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Abstract

In this paper, we study the problem of inventory management for a buyer whose ordering is governed by a multi-period *adjustment contract* that allows the buyer to adjust his inventory level upwards or downwards during an ordering period. Under this type of arrangement, the buyer has added flexibility in inventory control so as to reduce his inventory risk. The flexibility, however, likely comes at a cost, i.e., at a higher price per unit of inventory bought and a lower price per unit of inventory sold during an ordering period. This type of supply contract mimics the flexibility obtained from purchasing on a spot market by offering the buyer multiple adjustment opportunities in each ordering period, but with fixed prices. We prove the structure of the optimal inventory policy for a buyer who procures inventory under a portfolio of supply contracts consisting of a wholesale price contract and an adjustment contract. We also quantify the benefits of using the adjustment contract for the buyer.

Keywords: inventory, supply chain management, dynamic programming.

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1 Introduction

Given the complexity and increasing globalization of modern supply chains, and despite the trend toward long-term supplier partnerships, it is not uncommon for buyers (including retailers and manufacturers) to acquire identical goods from multiple sources; see, e.g., McMillan (1990), Richardson (1993) and Latour (2001). The use of multiple sources is particularly important in industries for which supply chain reliability and resiliency are critical (Sheffi et al. 2003). In addition, multiple supply sources can be used to balance the various risks in a supply chain, e.g., demand and price uncertainty (Martinez-de-Albeniz and Simchi-Levi, 2005).

When using multi-sourcing, buyers must balance competing priorities, such as cost, lead time, reliability and flexibility. For example, Hewlett Packard (HP) uses a multi-sourcing strategy that allows them to spread inventory risk over a number of suppliers. Some suppliers operate under long-term contracts, which are used to meet about 90% of expected demand. These contracts are supplemented with suppliers who operate under short-term contracts with slightly higher unit prices but guaranteed availability to cover uncertainties in demand. In 2001, of HP's commodity purchasing expense, 50% was through long-term contracts negotiated with suppliers, 35% was through nonbinding agreements which provided high flexibility, and 15% was used for purchases on spot markets. For details, see Billington (2002) and Carbone (2001).

As the above example demonstrates, the use of multi-sourcing generally involves the management of different types of supply contracts for different suppliers. In the existing literature, many types of supply contracts have been considered, some of which are also commonly used in practice, including buyback, quantity flexibility, revenue sharing and option contracts. More recently, researchers have studied portfolios of contracts, i.e., a buyer who makes procurement decisions under a set of supply contracts or a combination of supply contracts and spot market procurement (see, e.g., Martínez-de-Albéniz and Simchi-Levi (2002), Wu et al. (2002)).

Spot market participation theoretically provides the buyer the most flexibility in inventory control but has high price uncertainty due to fluctuations in supply and demand (see Kleindorfer and Wu (2003) for a review of the literature on combining supply contracting and spot markets). To mimic the flexibility of the spot market, we consider a buyer who procures inventory, in part, under an *adjustment contract*. Such a contract allows a buyer to frequently adjust his inventory level upwards or downwards at a corresponding buying or selling price. The flexibility provided by the adjustment contract is especially valuable for products with high demand uncertainty and large shortage costs, e.g., critical machine parts, high-tech products, fashion items. This flexibility, however, comes at a cost, i.e., a (relatively) high price per unit of inventory purchased and a (relatively) low price per unit of inventory returned. We thus consider a buyer who procures inventory from two sources, one of which operates under a standard wholesale price contract, while the other operates under the adjustment contract. The buyer can thus take advantage of the low purchase price of the wholesale price contract and the inventory flexibility of the adjustment contract. In this paper, our goal is to determine how the buyer should operate, i.e., manage its ordering and inventory decisions, given this portfolio of contracts.

The paper is organized as follows. Section 2 provides a review of the related literature. Section 3 presents the optimal inventory policy for a buyer for multi-period model with multiple adjustment opportunities. In Section 4 we study the value of the adjustment contract to the buyer. In Section 5 we consider several extensions, including fixed order costs, order bounds and positive lead times. In Section 6 we present the results of an extensive computational study. Section 7 provides a conclusion and a discussion of future research.

