cox transformation is used. The values of X (with $n = 60$) used in this computation are uniformly spaced in [0, 100]. The results suggest the importance of making corrections to the plug-in method. We can see that the required variance correction $c(\psi)^2$ is generally quite large, especially when x_0 is far from the in-sample values.

Tables 2 to 8 report the empirical relative frequency of coverage of the true quantiles using the three methods. Tables 2 to 6 summarize the results for Model 1 (homoscedastic errors), and Tables 7 and 8 present the results for Model 2 (heteroscedastic errors). It is obvious from the results that the PQL method has the poorest performance. The empirical coverage probabilities do not converge to the nominal value of 95% when the sample size increases, as expected from the asymptotic theory. The CPQL method has the best performance in all cases, and the empirical coverage probabilities are very close to the nominal value even for a small sample size such as 15. In what follows we summarize some regularities that seem to have emerged from the experiment. Our discussion will focus on the following aspects: (i) the effect of the nonlinearity parameter λ, (ii) the sample size n, (iii) the design region and the x_0 value, (iv) out-of-sample forecast, and (v) the effect of heteroscedasticity.

Degree of nonlinearity. For transformations indexed by a power parameter λ, the smaller the λ, the more nonlinear the transformation is. The Monte Carlo results show that the λ value has a dramatic effect on the performance of the delta method, especially when p is large. For example, for Model 1 under the Box-Cox power transformation (Tables 2 to 4), the coverage probabilities for the nominal 95% CI of the 99th quantile at $x_0 = 0$ and $n = 15$ are 0.8149, 0.8542 and 0.9067 for $\lambda = 0.01$, 0.1 and 0.5, respectively. For Model 1 under the dual-power transformation (Tables 5 and 6), the coverage probabilities for the same quantile are 0.7848 and 0.8980 for $\lambda = 0.1$ and 0.5, respectively. For Model 2 under the Box-Cox power transformation (Tables 7 and 8), the coverage probabilities for the nominal 95% CI of the 99th quantile at $x_0 = 0$ and $n = 30$ are 0.8716 and 0.9079 for $\lambda = 0.1$ and 0.25, respectively. Generally speaking, when the degree of nonlinearity is high, the empirical coverage of the delta method can be far below the nominal level. In contrast, the empirical coverage probabilities of the CPQL method are very close to the nominal 95% value.

Sample size. The empirical coverage of the delta method improves as the sample size increases. However, in many situations the empirical coverage is still significantly below its nominal level even when the sample size is as large as 60 (see, e.g., the case of $p = 0.99$, $x_0 = 0$, $n = 60$). In contrast, the empirical coverage of the CPQL method are close to the nominal value even when the sample size is only 30, and this appears to be true for all cases reported.

Design region and the x_0 value. Simulation results (not reported here) show that the range of values of X_i also effects the performance of the delta method. A larger range seems to have adverse effects on the performance of the delta method. In addition, the value of x_0 matters greatly to the magnitude of the correction and to the performance of the delta method.

Out-of-sample forecast. Figure 1 shows the performance of the three methods for Model 1 with $n = 60$. Various ranges of X_i values are examined. The purpose is to examine the performance of the three methods when x_0 deviates from the in-sample values. The vertical dotted lines represent the maximum value X_m of X_i in each case. For the delta method, the empirical coverage probability drops significantly when x_0 moves away from the design region.

Heteroscedasticity. Tables 7 and 8 show that estimating the weighting parameter does not have significant effect on the performance of the CPQL method. In contrast, the PQL method may perform very poorly. Once again, the performance of the delta method may be quite poor if λ is small. In all cases, the performance of the delta method is dominated by the CPQL method.

We should add that another undesirable property of the delta method is that it could end up with a negative lower quantile limit. This phenomenon, however, cannot happen for the CPQL method under the dual-power transformation or other transformations that are free of the truncation problem. For the case of the Box-Cox power transformation, the limits before the inverse transformation L^* or U^* could end up being negative, depending on whether λ is positive or negative. When this happens either the lower limit of the CPQL should be set to zero or the upper limit of the CPQL should be set to ∞. However, the chance of this happening is negligible if the truncation probability is small.[9]

6 Conclusions

The Box-Cox transformation is a popular technique for analyzing skewed data found in many economic applications. When the transformation is known, the usual inference theories can be applied and simple inverse transformations can be used for the original response. However, the transformation parameter is often unknown and has to be estimated from the data. A

[9] See Yang (1999) for the details.

common practice in this case is to use the so-called 'plug-in' method, i.e., plugging the estimated unknown parameters into the confidence-interval formula. This practice ignores the effect of estimating the transformation and/or the weighting. This paper shows that there is a serious consequence if these effects are ignored.

We show that the plug-in method can be modified to obtain an analytically calibrated confidence interval which is very accurate even when the sample size is small. The gains of introducing the corrected plug-in method are significant as the Monte Carlo results show that its performance dominates uniformly that of the delta method.

There are important implications of the corrected plug-in method. It may be applied to solve problems of a similar nature. For example, it can be applied to a model where both the response and regressors are transformed. Generally speaking, the methodology can be applied to any situation where standard inference theory is available when certain parameters (such as the transformation parameters) are known, but not available when these parameters are unknown and have to be estimated from the same set of data.

APPENDIX A: Proof of the Theorems

Proof of Theorem 1. First-order Taylor expansions of $\hat{\beta}(\hat{\lambda})$, $h(y_p, \hat{\lambda})$ and $\hat{\sigma}^*(\hat{\lambda})$ give

$$\begin{aligned}
\hat{\beta}(\hat{\lambda}) &= \hat{\beta}(\lambda) + \hat{\beta}_\lambda(\lambda)(\hat{\lambda} - \lambda) + O_p(n^{-1}), \\
h(y_p, \hat{\lambda}) &= h(y_p, \lambda) + h_\lambda(y_p, \lambda)(\hat{\lambda} - \lambda) + O_p(n^{-1}), \\
\hat{\sigma}^*(\hat{\lambda}) &= \hat{\sigma}^*(\lambda) + \hat{\sigma}^*_\lambda(\lambda)(\hat{\lambda} - \lambda) + O_p(n^{-1}).
\end{aligned}$$

Combining the above gives

$$\begin{aligned}
& x_0'\hat{\beta}(\hat{\lambda}) + \hat{\sigma}^*(\hat{\lambda})z_p - h(y_p, \hat{\lambda}) \\
= \; & x_0'\hat{\beta}(\lambda) + \hat{\sigma}^*(\lambda)z_p - h(y_p, \lambda) + (\hat{\lambda} - \lambda)[x_0'\hat{\beta}_\lambda(\lambda) + \hat{\sigma}^*_\lambda(\lambda)z_p - h_\lambda(y_p, \lambda)] + O_p(n^{-1}).
\end{aligned}$$

It is easy to see that $1/\hat{\sigma}^*(\hat{\lambda}) = 1/\hat{\sigma}^*(\lambda) + O_p(n^{-1/2}) = 1/\sigma + O_p(n^{-1/2})$. Summarizing the above, we obtain the first result of Theorem 1. The rest of the proof is straightforward.

Proof of Theorem 2. Yang (1999) obtained an accurate approximation to $\tau^2(\psi)$ as follows:

$$\tau^2(\psi) \approx \begin{cases} \frac{n\lambda^2}{\|Q(\theta^{-1}\odot\phi+\frac{1}{2}\theta)\|^2+2\|\phi-\bar{\phi}\|^2+\frac{3}{2}\|\theta\|^2}, & \text{if } \lambda \neq 0, \\ \frac{4n\sigma^2}{\|Q\eta^2\|^2+8\sigma^2\|\eta-\bar{\eta}\|^2+6n\sigma^4}, & \text{if } \lambda = 0. \end{cases}$$

For the power transformation, we have $h_\lambda(y, \lambda) = [y^\lambda \log y - h(y, \lambda)]/\lambda$ for $\lambda \neq 0$, and $(\log y)^2/2$ for $\lambda = 0$. The latter leads directly to equation (24) of Theorem 2 for the case of $\lambda = 0$. When $\lambda \neq 0$, using the following approximation:

$$\lambda \log Y_i \approx \phi_i + \theta_i e_i - \frac{1}{2}\theta_i^2 e_i^2,$$

we obtain the approximations to $\mathrm{E}[h_\lambda(Y_i, \lambda)]$ and $h_\lambda(y_p, \lambda)$, which lead to the result in equation (23) of Theorem 2 after some algebra.