2 Literature Review

Since this paper considers the procurement decisions of a buyer who purchases inventory from multiple sources under a portfolio of multi-period supply contracts, we will focus our literature review on the previous research considering procurement decisions with multiple supply options. In addition, since the adjustment contract considered in this paper has some similarities to spot market procurement, we will briefly review the literature considering the use of spot market purchases to supplement procurement under more conventional supply contracts.

2.1 Procurement Decisions with Multiple Supply Options

There is an extensive body of research on inventory management for a buyer who procures inventory from multiple sources. See Minner (2003) for a review. The papers in this area make varying assumptions regarding the differences between the supply sources in terms of costs, lead times, prices, etc. For example, numerous authors have considered the problem of inventory management for a buyer who procures inventory from multiple sources with differing price and lead time combinations, e.g., Whittmore and Saunders (1977), Moinzadeh and Nahmias (1988), Martínez-de-Albéniz (2006), Veeraraghavan and Sheller-Wolf (2006), Yazlali and Erhun (2007), etc. Several other authors have considered the problem of a buyer who can place emergency or expedited orders, e.g., Chiang and Gutierrez (1998), Lawson and Porteus (2000) and Tagaras and Vlachos (2001). Other authors have considered the value of multi-sourcing for coping with the risk of a supply disruption, e.g., Parlar and Perry (1996) and Tomlin (2006), or for coping with yield uncertainty, e.g., Gerchak and Parlar (1990) and Anupindi and Akella (1993), etc.

The current paper differs from this literature in that we consider suppliers who differ in the degree of *procurement flexibility* they provide to the buyer. In addition to purchasing under a standard wholesale price contract, which allows inventory to be ordered at fixed intervals at a given price, the buyer also has the option of procuring inventory under an adjustment contract that allows more frequent replenishments and lets the buyer return inventory to the supplier. This additional flexibility comes at a higher procurement cost. While there has been some literature considering suppliers with differing flexibility, the specific models differ from that considered in this paper. Janssen and Kok (1999) consider a buyer who procures inventory from two suppliers under two different types of contracts. The main supplier provides a fixed quantity, Q, each period. The buyer orders inventory from a second supplier using a periodic review base-stock policy with base-stock level S. Moinzadeh and Nahmias (2000) study a fixed delivery contract with adjustments. Under such a contract, a buyer orders a fixed quantity, Q, each period. Before delivery, the buyer has an opportunity to adjust Q upwards by paying a fixed cost and an additional cost per unit. The authors assume an (s, S) policy is used for the adjustments. Our model is similar, but we do not assume a fixed order quantity and we allow for the return of inventory, rather than the adjustment of the order quantity prior to delivery.

Another stream of literature considers competition between suppliers when the buyer chooses among them based on a number of criteria, e.g., price and service level. See, for example, Ha et al. (2003), Cachon and Zhang (2007) and Benjaafar et al. (2007).

2.2 Procurement Decisions with a Spot Market Option

Several authors have considered a buyer who procures inventory using a combination of supply contracts and spot market procurement. Bonser and Wu (2001) study a procurement problem in which the buyer may purchase under several supply contracts, each of which specifies a price and minimum and maximum annual quantities, as well as from a spot market. Martínez-de-Albéniz and Simchi-Levi (2002) consider a buyer who purchases goods using a portfolio of options and flexibility contracts, as well as spot market procurement. Kleinknecht et al. (2002)

determine the optimal policy for a buyer who procures inventory using both option contracts and spot market procurement. Lee and Whang (2002) consider a set of buyers who each have two purchase opportunities. In the first, each buyer can purchase at a fixed price set by the manufacturer. In the second, the buyers may buy or sell on the spot market given a price determined endogenously to equate supply and demand. Goel and Gutierrez (2006) consider incorporating spot market and futures price information into procurement decisions. Two options are considered: spot market and forward contract procurement. Both allow the buyer to buy or sell inventory in each period and include transaction costs, so that the buying and selling prices may differ. Cohen and Agrawal (1999) consider a buyer who can choose between short and long term contracts with its supplier. A long term contract reduces delivery lead time and sets a fixed price, but includes a fixed cost at the start of the relationship. The short term contract includes no fixed cost, but requires the buyer to pay the random market price. Peleg et al. (2002) consider a buyer who can choose between three procurement strategies: (1) a long term contract with a single supplier; (2) an on-line search; and (3) a combined strategy. The authors develop conditions under which each alternative is preferred.

3 Multi-Period Model with Multiple Adjustment Opportunities

In this section, we consider a general multi-period inventory problem for a buyer who procures inventory using a portfolio of supply contracts consisting of a wholesale price contract and an adjustment contract. Under the wholesale price contract, the buyer can order inventory at the start of each review period. Under the adjustment contract, the buyer has several opportunities to adjust his inventory level, i.e., to purchase or sell inventory, during each ordering period.

3.1 Problem Description

We consider an inventory planning horizon of length T, say $[0, T]$. Let $m \in \mathbb{N}$ denote the number of periods and $n \in \mathbb{N}$ the number of adjustment opportunities per period. Thus, we consider a model with $n+1$ subperiods in each of the m periods. Let $[0, T_1)$, $[T_1, T_2)$, $\ldots$, $[T_{m-1}, T]$ denote the m periods. We call $0, T_1, \ldots, T_{m-1}$ the *regular order opportunities*, when the buyer may purchase under the wholesale price contract. For period j, i.e., $[T_{j-1}, T_j)$, $j = 1, \ldots, m$, let $t_1^j, \ldots, t_n^j$ denote the *adjustment opportunities*, when the buyer may buy or sell under the adjustment contract, where $T_{j-1} \triangleq t_0^j < t_1^j < \cdots < t_n^j < t_{n+1}^j \triangleq T_j$. We call time interval

$[t_{i-1}^j, t_i^j)$ subperiod i (or the ith subperiod) of period j. The length of the subperiods within a period and the length of the same subperiod within different periods can differ. For all orders, we assume instantaneous delivery, i.e., zero lead time. Unsatisfied demand is backlogged. Fixed order costs are negligible. Below, we list additional assumptions and notation for this model:

1. Let $\alpha_{i(j)} \in (0,1)$ denote the discount factor for the ith subperiod of period j.

2. Let $D_{i(j)} \geq 0$ denote the demand during subperiod i in period j with probability density function (pdf) $f_{i(j)}(s)$ and cumulative distribution function (cdf) $F_{i(j)}(s)$, $s \in \mathbb{R}$, where $f_{i(j)}(s) = 0$, if $s < 0$, for any i, j. We assume that all the subperiod demands are mutually independent. However, the discussion can be applied to correlated demands as well.

3. Let $w_j > 0$, $j = 1, \ldots, m$, denote the wholesale price charged at the start of period j.

4. At the ith adjustment opportunity in period j, the buyer may procure inventory at the unit cost $P_{Bi(j)}$ or may return inventory to the supplier at the price $P_{Si(j)}$ per unit. We assume $P_{Bi(j)} \geq P_{Si(j)}$, for all i, j.

5. The unit penalty cost is $p > 0$ and $h_{i(j)} > 0$ is the unit holding cost for the ith subperiod of period j. We assume that $p > h_{i(j)}$, $p > P_{Bi(j)}$ and $p > w_j$ for all i, j.

3.2 Model Formulation and Results

Let $C(x)$ denote the minimum expected cost for the entire planning horizon, given on-hand inventory level $x \in \mathbb{R}$ at time 0. To formulate $C(x)$, we treat each regular order opportunity as a special adjustment opportunity, referred to as the 0th adjustment opportunity in the period, with $P_{B0(j)} = w_j$ and $P_{S0(j)} = -\infty$, so that the selling opportunity will never be used.

Utilizing this idea, we have the following dynamic programming (DP) formulation for $C(x)$:

$$C(x) \triangleq C_{0(1)}(x) = \min_{y \geq x} \left\{ w_1(y - x) + L_{1(1)}(y) + \alpha_{1(1)} \int_{-\infty}^{\infty} C_{1(1)}(y - s) f_{1(1)}(s) ds \right\}, \qquad (1)$$

where $L_{i(j)}(y)$ is the usual expected holding and penalty cost for the ith subperiod of period j with starting inventory level y, for $i = 1, \ldots, n+1, j = 1, \ldots, m$ and $C_{i(j)}(z)$ is the minimum expected cost for the remainder of the horizon after the ith adjustment in period j, given the on-hand inventory level z, $z \in \mathbb{R}$. Thus, $C_{i(j)}(z)$, $i = 0, 1, \ldots, n$, $j = 1, \ldots, m$, satisfies:

$$\begin{aligned} C_{i(j)}(z) = & \left\{ \min_{Z_B \geq z} P_{Bi(j)}(Z_B - z) + L_{i+1(j)}(Z_B) + \alpha_{i+1(j)} \int_{-\infty}^{\infty} C_{i+1(j)}(Z_B - s) f_{i+1(j)}(s) ds \right\} \wedge \\ & \left\{ \min_{Z_S \leq z} -P_{Si(j)}(z - Z_S) + L_{i+1(j)}(Z_S) + \alpha_{i+1(j)} \int_{-\infty}^{\infty} C_{i+1(j)}(Z_S - s) f_{i+1(j)}(s) ds \right\} \\ \triangleq & \left\{ \min_{Z_B \geq z} G_{Bi(j)}(z, Z_B) \right\} \wedge \left\{ \min_{Z_S \leq z} G_{Si(j)}(z, Z_S) \right\}, \end{aligned} \tag{2}$$

where $C_{n+1(j)}(z) = C_{0(j+1)}(z)$, $C_{n+1(m)}(z) = 0$, for all $z \in \mathbb{R}$ and $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

We can prove that the optimal policy is a *buy-up-to or sell-down-to policy* at the adjustment opportunities, along with a standard *order-up-to-policy* at the regular order opportunities. The buy-up-to or sell-down-to policy is similar to the critical interval policy studied by Goel and Gutierrez (2006). Define $Z^*_{Bi(j)}$ and $Z^*_{Si(j)}$ to be the values of Z_B and Z_S that minimize $G_{Bi(j)}(z, Z_B)$ and $G_{Si(j)}(z, Z_S)$, respectively. Then (all the proofs are found in Appendix I):

Proposition 3.1 *The optimal ith adjustment policy for period j is:*

$$\begin{cases} Buy-up-to\ Z^*_{Bi(j)} & \text{if the on-hand inventory level is below } Z^*_{Bi(j)}, \\ Neither\ buy\ nor\ sell & \text{if the on-hand inventory level is between } Z^*_{Bi(j)} \text{ and } Z^*_{Si(j)}, \\ Sell-down-to\ Z^*_{Si(j)} & \text{if the on-hand inventory level is above } Z^*_{Si(j)}, \end{cases}$$

and the optimal regular order policy for period j is the order-up-to $Z^*_{B0(j)}$ *policy, where* $Z^*_{Si(j)} \geq Z^*_{Bi(j)} > 0$, $Z^*_{Bn+1(m)} = F^{-1}_{n+1(m)}\left(\frac{p - P_{Bn+1(m)}}{h_{n+1(m)} + p}\right)$ *and* $Z^*_{Sn+1(m)} = F^{-1}_{n+1(m)}\left(\frac{p - P_{Sn+1(m)}}{h_{n+1(m)} + p}\right)$.

3.3 Sensitivity Analysis on the Cost Parameters

Next, we study the behavior of the optimal inventory control parameters. We focus on the single period, single adjustment case, i.e., $m = n = 1$. For notational convenience, we remove all the subscripts for periods and subperiods in this section. Specifically, we study the sensitivity of Y^*, Z^*_B and Z^*_S, where $Y^* = Z^*_{B0}$ is the optimal order-up-to level, with respect to the cost parameters, w, P_B and P_S. Our main results are as follows:

Proposition 3.2

(i) Y^* *is a nonincreasing function of* w, *while* Z^*_B *and* Z^*_S *do not change as* w *changes.*

(ii) Y^* *is a nondecreasing function of* P_B, Z^*_B *is a nonincreasing function of* P_B, *while* Z^*_S *does not change as* P_B *changes.*

(iii) Y^* is a nondecreasing function of P_S, Z_S^* is a nonincreasing function of P_S, while Z_B^* does not change as P_S changes.

Intuitively, if the wholesale price, w, increases, the buyer should not order more inventory at the start of the period. However, w will not affect the optimal parameters at the adjustment time, i.e., Z_B^* and Z_S^*. If the buying price, P_B, increases, the buyer should not buy more at the adjustment time. However, in order to take advantage of the smaller relative value of w, the buyer may increase his inventory level at the start of the period. This additional inventory can be used to fill demand in the second subperiod. If the selling price, P_S, increases, the buyer should not sell less at the adjustment time. However, in order to take advantage of the smaller relative value of w, the buyer may increase his inventory level at the start of the period. Any excess can be sold at the adjustment time at the higher selling price, P_S.