Proof of Theorem 3. First-order Taylor expansion of the numerator of $T_p(\hat{\gamma}, \hat{\lambda})$ leads to

$$\begin{aligned}
& x_0'\hat{\beta}(\hat{\gamma}, \hat{\lambda}) + \hat{\sigma}^*(\hat{\gamma}, \hat{\lambda})\omega_0(\hat{\gamma})z_p - h(y_p, \hat{\lambda}) \\
= \; & x_0'\hat{\beta}(\gamma, \lambda) + \hat{\sigma}^*(\gamma, \lambda)\omega_0(\gamma)z_p - h(y_p, \lambda) \\
& +[x_0'\hat{\beta}_\lambda(\gamma, \lambda) + \hat{\sigma}^*_\lambda(\gamma, \lambda)\omega_0(\gamma)z_p - h_\lambda(y_p, \lambda)](\hat{\lambda} - \lambda) \\
& +[x_0'\hat{\beta}_\gamma(\gamma, \lambda) + \hat{\sigma}^*_\gamma(\gamma, \lambda)\omega_0(\gamma)z_p + \hat{\sigma}^*(\gamma, \lambda)\omega_{0\gamma}(\gamma)z_p](\hat{\lambda} - \lambda) + O_p(n^{-1}).
\end{aligned}$$

As $1/\hat{\sigma}^*(\hat{\gamma},\hat{\lambda}) = 1/\hat{\sigma}^*(\gamma,\lambda) + O_p(n^{-1/2}) = 1/\sigma + O_p(n^{-1/2})$, the above leads to the first part of the theorem. The rest of the proof is straightforward.

APPENDIX B: Delta Method Quantile Limits

Carroll and Ruppert (1991) recommended using the delta method or the likelihood ratio test method to construct confidence intervals for y_p. Implementation of the delta method is straightforward. Implementation of the likelihood ratio test method requires constrained maximization. To apply the delta method, let $\hat{\psi}$ be the MLE of ψ where ψ may or may not contain γ. Write $\hat{\beta}(\hat{\lambda})$ or $\hat{\beta}(\hat{\gamma},\hat{\lambda})$ as $\hat{\beta}$, and $\hat{\sigma}(\hat{\lambda})$ or $\hat{\sigma}(\hat{\gamma},\hat{\lambda})$ as $\hat{\sigma}$. Suppose that the distribution of $\hat{\psi}$ is asymptotically normal with mean ψ and variance matrix $I^{-1}(\psi)$. Then the distribution of the MLE of y_p, $\hat{y}_p = h^{-1}[(x_0'\hat{\beta} + \hat{\sigma}\omega_0(\hat{\gamma})z_p), \hat{\lambda}] \equiv g(\hat{\psi})$, is asymptotically normal with mean $g(\psi)$ and variance $g_\psi'(\psi)I^{-1}(\psi)g_\psi(\psi)$. The variance can be consistently estimated by $g_\psi'(\hat{\psi})J^{-1}(\hat{\psi})g_\psi(\hat{\psi})$ with $J(\hat{\psi})$ being the observed information matrix evaluated at $\hat{\psi}$. Thus, a $100(1-\alpha)\%$ large-sample confidence interval for y_p is given by

$$\widehat{y_p} \pm z_{\alpha/2}\sqrt{g_\psi'(\hat{\psi})J^{-1}(\hat{\psi})g_\psi(\hat{\psi})}.$$

Implementation of the delta-method QL requires the quantile function and its partial derivatives, as well as the observed information matrix. We provide the details of the delta method below.

The quantile function and partial derivatives. The quantile function for the general transformation model with heteroscedastic errors is defined as

$$g(\psi) = h^{-1}[x_0'\beta + \sigma\omega_0(\gamma)z_p, \lambda].$$

For the Box-Cox transformation model with heteroscedastic errors, $g(\psi) = (1+\lambda\mu_0)^{1/\lambda}$, where $\mu_0 = x_0'\beta + \sigma\omega_0(\gamma)z_p$. Hence, the partial derivatives of $g(\psi)$ are

$$\begin{aligned} g_\beta(\psi) &= (1+\lambda\mu_0)^{1/\lambda-1}x_0 \\ g_{\sigma^2}(\psi) &= \frac{1}{2\sigma}(1+\lambda\mu_0)^{1/\lambda-1}\omega_0(\gamma)z_p \end{aligned}$$

$$g_\gamma(\psi) = (1+\lambda\mu_0)^{1/\lambda-1}\sigma z_p \omega_{0\gamma}(\gamma)$$

$$g_\lambda(\psi) = g(\psi)\left[\mu_0\lambda^{-1}(1+\lambda\mu_0)^{-1} - \lambda^{-2}\log(1+\lambda\mu_0)\right].$$

Setting $\omega_0(\gamma) = 1$ in the above expressions and removing the $g_\gamma(\psi)$ element gives the partial derivatives of the quantile function for the Box-Cox transformation model with homoscedastic errors.

For the dual-power transformation model with heteroscedastic errors, we have $g(\psi) = [\lambda\mu_0 + (1+\lambda^2\mu_0^2)^{1/2}]^{1/\lambda}$, and the partial derivatives are

$$\begin{aligned}
g_\beta(\psi) &= g(\psi)x_0(1+\lambda^2\mu_0^2)^{-1/2}\\
g_{\sigma^2}(\psi) &= \frac{1}{2\sigma}g(\psi)\omega_0(\gamma)z_p(1+\lambda^2\mu_0^2)^{-1/2}\\
g_\gamma(\psi) &= g(\psi)\sigma\omega_{0\gamma}(\gamma)z_p(1+\lambda^2\mu_0^2)^{-1/2}\\
g_\lambda(\psi) &= g(\psi)\left[\mu_0\lambda^{-1}(1+\lambda^2\mu_0^2)^{-1/2} - \lambda^{-2}\log(\lambda\mu_0 + (1+\lambda^2\mu_0^2)^{1/2})\right].
\end{aligned}$$

Again, setting $\omega_0(\gamma) = 1$ and removing the $g_\gamma(\psi)$ gives the partial derivatives of the quantile function in the dual-power transformation model with homoscedastic errors.

The observed information matrix. For models with general transformation and weighting function, the elements of the observed information matrix are given by:

$$\begin{aligned}
J_{\beta\beta} &= \frac{1}{\sigma^2}\sum_{i=1}^n \frac{x_i x_i'}{\omega_i^2(\gamma)}\\
J_{\sigma^2\sigma^2} &= \frac{1}{\sigma^6}\sum_{i=1}^n \frac{[h(Y_i,\lambda)-x_i'\beta]^2}{\omega_i^2(\gamma)} - \frac{n}{\sigma^4}\\
J_{\gamma\gamma} &= \sum_{i=1}^n\left(\frac{\omega_{i\gamma\gamma}^2(\gamma)}{\omega_i(\gamma)} - \frac{\omega_{i\gamma}^2(\gamma)}{\omega_i^2(\gamma)}\right) - \frac{1}{\sigma^2}\sum_{i=1}^n [h(Y_i,\lambda)-x_i'\beta]^2\left(\frac{\omega_{i\gamma\gamma}(\gamma)}{\omega_i^3(\gamma)} - \frac{3\omega_{i\gamma}^2(\gamma)}{\omega_i^4(\gamma)}\right)\\
J_{\lambda\lambda} &= \frac{1}{\sigma^2}\sum_{i=1}^n \frac{h_\lambda^2(Y_i,\lambda) + [h(Y_i,\lambda)-x_i'\beta]h_{\lambda\lambda}(Y_i,\lambda)}{\omega_i^2(\gamma)} - \sum_{i=1}^n\left(\frac{h_{y\lambda\lambda}(Y_i,\lambda)}{h_y(Y_i,\lambda)} - \frac{h_{y\lambda}^2(Y_i,\lambda)}{h_y^2(Y_i,\lambda)}\right)\\
J_{\beta\sigma^2} &= \frac{1}{\sigma^4}\sum_{i=1}^n \frac{[h(Y_i,\lambda)-x_i'\beta]x_i}{\omega_i^2(\gamma)}\\
J_{\beta\gamma} &= \frac{2}{\sigma^2}\sum_{i=1}^n \frac{[h(Y_i,\lambda)-x_i'\beta]x_i\omega_{i\gamma}(\gamma)}{\omega_i^3(\gamma)}\\
J_{\beta\lambda} &= \frac{1}{\sigma^2}\sum_{i=1}^n \frac{h_\lambda(Y_i,\lambda)x_i}{\omega_i^2(\gamma)}
\end{aligned}$$

$$
\begin{aligned}
J_{\sigma^2\gamma} &= \frac{1}{\sigma^4}\sum_{i=1}^{n}\frac{[h(Y_i,\lambda)-x_i'\beta]^2\omega_{i\gamma}(\gamma)}{\omega_i^3(\gamma)} \\
J_{\sigma^2\lambda} &= -\frac{1}{\sigma^4}\sum_{i=1}^{n}\frac{[h(Y_i,\lambda)-x_i'\beta]h_\lambda(Y_i,\lambda)}{\omega_i^2(\gamma)} \\
J_{\gamma\lambda} &= -\frac{2}{\sigma^2}\sum_{i=1}^{n}\frac{[h(Y_i,\lambda)-x_i'\beta]h_\lambda(Y_i,\lambda)\omega_{i\gamma}(\gamma)}{\omega_i^3(\gamma)}.
\end{aligned}
$$