The optimal buy-up-to level, Z_B^*, does not depend on the selling price, P_S. Similarly, the optimal sell-down-to level, Z_S^*, does not depend on the buying price, P_B. Intuitively, the buyer decides whether to buy or sell at the adjustment opportunity given his on-hand inventory level, the buying price and the selling price. If the buyer decides to buy, he determines the optimal buy-up-to level, Z_B^*, by minimizing the corresponding inventory cost for the second subperiod, which does not depend on the selling price, P_S. Similarly, if the buyer decides to sell, he determines the optimal sell-down-to level, Z_S^*, without considering the buying price P_B.

3.4 Infinite Horizon Model

We next study an infinite horizon version of the model. To simplify the analysis, we consider homogenous periods and subperiods. Thus, subscripts for periods and subperiods are removed.

Let $C_i(z)$, for all $z \in \mathbb{R}$, denote the minimum expected cost for the remainder of the horizon after the ith adjustment in any period with initial on-hand inventory level z, $i = 0, \ldots, n$. For this homogenous infinite horizon model, the rest of the planning horizon after any ith subperiod looks the same, i.e., $C_i(\cdot)$ does not depend on which period we are in. Thus, we have the following DP formulation for $C_i(z)$:

$$C_0(z) = \min_{Z_B \geq z} \left\{ w(Z_B - z) + L(Z_B) + \alpha \int_{-\infty}^{\infty} C_1(Z_B - s) f(s) ds \right\} = \min_{Z_B \geq z} G_0(z, Z_B), \tag{3}$$

and for $i = 1, \ldots, n$

$$\begin{aligned} C_i(z) = &\left\{\min_{Z_B \geq z} P_B(Z_B - z) + L(Z_B) + \alpha \int_{-\infty}^{\infty} C_{i+1}(Z_B - s) f(s) ds\right\} \wedge \\ &\left\{\min_{Z_S \leq z} -P_S(z - Z_S) + L(Z_S) + \alpha \int_{-\infty}^{\infty} C_{i+1}(Z_S - s) f(s) ds\right\} \\ \triangleq &\left\{\min_{Z_B \geq z} G_{Bi}(z, Z_B)\right\} \wedge \left\{\min_{Z_S \leq z} G_{Si}(z, Z_S)\right\}, \end{aligned} \tag{4}$$

where $C_{n+1}(x) = C_0(x)$, for all $x \in \mathbb{R}$.

This formulation differs from the usual formulation for an infinite horizon inventory control problem. Although the remainder of the planning horizon after any ith adjustment opportunity looks the same, the remainder of the planning horizon after any two *different* adjustment opportunities, say the i_1th and i_2th adjustment opportunities, $i_1 \neq i_2$, looks different. However, the cycle from any ith adjustment opportunity to the next ith adjustment opportunity repeats again and again towards the end of the planning horizon. Therefore, (3)-(4) describe the iterative relationship among the $C_i(\cdot)$'s, $i = 0, 1, \ldots, n$.

Given this formulation, we can show that the buy-up-to or sell-down-to policy is still optimal for the infinite horizon case. Let Z_{Bi}^* and Z_{Si}^* be the optimal buy-up-to and sell-down-to inventory levels at the start of any subperiod i. In general, little can be said about the relationship between Z_{Bi}^* and Z_{Si}^*, $i = 0, 1, \ldots, n$. However, if $P_B = w$, we have the following result:

Proposition 3.3 *If $P_B = w$, then $Z_{B0}^* = Z_{B1}^* = \cdots = Z_{Bn}^* = F^{-1}\left(\frac{p-(1-\alpha)w}{h+p}\right)$.*

This result is somewhat counter intuitive. In our model, even when $P_B = w$, the regular order opportunities are one-way adjustments, while the adjustment opportunities allow for two-way adjustments. Therefore, the optimal buy-up-to levels are not necessarily equal, as would be the case in a traditional infinite horizon model. However, the fact that $Z_{Si}^* \geq Z_{Bi}^*$ eliminates the impact of the downward adjustment opportunities on the optimal buy-up-to levels. Therefore, if the buyer starts the planning horizon with inventory less than or equal to Z_{B0}^*, the buyer will never sell at an adjustment opportunity. Thus, the order-up-to, buy-up-to or sell-down-to inventory policy will reduce to an order-up-to policy with a constant order-up-to level.