Now, for the Box-Cox transformation, we have $h_y(y,\lambda) = y^{\lambda-1}$, $h_{y\lambda}(y,\lambda) = y^{\lambda-1}\log y$, $h_{y\lambda\lambda}(y,\lambda) = y^{\lambda-1}(\log y)^2$, and (for $y > 0$)

$$
h_\lambda(y,\lambda) = \begin{cases} \frac{1}{\lambda}[1+\lambda h(y,\lambda)]\log y - \frac{1}{\lambda}h(y,\lambda), & \lambda \neq 0, \\ \frac{1}{2}(\log y)^2, & \lambda = 0, \end{cases}
$$

$$
h_{\lambda\lambda}(y,\lambda) = \begin{cases} h_\lambda(y,\lambda)(\log y - \frac{1}{\lambda}) + \frac{1}{\lambda^2}[h(y,\lambda) - \log y], & \lambda \neq 0, \\ \frac{1}{3}(\log y)^3, & \lambda = 0. \end{cases}
$$

For the dual-power transformation, we have

$$
\begin{aligned}
h_\lambda(y,\lambda) &= \frac{1}{2\lambda}(y^\lambda + y^{-\lambda})\log y - \frac{1}{\lambda}h(y,\lambda) \\
h_{\lambda\lambda}(y,\lambda) &= h(y,\lambda)(\frac{2}{\lambda^2} + (\log y)^2) - \frac{1}{\lambda^2}(y^\lambda + y^{-\lambda})\log y \\
h_y(y,\lambda) &= \frac{1}{2}[y^{\lambda-1} + y^{-\lambda-1}], \\
h_{y\lambda}(y,\lambda) &= \frac{1}{2}(y^{\lambda-1} - y^{-\lambda-1})\log y \\
h_{y\lambda\lambda}(y,\lambda) &= \frac{1}{2}(y^{\lambda-1} + y^{-\lambda-1})(\log y)^2.
\end{aligned}
$$

Setting $\omega_i(\gamma) \equiv 1$ gives the information matrix for models with homoscedastic errors. Furthermore, for the Box-Cox transformation model with homoscedastic errors, the observed information matrix reduces to

$$
\begin{aligned}
J_{\beta\beta} &= \frac{1}{\sigma^2}\mathbf{X}'\mathbf{X} \\
J_{\sigma^2\sigma^2} &= -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6}[h(\mathbf{Y},\lambda) - \mathbf{X}\beta]'[h(\mathbf{Y},\lambda) - \mathbf{X}\beta] \\
J_{\lambda\lambda} &= \frac{1}{\sigma^2}[h_\lambda'(\mathbf{Y},\lambda)h_\lambda(\mathbf{Y},\lambda) + (h(\mathbf{Y},\lambda) - \mathbf{X}\beta)'h_{\lambda\lambda}(\mathbf{Y},\lambda)] \\
J_{\beta\sigma^2} &= \frac{1}{\sigma^4}\mathbf{X}'[h(\mathbf{Y},\lambda) - \mathbf{X}\beta] \\
J_{\beta\lambda} &= -\frac{1}{\sigma^2}\mathbf{X}'h_\lambda(\mathbf{Y},\lambda) \\
J_{\sigma^2\lambda} &= -\frac{1}{\sigma^4}[h(\mathbf{Y},\lambda) - \mathbf{X}\beta]'h_\lambda(\mathbf{Y},\lambda).
\end{aligned}
$$

For the dual-power transformation model with homoscedastic errors, the information matrix takes identical forms as the above except that the quantity $4\sum_{i=1}^{n}\log^2 Y_i/(Y_i^{\lambda}+Y_i^{-\lambda})^2$ has to be subtracted from $J_{\lambda\lambda}$.

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Table 1. Summary of $c(\psi)$ Values: Model 1, Box-Cox Transformation, $n = 60$, $\psi' = (5, 1, 1, 0.1)$, $X_i \in [0, 100]$

x_0	κ_{n0}	$c(\psi)$						
		$p = .01$	$p = .05$	$p = .25$	$p = .50$	$p = .75$	$p = .95$	$p = .99$
0	3.8247	-1.6936	-1.6133	-1.5043	-1.4319	-1.3618	-1.2645	-1.1982
10	4.4711	-0.8058	-0.7796	-0.7010	-0.6427	-0.5874	-0.5128	-0.4775
20	5.2936	-0.1097	-0.0717	-0.0136	0.0181	0.0568	0.0938	0.1172
30	6.2808	0.4867	0.5102	0.5477	0.5721	0.5887	0.6117	0.6436
40	7.2520	0.9338	0.9498	0.9595	0.9639	0.9717	0.9770	0.9794
50	7.7427	1.1070	1.0972	1.0846	1.0729	1.0593	1.0438	1.0250
60	7.3826	0.9085	0.8830	0.8535	0.8267	0.8094	0.7691	0.7374
70	6.4528	0.4947	0.4681	0.4119	0.3862	0.3467	0.2940	0.2551
80	5.4485	0.0212	-0.0201	-0.0677	-0.0959	-0.1333	-0.1816	-0.2107
90	4.5953	-0.3964	-0.4493	-0.4951	-0.5316	-0.5726	-0.6085	-0.6939
100	3.9215	-0.8147	-0.8378	-0.8795	-0.9125	-0.9533	-1.0005	-1.0364
120	2.9831	-1.4352	-1.4691	-1.5178	-1.5518	-1.5861	-1.6357	-1.6709
140	2.3855	-1.9793	-2.0046	-2.0445	-2.0838	-2.1087	-2.1556	-2.1827
160	1.9800	-2.3971	-2.4233	-2.4691	-2.5241	-2.5595	-2.5584	-2.5932
180	1.6892	-2.8127	-2.8291	-2.8347	-2.8595	-2.9067	-2.9962	-2.9795

Table 2. Empirical Coverage Probabilities for 95% QL: Model 1, Box-Cox Transformation, $\psi' = (5, 1, 1, 0.01)$, $X_i \in [0, 100]$

		$n = 15$			$n = 30$			$n = 60$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9183	0.7876	0.8882	0.9335	0.7970	0.9073	0.9426	0.8162	0.9329
	10	0.9283	0.8835	0.9275	0.9420	0.8996	0.9377	0.9489	0.9108	0.9476
	20	0.9382	0.9342	0.9385	0.9444	0.9418	0.9431	0.9504	0.9483	0.9499
	40	0.9374	0.9231	0.9363	0.9480	0.9304	0.9477	0.9441	0.9191	0.9452
	50	0.9352	0.9094	0.9355	0.9380	0.9076	0.9403	0.9479	0.9124	0.9483
	60	0.9352	0.9092	0.9330	0.9421	0.9086	0.9415	0.9474	0.9221	0.9477
	80	0.9454	0.9443	0.9346	0.9483	0.9480	0.9440	0.9504	0.9502	0.9485
	100	0.9312	0.9012	0.9359	0.9414	0.8965	0.9449	0.9465	0.8946	0.9479
0.05	0	0.9172	0.7596	0.8833	0.9333	0.7809	0.9111	0.9444	0.7933	0.9324
	10	0.9321	0.8786	0.9237	0.9431	0.8922	0.9388	0.9477	0.9029	0.9452
	20	0.9393	0.9339	0.9348	0.9427	0.9401	0.9393	0.9501	0.9478	0.9475
	40	0.9360	0.9101	0.9344	0.9408	0.9050	0.9397	0.9445	0.9062	0.9433
	50	0.9233	0.8762	0.9222	0.9371	0.8789	0.9360	0.9475	0.8890	0.9482
	60	0.9310	0.8917	0.9282	0.9402	0.8923	0.9394	0.9477	0.8998	0.9473
	80	0.9425	0.9411	0.9300	0.9504	0.9502	0.9462	0.9466	0.9465	0.9432
	100	0.9269	0.8794	0.9259	0.9414	0.8797	0.9404	0.9476	0.8783	0.9472
0.25	0	0.9259	0.7499	0.8773	0.9411	0.7657	0.9073	0.9402	0.7682	0.9213
	10	0.9298	0.8591	0.9068	0.9437	0.8853	0.9318	0.9465	0.8973	0.9439
	20	0.9344	0.9284	0.9167	0.9469	0.9445	0.9368	0.9494	0.9487	0.9473
	40	0.9328	0.8824	0.9225	0.9405	0.8654	0.9353	0.9461	0.8635	0.9446
	50	0.9223	0.8300	0.9172	0.9391	0.8322	0.9361	0.9434	0.8358	0.9428
	60	0.9275	0.8474	0.9193	0.9403	0.8588	0.9364	0.9490	0.8662	0.9449
	80	0.9433	0.9423	0.9219	0.9457	0.9456	0.9362	0.9443	0.9442	0.9408
	100	0.9303	0.8641	0.9199	0.9440	0.8549	0.9371	0.9452	0.8449	0.9431
0.50	0	0.9242	0.7509	0.8601	0.9408	0.7653	0.8954	0.9516	0.7852	0.9285
	10	0.9308	0.8642	0.8891	0.9438	0.8898	0.9249	0.9479	0.8968	0.9367
	20	0.9416	0.9366	0.9108	0.9393	0.9363	0.9236	0.9500	0.9497	0.9445
	40	0.9349	0.8610	0.9119	0.9428	0.8497	0.9289	0.9453	0.8409	0.9385
	50	0.9274	0.8034	0.9078	0.9389	0.8100	0.9277	0.9505	0.8087	0.9451
	60	0.9289	0.8309	0.9105	0.9407	0.8461	0.9327	0.9451	0.8478	0.9406
	80	0.9355	0.9351	0.9103	0.9440	0.9440	0.9333	0.9479	0.9477	0.9380
	100	0.9336	0.8525	0.9109	0.9457	0.8433	0.9323	0.9482	0.8265	0.9380
0.75	0	0.9196	0.7597	0.8488	0.9367	0.7870	0.8934	0.9403	0.7989	0.9191
	10	0.9340	0.8801	0.8810	0.9419	0.8975	0.9125	0.9437	0.9078	0.9325
	20	0.9396	0.9361	0.8914	0.9456	0.9452	0.9212	0.9485	0.9485	0.9324
	40	0.9275	0.8662	0.8894	0.9401	0.8600	0.9235	0.9462	0.8574	0.9321
	50	0.9240	0.8356	0.8975	0.9409	0.8342	0.9288	0.9478	0.8354	0.9375
	60	0.9258	0.8531	0.8990	0.9390	0.8622	0.9229	0.9462	0.8716	0.9386
	80	0.9424	0.9422	0.9005	0.9475	0.9474	0.9279	0.9506	0.9495	0.9427
	100	0.9267	0.8425	0.8948	0.9400	0.8356	0.9211	0.9463	0.8271	0.9391
0.95	0	0.9206	0.8055	0.8278	0.9401	0.8378	0.8785	0.9474	0.8421	0.9146
	10	0.9273	0.8930	0.8369	0.9433	0.9186	0.8988	0.9417	0.9180	0.9209
	20	0.9384	0.9379	0.8575	0.9465	0.9464	0.9020	0.9451	0.9450	0.9251
	40	0.9299	0.8967	0.8635	0.9437	0.8979	0.9037	0.9474	0.8981	0.9276
	50	0.9223	0.8748	0.8650	0.9393	0.8875	0.9089	0.9485	0.8938	0.9266
	60	0.9296	0.8951	0.8653	0.9410	0.9049	0.9098	0.9423	0.9044	0.9224
	80	0.9446	0.9446	0.8703	0.9418	0.9405	0.9041	0.9501	0.9492	0.9327
	100	0.9288	0.8631	0.8773	0.9422	0.8503	0.9062	0.9440	0.8434	0.9256
0.99	0	0.9193	0.8323	0.8149	0.9395	0.8601	0.8642	0.9455	0.8697	0.9068
	10	0.9325	0.9122	0.8314	0.9446	0.9278	0.8833	0.9482	0.9334	0.9161
	20	0.9413	0.9407	0.8431	0.9442	0.9442	0.8893	0.9451	0.9450	0.9135
	40	0.9366	0.9165	0.8451	0.9406	0.9134	0.8910	0.9477	0.9193	0.9205
	50	0.9311	0.9064	0.8458	0.9403	0.9075	0.8921	0.9475	0.9107	0.9201
	60	0.9359	0.9141	0.8507	0.9438	0.9212	0.8990	0.9445	0.9239	0.9214
	80	0.9423	0.9421	0.8544	0.9458	0.9454	0.8960	0.9486	0.9478	0.9254
	100	0.9306	0.8763	0.8612	0.9399	0.8637	0.9001	0.9443	0.8581	0.9234

Table 3. Empirical Coverage Probabilities for 95% QL: Model 1, Box-Cox Transformation, $\psi' = (5, 1, 1, 0.1)$, $X_i \in [0, 100]$

		$n=15$			$n=30$			$n=60$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9127	0.7214	0.8925	0.9330	0.7474	0.9158	0.9470	0.7679	0.9384
	10	0.9257	0.8753	0.9273	0.9434	0.8974	0.9424	0.9481	0.9067	0.9468
	20	0.9405	0.9370	0.9211	0.9467	0.9457	0.9363	0.9499	0.9495	0.9450
	40	0.9366	0.9193	0.9126	0.9459	0.9218	0.9343	0.9461	0.9165	0.9395
	50	0.9313	0.9049	0.9072	0.9424	0.9079	0.9324	0.9468	0.9099	0.9413
	60	0.9328	0.9090	0.9048	0.9457	0.9202	0.9261	0.9431	0.9149	0.9356
	80	0.9413	0.9404	0.9045	0.9459	0.9459	0.9309	0.9470	0.9470	0.9393
	100	0.9324	0.9061	0.9148	0.9444	0.9109	0.9413	0.9471	0.9052	0.9450
0.05	0	0.9168	0.7058	0.8931	0.9379	0.7351	0.9207	0.9435	0.7525	0.9361
	10	0.9316	0.8660	0.9294	0.9411	0.8893	0.9384	0.9420	0.8945	0.9407
	20	0.9395	0.9362	0.9214	0.9476	0.9464	0.9373	0.9497	0.9494	0.9440
	40	0.9309	0.9010	0.9109	0.9401	0.8992	0.9321	0.9478	0.9018	0.9446
	50	0.9316	0.8884	0.9124	0.9430	0.8888	0.9341	0.9464	0.8885	0.9419
	60	0.9290	0.8938	0.9103	0.9417	0.9030	0.9310	0.9508	0.9107	0.9442
	80	0.9417	0.9412	0.9107	0.9437	0.9437	0.9292	0.9529	0.9529	0.9455
	100	0.9312	0.8907	0.9208	0.9415	0.8867	0.9361	0.9455	0.8855	0.9431
0.25	0	0.9266	0.6888	0.8911	0.9405	0.7182	0.9206	0.9444	0.7307	0.9347
	10	0.9287	0.8576	0.9177	0.9404	0.8778	0.9348	0.9474	0.8941	0.9450
	20	0.9422	0.9396	0.9221	0.9475	0.9472	0.9374	0.9493	0.9492	0.9451
	40	0.9281	0.8591	0.9126	0.9392	0.8593	0.9341	0.9468	0.8613	0.9441
	50	0.9230	0.8228	0.9121	0.9407	0.8386	0.9369	0.9475	0.8427	0.9449
	60	0.9257	0.8533	0.9083	0.9438	0.8682	0.9380	0.9508	0.8801	0.9455
	80	0.9392	0.9389	0.9154	0.9498	0.9497	0.9402	0.9529	0.9525	0.9459
	100	0.9358	0.8758	0.9241	0.9416	0.8653	0.9395	0.9472	0.8623	0.9457
0.50	0	0.9250	0.6931	0.8849	0.9374	0.7295	0.9216	0.9444	0.7250	0.9308
	10	0.9311	0.8569	0.9121	0.9409	0.8886	0.9337	0.9486	0.8982	0.9427
	20	0.9376	0.9358	0.9115	0.9492	0.9491	0.9379	0.9441	0.9441	0.9404
	40	0.9338	0.8534	0.9186	0.9409	0.8437	0.9356	0.9458	0.8414	0.9426
	50	0.9321	0.8186	0.9207	0.9373	0.8137	0.9328	0.9447	0.8185	0.9423
	60	0.9353	0.8443	0.9225	0.9386	0.8572	0.9306	0.9465	0.8707	0.9439
	80	0.9349	0.9349	0.9100	0.9461	0.9459	0.9360	0.9474	0.9464	0.9431
	100	0.9292	0.8583	0.9151	0.9407	0.8527	0.9347	0.9462	0.8519	0.9444
0.75	0	0.9162	0.7129	0.8756	0.9404	0.7503	0.9156	0.9437	0.7651	0.9349
	10	0.9292	0.8754	0.9069	0.9426	0.9006	0.9340	0.9449	0.9090	0.9420
	20	0.9390	0.9382	0.9095	0.9450	0.9450	0.9280	0.9477	0.9475	0.9402
	40	0.9267	0.8591	0.9127	0.9430	0.8577	0.9330	0.9485	0.8576	0.9444
	50	0.9243	0.8302	0.9065	0.9396	0.8417	0.9349	0.9415	0.8409	0.9423
	60	0.9283	0.8624	0.9126	0.9442	0.8765	0.9338	0.9467	0.8849	0.9441
	80	0.9415	0.9415	0.9153	0.9451	0.9447	0.9288	0.9483	0.9462	0.9415
	100	0.9297	0.8630	0.9111	0.9397	0.8562	0.9357	0.9409	0.8430	0.9377
0.95	0	0.9174	0.7722	0.8664	0.9353	0.8002	0.9068	0.9440	0.8147	0.9281
	10	0.9368	0.9035	0.8888	0.9394	0.9161	0.9184	0.9471	0.9285	0.9334
	20	0.9478	0.9475	0.8985	0.9470	0.9466	0.9233	0.9444	0.9440	0.9336
	40	0.9322	0.8988	0.9026	0.9415	0.8964	0.9260	0.9488	0.9036	0.9420
	50	0.9304	0.8903	0.9083	0.9429	0.8976	0.9288	0.9495	0.8964	0.9406
	60	0.9336	0.9026	0.9008	0.9422	0.9112	0.9246	0.9464	0.9172	0.9385
	80	0.9380	0.9378	0.8926	0.9476	0.9465	0.9271	0.9476	0.9460	0.9381
	100	0.9308	0.8715	0.9059	0.9403	0.8644	0.9276	0.9448	0.8592	0.9394
0.99	0	0.9155	0.8030	0.8524	0.9310	0.8305	0.9008	0.9391	0.8445	0.9238
	10	0.9326	0.9102	0.8781	0.9466	0.9289	0.9160	0.9463	0.9345	0.9333
	20	0.9431	0.9429	0.8825	0.9459	0.9457	0.9194	0.9481	0.9469	0.9328
	40	0.9342	0.9124	0.8903	0.9473	0.9206	0.9260	0.9461	0.9150	0.9364
	50	0.9348	0.9079	0.8922	0.9416	0.9105	0.9246	0.9442	0.9096	0.9341
	60	0.9355	0.9177	0.8929	0.9428	0.9271	0.9233	0.9510	0.9332	0.9379
	80	0.9404	0.9402	0.8926	0.9464	0.9450	0.9226	0.9506	0.9493	0.9408
	100	0.9323	0.8854	0.9012	0.9406	0.8758	0.9249	0.9452	0.8725	0.9397