4 Value of the Adjustment Contract to the Buyer

Clearly, the additional flexibility provided by the adjustment contract should not increase the expected costs at the buyer. In this section, we demonstrate that, under certain conditions, the expected cost for the model with the adjustment contract will be strictly less than for the model with no adjustment contract. In addition, we show that the expected cost will be strictly decreasing in the number adjustment opportunities.

4.1 Model with Adjustment vs Model without Adjustment

We consider a single period model and compare the total expected cost with adjustments, $C(x)$, to the total expected cost for a similar single period model without adjustments, $C_{NA}(x)$. Note that for this single period model, the subscripts for periods can be removed. The analysis below can easily be extended to study the multi-period model.

In order to compare the costs for these two models, we need to ensure that the methods used to assess costs are the same. Note that the holding and shortage penalty costs for the model with adjustments are assessed at the end of each subperiod, while the holding and penalty costs for a model without adjustments (i.e., a standard single period inventory model) are usually assessed at the end of the period. In order to ensure a fair comparison, we will modify the standard single period (no adjustment) model to charge holding and shortage penalty costs at the end of each subperiod. Therefore, the no adjustment model is a special case of the adjustment model with $P_{Bi} = \infty$ and $P_{Si} = -\infty$. Let Y^*_{NA} and Y^* denote the optimal order-up-to levels for the no adjustment and adjustment models, respectively. Comparing $C_{NA}(x)$ to $C(x)$, assuming the optimal adjustment policy is used, we obtain the following result.

Proposition 4.1

$C(x) < C_{NA}(x)$ *if and only if* $P\left\{x \vee Y^*_{NA} - \sum_{k=1}^{i} D_k \in (\infty, Z^*_{Bi}) \bigcup (Z^*_{Si}, \infty),\ \textit{for some } i\right\} > 0.$

Note that the above probability condition implies that, with positive probability, some adjustment opportunity(s) will be utilized. This proposition says that the adjustment contract always brings positive expected savings to the buyer when compared to a contract with no adjustment opportunities, as long as with a positive probability the subperiod demands will be high enough or low enough such that an upward or downward adjustment would be used at least for once. For example, if the support of the subperiod demands is $[0, \infty)$, each upward adjustment opportunity will be used with a positive probability. Furthermore, if the buying prices are higher or the selling prices are lower, the buyer is less likely to make use of the adjustment opportunities (i.e., the probability condition above is less likely to hold).

4.2 Impact of the Number of Adjustment Opportunities

We next study the impact of the number of adjustment opportunities in a single period. Recall that n is the number of the adjustment opportunities per period. We study the impact on the expected cost of having one more adjustment opportunity, which we refer to as an "extra adjustment". With this extra adjustment, there are $n+1$ adjustment opportunities. Let $C^{extra}(x)$ denote the minimum expected cost for the model with an extra adjustment opportunity. Suppose the extra adjustment occurs in subperiod i. Let D_{i1} denote the demand in subperiod i before the extra adjustment and Z^*_{li1}, $l = B, S$, the optimal adjustment level for the extra adjustment opportunity. The comparison between $C^{extra}(x)$ and $C(x)$ can be analyzed similarly as the comparison between $C_{NA}(x)$ and $C(x)$ in the previous section.

Proposition 4.2
$C^{extra}(x) < C(x)$ *if and only if* $P\left\{x \vee Y^* - \sum_{k=1}^{i-1} D_k - D_{i1} \in (-\infty, Z^*_{Bi1}) \bigcup (Z^*_{Si1}, \infty)\right\} > 0.$

The proposition implies that the more adjustment opportunities the supplier provides, the greater the expected savings at the buyer if, with positive probability, that the extra adjustment will be utilized. In practice, more adjustment opportunities would likely be associated with less favorable adjustment prices. Therefore, the buyer would need to trade-off this additional cost with the benefits of having more adjustment opportunities.

Finally, the computational study presented in Section 6 considers the *marginal* benefit to the buyer of having more adjustment opportunities. The study indicates that the marginal benefit decreases in the number of opportunities. Thus, having a small number of adjustment opportunities may be sufficient for the buyer to achieve a large percentage of the maximum benefit, i.e., the benefit achieved when the buyer has infinitely many adjustment opportunities.