Table 4. Empirical Coverage Probabilities for 95% QL: Model 1, Box-Cox Transformation, $\psi' = (5, 1, 0.5, 0.5)$, $X_i \in [0, 100]$

		$n = 15$			$n = 30$			$n = 60$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9155	0.7270	0.9194	0.9357	0.7493	0.9371	0.9464	0.7749	0.9463
	10	0.9304	0.8881	0.9126	0.9402	0.9049	0.9318	0.9489	0.9157	0.9429
	20	0.9394	0.9382	0.9051	0.9466	0.9466	0.9273	0.9482	0.9482	0.9403
	40	0.9366	0.9183	0.9013	0.9447	0.9190	0.9315	0.9486	0.9203	0.9399
	50	0.9328	0.9048	0.8966	0.9431	0.9107	0.9269	0.9473	0.9110	0.9381
	60	0.9335	0.9128	0.8990	0.9456	0.9235	0.9255	0.9470	0.9224	0.9376
	80	0.9426	0.9425	0.9016	0.9501	0.9501	0.9298	0.9495	0.9495	0.9393
	100	0.9311	0.9057	0.9124	0.9415	0.9025	0.9356	0.9446	0.8974	0.9407
0.05	0	0.9156	0.6951	0.9187	0.9328	0.7271	0.9313	0.9439	0.7345	0.9462
	10	0.9310	0.8763	0.9180	0.9440	0.9006	0.9374	0.9430	0.9047	0.9425
	20	0.9422	0.9401	0.9083	0.9452	0.9451	0.9313	0.9510	0.9510	0.9423
	40	0.9259	0.8941	0.9013	0.9422	0.9009	0.9315	0.9464	0.8970	0.9381
	50	0.9290	0.8836	0.9026	0.9395	0.8881	0.9282	0.9479	0.8882	0.9405
	60	0.9307	0.8982	0.9006	0.9416	0.9048	0.9280	0.9479	0.9112	0.9408
	80	0.9408	0.9407	0.9059	0.9426	0.9426	0.9234	0.9486	0.9484	0.9403
	100	0.9301	0.8863	0.9114	0.9428	0.8884	0.9383	0.9445	0.8850	0.9439
0.25	0	0.9230	0.6656	0.9175	0.9319	0.6895	0.9312	0.9450	0.7213	0.9445
	10	0.9303	0.8570	0.9152	0.9470	0.8916	0.9393	0.9502	0.9049	0.9447
	20	0.9405	0.9389	0.9159	0.9457	0.9457	0.9324	0.9481	0.9481	0.9428
	40	0.9263	0.8620	0.9104	0.9402	0.8627	0.9319	0.9450	0.8525	0.9413
	50	0.9254	0.8363	0.9146	0.9394	0.8353	0.9320	0.9397	0.8386	0.9381
	60	0.9270	0.8609	0.9110	0.9374	0.8723	0.9318	0.9465	0.8764	0.9415
	80	0.9449	0.9446	0.9154	0.9471	0.9467	0.9359	0.9483	0.9479	0.9427
	100	0.9327	0.8697	0.9185	0.9397	0.8682	0.9364	0.9445	0.8619	0.9427
0.50	0	0.9217	0.6674	0.9115	0.9405	0.6984	0.9371	0.9457	0.7184	0.9429
	10	0.9312	0.8613	0.9175	0.9431	0.8855	0.9362	0.9485	0.9036	0.9445
	20	0.9398	0.9383	0.9190	0.9457	0.9457	0.9352	0.9478	0.9474	0.9416
	40	0.9332	0.8512	0.9177	0.9412	0.8417	0.9349	0.9445	0.8405	0.9412
	50	0.9314	0.8148	0.9182	0.9411	0.8153	0.9343	0.9473	0.8241	0.9452
	60	0.9331	0.8505	0.9167	0.9398	0.8666	0.9342	0.9456	0.8687	0.9421
	80	0.9386	0.9386	0.9129	0.9450	0.9447	0.9356	0.9470	0.9453	0.9425
	100	0.9336	0.8640	0.9179	0.9425	0.8567	0.9354	0.9479	0.8584	0.9447
0.75	0	0.9189	0.6766	0.9109	0.9384	0.7161	0.9365	0.9436	0.7353	0.9431
	10	0.9368	0.8790	0.9195	0.9402	0.8934	0.9345	0.9450	0.9033	0.9403
	20	0.9418	0.9412	0.9152	0.9443	0.9443	0.9340	0.9461	0.9456	0.9398
	40	0.9313	0.8679	0.9199	0.9403	0.8549	0.9344	0.9457	0.8585	0.9428
	50	0.9300	0.8352	0.9177	0.9358	0.8371	0.9342	0.9445	0.8397	0.9432
	60	0.9287	0.8605	0.9142	0.9397	0.8756	0.9332	0.9492	0.8857	0.9455
	80	0.9403	0.9403	0.9165	0.9448	0.9439	0.9323	0.9432	0.9417	0.9396
	100	0.9274	0.8631	0.9114	0.9383	0.8577	0.9325	0.9452	0.8583	0.9421
0.95	0	0.9147	0.7386	0.9174	0.9335	0.7679	0.9298	0.9486	0.7902	0.9463
	10	0.9342	0.8948	0.9140	0.9446	0.9149	0.9341	0.9478	0.9252	0.9441
	20	0.9410	0.9402	0.9036	0.9491	0.9490	0.9319	0.9471	0.9468	0.9401
	40	0.9339	0.9019	0.9185	0.9408	0.8974	0.9332	0.9435	0.8924	0.9411
	50	0.9300	0.8866	0.9115	0.9414	0.8896	0.9303	0.9465	0.8939	0.9402
	60	0.9306	0.9009	0.9058	0.9418	0.9122	0.9304	0.9425	0.9128	0.9367
	80	0.9418	0.9418	0.9054	0.9433	0.9429	0.9239	0.9492	0.9479	0.9421
	100	0.9282	0.8777	0.9082	0.9444	0.8771	0.9340	0.9413	0.8751	0.9380
0.99	0	0.9101	0.7673	0.9067	0.9379	0.8079	0.9321	0.9432	0.8157	0.9409
	10	0.9306	0.9002	0.9030	0.9459	0.9249	0.9307	0.9487	0.9294	0.9384
	20	0.9430	0.9427	0.9054	0.9436	0.9434	0.9291	0.9495	0.9490	0.9431
	40	0.9333	0.9118	0.9061	0.9448	0.9178	0.9271	0.9453	0.9164	0.9403
	50	0.9338	0.9086	0.9010	0.9410	0.9112	0.9303	0.9492	0.9174	0.9412
	60	0.9354	0.9186	0.9024	0.9430	0.9264	0.9271	0.9472	0.9301	0.9407
	80	0.9427	0.9427	0.8981	0.9474	0.9466	0.9274	0.9505	0.9490	0.9410
	100	0.9313	0.8912	0.9077	0.9431	0.8909	0.9304	0.9461	0.8861	0.9406

Table 5. Empirical Coverage Probabilities for 95% QL: Model 1, Dual-Power Transformation, $\psi' = (5, 1, 2, 0.1)$, $X_i \in [0, 100]$

		$n = 15$			$n = 30$			$n = 60$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9183	0.6583	0.7826	0.9339	0.6893	0.8236	0.9474	0.6999	0.8600
	10	0.9342	0.8460	0.9023	0.9415	0.8708	0.9250	0.9480	0.8807	0.9380
	20	0.9414	0.9329	0.9385	0.9468	0.9422	0.9435	0.9456	0.9432	0.9483
	40	0.9379	0.9202	0.9250	0.9450	0.9184	0.9397	0.9480	0.9201	0.9466
	50	0.9352	0.9063	0.9164	0.9424	0.9086	0.9353	0.9443	0.9061	0.9409
	60	0.9355	0.9093	0.9182	0.9401	0.9100	0.9254	0.9456	0.9169	0.9410
	80	0.9404	0.9390	0.9149	0.9484	0.9476	0.9356	0.9493	0.9485	0.9436
	100	0.9390	0.9216	0.9318	0.9404	0.9180	0.9366	0.9477	0.9199	0.9471
0.05	0	0.9201	0.6361	0.7832	0.9355	0.6777	0.8311	0.9470	0.6980	0.8664
	10	0.9312	0.8397	0.9102	0.9416	0.8675	0.9249	0.9466	0.8827	0.9383
	20	0.9366	0.9304	0.9306	0.9494	0.9478	0.9458	0.9477	0.9470	0.9447
	40	0.9324	0.8984	0.9134	0.9402	0.8972	0.9340	0.9474	0.9058	0.9455
	50	0.9335	0.8809	0.9140	0.9395	0.8846	0.9354	0.9454	0.8857	0.9421
	60	0.9316	0.8933	0.9140	0.9451	0.9031	0.9357	0.9478	0.9075	0.9435
	80	0.9434	0.9419	0.9163	0.9463	0.9457	0.9291	0.9494	0.9490	0.9429
	100	0.9396	0.9118	0.9320	0.9444	0.9088	0.9454	0.9486	0.9044	0.9475
0.25	0	0.9288	0.6510	0.7899	0.9387	0.6883	0.8470	0.9461	0.7077	0.8819
	10	0.9345	0.8549	0.9069	0.9427	0.8776	0.9308	0.9488	0.8927	0.9398
	20	0.9394	0.9360	0.9235	0.9449	0.9447	0.9392	0.9448	0.9448	0.9388
	40	0.9348	0.8664	0.9174	0.9451	0.8673	0.9379	0.9468	0.8590	0.9439
	50	0.9324	0.8355	0.9186	0.9462	0.8435	0.9397	0.9451	0.8398	0.9430
	60	0.9335	0.8578	0.9146	0.9432	0.8778	0.9382	0.9455	0.8801	0.9423
	80	0.9402	0.9398	0.9190	0.9497	0.9497	0.9399	0.9474	0.9473	0.9432
	100	0.9375	0.8792	0.9281	0.9447	0.8803	0.9396	0.9475	0.8822	0.9450
0.50	0	0.9289	0.6712	0.7868	0.9360	0.7109	0.8501	0.9443	0.7225	0.8834
	10	0.9364	0.8607	0.8924	0.9461	0.8986	0.9274	0.9471	0.9033	0.9379
	20	0.9364	0.9346	0.9082	0.9485	0.9485	0.9323	0.9502	0.9501	0.9442
	40	0.9319	0.8496	0.9155	0.9415	0.8434	0.9335	0.9502	0.8440	0.9437
	50	0.9341	0.8156	0.9146	0.9452	0.8280	0.9358	0.9455	0.8277	0.9416
	60	0.9367	0.8539	0.9188	0.9458	0.8671	0.9382	0.9477	0.8800	0.9456
	80	0.9417	0.9417	0.9167	0.9473	0.9465	0.9357	0.9478	0.9467	0.9433
	100	0.9368	0.8715	0.9203	0.9464	0.8660	0.9404	0.9505	0.8667	0.9476
0.75	0	0.9250	0.7141	0.7945	0.9376	0.7512	0.8529	0.9425	0.7696	0.8912
	10	0.9277	0.8758	0.8769	0.9423	0.9083	0.9098	0.9506	0.9232	0.9320
	20	0.9405	0.9402	0.8966	0.9476	0.9473	0.9273	0.9491	0.9476	0.9383
	40	0.9375	0.8620	0.9105	0.9434	0.8601	0.9298	0.9454	0.8583	0.9415
	50	0.9313	0.8398	0.9128	0.9395	0.8460	0.9319	0.9481	0.8532	0.9410
	60	0.9355	0.8800	0.9176	0.9443	0.8909	0.9351	0.9473	0.8964	0.9433
	80	0.9437	0.9436	0.9119	0.9437	0.9419	0.9323	0.9455	0.9415	0.9381
	100	0.9401	0.8683	0.9153	0.9462	0.8593	0.9346	0.9478	0.8554	0.9434
0.95	0	0.9163	0.7825	0.7875	0.9368	0.8153	0.8546	0.9449	0.8375	0.8892
	10	0.9379	0.9125	0.8537	0.9438	0.9277	0.8961	0.9497	0.9384	0.9267
	20	0.9372	0.9366	0.8656	0.9456	0.9443	0.9047	0.9498	0.9467	0.9327
	40	0.9401	0.9029	0.8935	0.9443	0.8968	0.9187	0.9475	0.9031	0.9378
	50	0.9369	0.8951	0.8966	0.9410	0.9003	0.9199	0.9481	0.9099	0.9398
	60	0.9347	0.9113	0.8991	0.9440	0.9216	0.9251	0.9501	0.9314	0.9415
	80	0.9423	0.9410	0.8877	0.9449	0.9416	0.9227	0.9515	0.9471	0.9413
	100	0.9353	0.8667	0.8951	0.9434	0.8565	0.9257	0.9464	0.8558	0.9386
0.99	0	0.9201	0.8327	0.7848	0.9399	0.8616	0.8479	0.9440	0.8724	0.8838
	10	0.9377	0.9247	0.8376	0.9424	0.9353	0.8887	0.9479	0.9421	0.9128
	20	0.9413	0.9405	0.8607	0.9447	0.9425	0.8980	0.9435	0.9416	0.9232
	40	0.9363	0.9140	0.8776	0.9486	0.9201	0.9114	0.9482	0.9171	0.9340
	50	0.9387	0.9165	0.8897	0.9430	0.9187	0.9184	0.9481	0.9234	0.9363
	60	0.9398	0.9274	0.8862	0.9455	0.9343	0.9160	0.9481	0.9381	0.9359
	80	0.9400	0.9376	0.8773	0.9444	0.9408	0.9177	0.9489	0.9437	0.9353
	100	0.9323	0.8725	0.8915	0.9448	0.8686	0.9193	0.9494	0.8693	0.9416

Table 6. Empirical Coverage Probabilities for 95% QL: Model 1, Dual-Power Transformation, $\psi' = (5, 1, 1, 0.5)$, $X_i \in [0, 100]$

		$n = 15$			$n = 30$			$n = 60$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9144	0.6741	0.8756	0.9388	0.7080	0.9166	0.9486	0.7293	0.9386
	10	0.9329	0.8772	0.9235	0.9429	0.8936	0.9301	0.9430	0.9033	0.9397
	20	0.9406	0.9387	0.9073	0.9465	0.9456	0.9276	0.9477	0.9477	0.9382
	40	0.9348	0.9140	0.8976	0.9396	0.9138	0.9249	0.9470	0.9147	0.9363
	50	0.9348	0.9022	0.9023	0.9407	0.9081	0.9247	0.9444	0.9083	0.9364
	60	0.9396	0.9153	0.9001	0.9433	0.9216	0.9296	0.9464	0.9238	0.9379
	80	0.9401	0.9394	0.8984	0.9452	0.9451	0.9294	0.9469	0.9468	0.9357
	100	0.9353	0.9094	0.9187	0.9429	0.9117	0.9377	0.9439	0.9046	0.9403
0.05	0	0.9278	0.6570	0.8977	0.9396	0.6815	0.9257	0.9484	0.7117	0.9421
	10	0.9318	0.8653	0.9232	0.9416	0.8909	0.9349	0.9493	0.9014	0.9432
	20	0.9424	0.9389	0.9121	0.9483	0.9475	0.9319	0.9494	0.9494	0.9427
	40	0.9349	0.9002	0.9052	0.9418	0.8932	0.9271	0.9447	0.8969	0.9378
	50	0.9295	0.8830	0.9025	0.9400	0.8873	0.9274	0.9483	0.8938	0.9406
	60	0.9333	0.8978	0.9048	0.9435	0.9047	0.9271	0.9490	0.9143	0.9426
	80	0.9436	0.9433	0.9069	0.9493	0.9492	0.9341	0.9495	0.9495	0.9416
	100	0.9352	0.8960	0.9214	0.9422	0.8932	0.9377	0.9479	0.8932	0.9435
0.25	0	0.9336	0.6488	0.9080	0.9415	0.6753	0.9316	0.9447	0.6973	0.9375
	10	0.9304	0.8543	0.9207	0.9458	0.8907	0.9404	0.9490	0.8964	0.9442
	20	0.9340	0.9321	0.9108	0.9446	0.9445	0.9325	0.9484	0.9482	0.9432
	40	0.9328	0.8696	0.9133	0.9437	0.8596	0.9337	0.9507	0.8610	0.9459
	50	0.9296	0.8321	0.9141	0.9407	0.8339	0.9299	0.9443	0.8445	0.9423
	60	0.9294	0.8573	0.9101	0.9426	0.8749	0.9352	0.9460	0.8860	0.9411
	80	0.9432	0.9429	0.9182	0.9421	0.9419	0.9299	0.9479	0.9475	0.9425
	100	0.9331	0.8677	0.9215	0.9422	0.8666	0.9354	0.9456	0.8676	0.9424
0.50	0	0.9316	0.6559	0.9050	0.9353	0.6835	0.9255	0.9467	0.7124	0.9424
	10	0.9334	0.8593	0.9205	0.9390	0.8827	0.9317	0.9444	0.8968	0.9405
	20	0.9403	0.9391	0.9169	0.9461	0.9461	0.9354	0.9497	0.9493	0.9446
	40	0.9389	0.8554	0.9248	0.9427	0.8431	0.9352	0.9440	0.8405	0.9418
	50	0.9343	0.8180	0.9203	0.9411	0.8163	0.9365	0.9450	0.8213	0.9420
	60	0.9363	0.8531	0.9226	0.9424	0.8618	0.9347	0.9436	0.8734	0.9399
	80	0.9390	0.9390	0.9153	0.9427	0.9420	0.9321	0.9518	0.9499	0.9453
	100	0.9372	0.8687	0.9221	0.9429	0.8622	0.9377	0.9479	0.8632	0.9446
0.75	0	0.9296	0.6911	0.9129	0.9416	0.7189	0.9359	0.9432	0.7311	0.9408
	10	0.9284	0.8669	0.9096	0.9425	0.8990	0.9332	0.9444	0.9093	0.9387
	20	0.9419	0.9414	0.9130	0.9452	0.9450	0.9341	0.9495	0.9487	0.9424
	40	0.9312	0.8578	0.9224	0.9435	0.8630	0.9366	0.9493	0.8608	0.9453
	50	0.9341	0.8394	0.9218	0.9423	0.8449	0.9389	0.9474	0.8489	0.9441
	60	0.9346	0.8690	0.9257	0.9429	0.8854	0.9380	0.9475	0.8924	0.9452
	80	0.9445	0.9445	0.9187	0.9450	0.9440	0.9336	0.9536	0.9525	0.9492
	100	0.9312	0.8611	0.9127	0.9402	0.8600	0.9332	0.9476	0.8639	0.9446
0.95	0	0.9163	0.7475	0.9028	0.9321	0.7794	0.9301	0.9418	0.7899	0.9370
	10	0.9304	0.8926	0.9043	0.9424	0.9188	0.9306	0.9468	0.9284	0.9372
	20	0.9400	0.9398	0.9062	0.9452	0.9450	0.9306	0.9460	0.9443	0.9411
	40	0.9374	0.9014	0.9169	0.9377	0.8932	0.9324	0.9430	0.8954	0.9368
	50	0.9291	0.8848	0.9160	0.9415	0.8935	0.9333	0.9390	0.8906	0.9359
	60	0.9318	0.9050	0.9130	0.9423	0.9162	0.9335	0.9477	0.9192	0.9422
	80	0.9392	0.9389	0.9034	0.9465	0.9452	0.9313	0.9488	0.9459	0.9389
	100	0.9346	0.8761	0.9075	0.9467	0.8775	0.9373	0.9440	0.8620	0.9393
0.99	0	0.9174	0.7823	0.8980	0.9360	0.8205	0.9271	0.9447	0.8357	0.9390
	10	0.9337	0.9120	0.9028	0.9415	0.9262	0.9267	0.9474	0.9348	0.9384
	20	0.9439	0.9438	0.9012	0.9469	0.9466	0.9270	0.9495	0.9479	0.9427
	40	0.9364	0.9138	0.9071	0.9454	0.9171	0.9323	0.9477	0.9164	0.9404
	50	0.9381	0.9111	0.9077	0.9494	0.9236	0.9355	0.9487	0.9184	0.9410
	60	0.9367	0.9206	0.9041	0.9484	0.9335	0.9301	0.9508	0.9360	0.9402
	80	0.9421	0.9416	0.9001	0.9446	0.9430	0.9280	0.9472	0.9449	0.9349
	100	0.9333	0.8875	0.9014	0.9417	0.8780	0.9300	0.9448	0.8804	0.9421

Table 7. Empirical Coverage Probabilities for 95% QL: Model 2, Box-Cox Transformation, $\beta' = (5, 1)$, $\sigma = 0.5$, $\gamma = 0.1$, $X_i \in [0, 25]$

		$n = 30$, $\lambda = 0.1$			$n = 60$, $\lambda = 0.1$			$n = 60$, $\lambda = 0.05$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta	CPQL	PQl	Delta
0.01	0	0.9482	0.7336	0.9542	0.9540	0.7674	0.9534	0.9388	0.7668	0.9442
	5	0.9354	0.8130	0.9408	0.9416	0.8292	0.9408	0.9384	0.8332	0.9460
	10	0.9314	0.8981	0.9392	0.9408	0.9130	0.9408	0.9456	0.9202	0.9492
	15	0.9498	0.9309	0.9472	0.9482	0.9324	0.9466	0.9504	0.9372	0.9526
	20	0.9409	0.7888	0.9085	0.9452	0.7914	0.9264	0.9446	0.7912	0.9038
	25	*	*	*	0.9490	0.6200	0.8475	0.9444	0.6050	0.8008
0.05	0	0.9493	0.7482	0.9471	0.9472	0.7784	0.9466	0.9474	0.7886	0.9434
	5	0.9304	0.8256	0.9306	0.9370	0.8340	0.9366	0.9418	0.8432	0.9416
	10	0.9344	0.8956	0.9344	0.9432	0.9090	0.9420	0.9400	0.9060	0.9424
	15	0.9488	0.9339	0.9430	0.9520	0.9398	0.9488	0.9460	0.9330	0.9462
	20	0.9380	0.7978	0.9213	0.9444	0.8130	0.9388	0.9380	0.7996	0.9280
	25	*	*	*	0.9376	0.6188	0.9006	0.9502	0.6292	0.8504
0.25	0	0.9476	0.7917	0.9436	0.9454	0.8204	0.9488	0.9416	0.8330	0.9374
	5	0.9327	0.8585	0.9228	0.9402	0.8668	0.9364	0.9404	0.8658	0.9420
	10	0.9359	0.8767	0.9270	0.9416	0.8934	0.9374	0.9434	0.8936	0.9418
	15	0.9444	0.9308	0.9374	0.9494	0.9420	0.9444	0.9540	0.9476	0.9446
	20	0.9264	0.8152	0.9300	0.9450	0.8430	0.9474	0.9446	0.8370	0.9350
	25	0.9212	0.6797	0.9108	0.9376	0.6806	0.9338	0.9324	0.6556	0.9014
0.50	0	0.9328	0.7956	0.9189	0.9368	0.8302	0.9326	0.9378	0.8292	0.9270
	5	0.9366	0.8585	0.9288	0.9428	0.8656	0.9372	0.9444	0.8678	0.9400
	10	0.9428	0.8738	0.9276	0.9426	0.8834	0.9352	0.9452	0.8884	0.9364
	15	0.9459	0.9295	0.9335	0.9498	0.9430	0.9410	0.9414	0.9344	0.9316
	20	0.9476	0.8420	0.9353	0.9496	0.8496	0.9440	0.9430	0.8492	0.9290
	25	0.9473	0.7052	0.9318	0.9424	0.7040	0.9356	0.9436	0.6978	0.9172
0.75	0	0.9275	0.7959	0.8998	0.9348	0.8246	0.9226	0.9300	0.8204	0.9162
	5	0.9490	0.8537	0.9323	0.9520	0.8598	0.9392	0.9500	0.8704	0.9368
	10	0.9421	0.8858	0.9254	0.9440	0.8966	0.9348	0.9498	0.8964	0.9342
	15	0.9461	0.9252	0.9147	0.9458	0.9372	0.9336	0.9480	0.9374	0.9232
	20	0.9614	0.8257	0.9266	0.9490	0.8394	0.9308	0.9508	0.8344	0.9172
	25	0.9689	0.6918	0.9251	0.9530	0.6884	0.9286	0.9510	0.6858	0.9042
0.95	0	0.9179	0.7664	0.8875	0.9378	0.8114	0.9208	0.9386	0.8138	0.9060
	5	0.9536	0.8301	0.9181	0.9496	0.8458	0.9298	0.9468	0.8458	0.9222
	10	0.9468	0.9066	0.9026	0.9476	0.9180	0.9276	0.9486	0.9198	0.9172
	15	0.9516	0.9253	0.8912	0.9478	0.9310	0.9172	0.9440	0.9248	0.9006
	20	0.9300	0.7249	0.8759	0.9572	0.8234	0.9234	0.9524	0.8182	0.9012
	25	0.9376	0.5652	0.8850	0.9604	0.6614	0.9068	0.9592	0.6427	0.8756
0.99	0	0.9168	0.7509	0.8716	0.9406	0.7894	0.9158	0.9364	0.7914	0.9064
	5	0.9554	0.8077	0.9062	0.9498	0.8378	0.9240	0.9424	0.8278	0.9054
	10	0.9407	0.9120	0.8905	0.9422	0.9184	0.9102	0.9430	0.9230	0.9014
	15	0.9509	0.9180	0.8659	0.9508	0.9266	0.9088	0.9500	0.9246	0.8860
	20	0.9289	0.7082	0.8585	0.9558	0.7952	0.9120	0.9478	0.7926	0.8800
	25	0.9417	0.5409	0.8648	0.9566	0.6464	0.8998	0.9578	0.6401	0.8602

Note: An asterisk denotes simulation failure due to nonconvergence of some cases.

Table 8. Empirical Coverage Probabilities for 95% QL: Model 2, Box-Cox Transformation, $\beta' = (5, 1)$, $\sigma = 0.2$, $\gamma = 0.1$, $X_i \in [0, 25]$

		$n = 30$, $\lambda = 0.25$			$n = 60$, $\lambda = 0.25$		
p	x_0	CPQL	PQL	Delta	CPQL	PQL	Delta
0.01	0	0.9418	0.7261	0.9370	0.9372	0.7594	0.9366
	5	0.9383	0.8001	0.9201	0.9424	0.8254	0.9348
	10	0.9256	0.8903	0.9099	0.9354	0.9030	0.9260
	15	0.9435	0.9309	0.9163	0.9468	0.9344	0.9326
	20	0.9461	0.8197	0.9327	0.9444	0.8142	0.9392
	25	*	*	*	0.9496	0.6532	0.9450
0.05	0	0.9327	0.7446	0.9287	0.9462	0.7884	0.9456
	5	0.9337	0.8169	0.9215	0.9446	0.8354	0.9368
	10	0.9325	0.8911	0.9199	0.9422	0.9090	0.9310
	15	0.9452	0.9315	0.9251	0.9502	0.9392	0.9418
	20	0.9424	0.8223	0.9348	0.9400	0.8164	0.9362
	25	0.9393	0.6468	0.9343	0.9426	0.6570	0.9420
0.25	0	0.9339	0.7704	0.9295	0.9404	0.8092	0.9386
	5	0.9306	0.8479	0.9206	0.9396	0.8486	0.9372
	10	0.9276	0.8795	0.9210	0.9436	0.8986	0.9376
	15	0.9512	0.9402	0.9388	0.9476	0.9402	0.9420
	20	0.9362	0.8199	0.9324	0.9436	0.8250	0.9380
	25	0.9205	0.6615	0.9283	0.9374	0.6770	0.9410
0.50	0	0.9251	0.7668	0.9175	0.9376	0.8176	0.9364
	5	0.9280	0.8503	0.9194	0.9416	0.8652	0.9392
	10	0.9367	0.8825	0.9301	0.9412	0.8912	0.9372
	15	0.9467	0.9311	0.9373	0.9476	0.9382	0.9422
	20	0.9399	0.8252	0.9347	0.9406	0.8378	0.9376
	25	0.9392	0.6759	0.9367	0.9418	0.6996	0.9386
0.75	0	0.9230	0.7705	0.9108	0.9380	0.8184	0.9358
	5	0.9351	0.8350	0.9267	0.9468	0.8590	0.9416
	10	0.9375	0.8888	0.9281	0.9432	0.9020	0.9350
	15	0.9479	0.9323	0.9301	0.9438	0.9296	0.9332
	20	0.9495	0.8163	0.9347	0.9506	0.8294	0.9464
	25	0.9489	0.6689	0.9479	0.9468	0.6846	0.9462
0.95	0	0.9214	0.7618	0.9096	0.9260	0.7936	0.9232
	5	0.9476	0.8143	0.9301	0.9454	0.8416	0.9356
	10	0.9394	0.8967	0.9232	0.9450	0.9114	0.9270
	15	0.9502	0.9325	0.9131	0.9558	0.9458	0.9400
	20	0.9554	0.8171	0.9284	0.9548	0.8150	0.9366
	25	0.9603	0.6469	0.9399	0.9518	0.6698	0.9440
0.99	0	0.9216	0.7462	0.9079	0.9376	0.7846	0.9328
	5	0.9469	0.8024	0.9190	0.9486	0.8256	0.9360
	10	0.9376	0.9019	0.9155	0.9496	0.9218	0.9342
	15	0.9479	0.9279	0.9013	0.9472	0.9324	0.9224
	20	0.9576	0.8069	0.9210	0.9562	0.8326	0.9400
	25	0.9619	0.6478	0.9328	0.9494	0.6604	0.9382

Note: An asterisk denotes simulation failure due to nonconvergence of some cases.

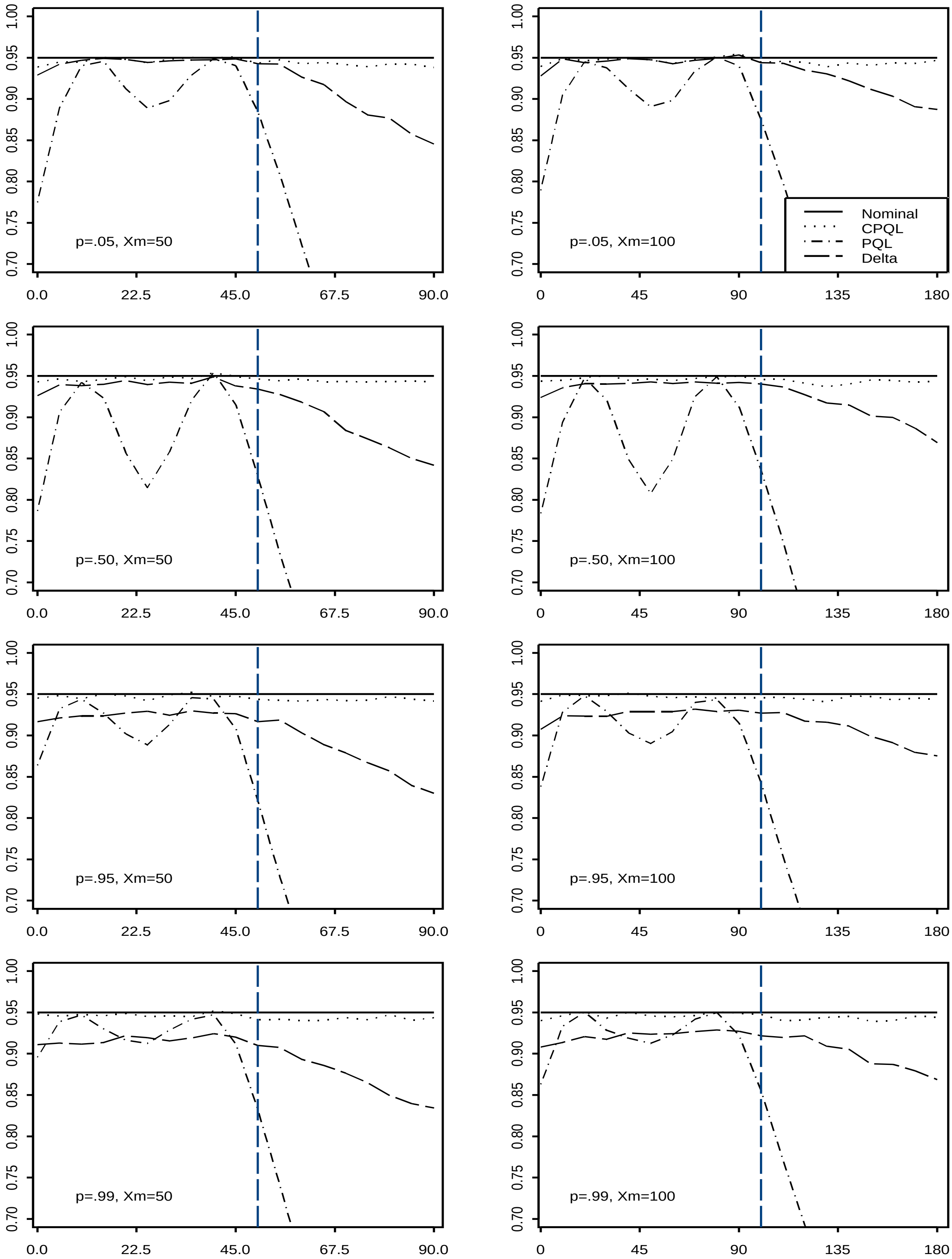

Figure 1: Plots of the coverage probabilities against x_0: Model 1, Box-Cox transformation, $n = 60$, $\psi' = (5, 1, 1, 0.01)$, and $X_i \in [0, X_m]$ where X_m is marked by the vertical dashed line