5 Model Extensions

In this section, we consider three extensions to the basic problem studied in this paper: (i) fixed costs for regular orders and/or adjustments; (ii) upper bounds on the quantity that can be ordered at any regular order opportunity and the amount of upward or downward adjustment; and (iii) a positive constant lead time for each regular order. Here we provide just a brief overview of the main results. For a more detailed analysis, see Li (2004).

5.1 Fixed Costs

In this section, we consider fixed costs for each regular order and each adjustment, including any costs associated with ordering or returning inventory, e.g., the transportation cost.

For an inventory model with a wholesale price contract only, an (s, S) policy is optimal if there is a fixed order cost. For our single period model with one adjustment opportunity, we will show that an (z_B^*, Z_B^*), (z_S^*, Z_S^*) inventory control policy, similar to an (s, S) policy, is optimal for the adjustment opportunity, where $z_B^* < Z_B^*$, $z_S^* > Z_S^*$. Under this policy, at the adjustment opportunity, the buyer will buy only if his on-hand inventory level is less than z_B^*, and then will order to raise his inventory level up to Z_B^*; the buyer will sell only if his on-hand inventory level is higher than z_S^*, and then will sell to decrease his inventory level to Z_S^*. For the regular order opportunity, we will demonstrate that an (s, S) type of policy is *not* optimal.

For the single period model with a single adjustment opportunity, we can write the DP formulation, including the fixed costs for the regular order and the upward and downward adjustments, denoted by K_O, K_B and K_S, respectively. Let $1_{\{.\}}$ be an indicator function.

$$C(x) = \min_{y \geq x} \left\{ K_O 1_{\{y>x\}} + w(y-x) + L_1(y) + \int_{-\infty}^{\infty} C_1(y-s) f_1(s) ds \right\} \triangleq \min_{y \geq x} H(x, y), \quad (5)$$

where

$$\begin{aligned} C_1(z) &= \left\{ \min_{Z_B \geq z} K_B 1_{\{Z_B > z\}} + P_B(Z_B - z) + L_2(Z_B) \right\} \wedge \left\{ \min_{Z_S \leq z} K_S 1_{\{Z_S < z\}} - P_S(z - Z_S) + L_2(Z_S) \right\} \\ &\triangleq \left\{ \min_{Z_B \geq z} H_B(z, Z_B) \right\} \wedge \left\{ \min_{Z_S \geq z} H_S(z, Z_S) \right\}. \end{aligned} \quad (6)$$

To determine the optimal policy, we need the following definitions:

Definition 5.1 *Let $K \geq 0$ and let $g(x)$ be a differentiable function. We say that $g(x)$ is "left" K-convex if $K + g(x+a) - g(x) - ag'(x) \geq 0$, for all $a > 0$ and all $x \in \mathbb{R}$.*

Definition 5.2 *Let $K \geq 0$ and let $g(x)$ be a differentiable function. We say that $g(x)$ is "right" K-convex if $K + g(x-b) - g(x) + bg'(x) \geq 0$, for all $b > 0$ and all $x \in \mathbb{R}$.*

The definition of "left" K-convexity is the same as the definition of K-convexity used in Scarf (1959). This "left" K-convexity is used to guarantee that the change in $g(\cdot)$ from any local maximum to the global minimum is less than K, where the local maximum is to the left of the global minimum. This condition is sufficient to prove the optimality of an (s, S) type of inventory policy. In contrast, the "right" K-convexity is used to guarantee that the change in $g(\cdot)$ from any local maximum to the global minimum is less than K, where the local maximum is to the right of the global minimum. This condition is sufficient to prove the optimality of a (z_S^*, Z_S^*) type of inventory (return) policy, where $z_S^* > Z_S^*$. This policy says that the inventory level will be reduced to Z_S^* via returns if and only if the inventory level is larger than z_S^*. We can now present the following results:

Proposition 5.3 *For the single period, single adjustment model with fixed costs for adjustments, a (z_B^*, Z_B^*), (z_S^*, Z_S^*) policy is optimal, where z_B^* and z_S^* satisfy:*

$$\begin{aligned} K_B + P_B(Z_B^*) + L_2(Z_B^*) &= P_B(z_B^*) + L_2(z_B^*), \ \ i.e., \ \ G_B(z_B^*, Z_B^*) = G_B(z_B^*, z_B^*), \\ K_S + P_S(Z_S^*) + L_2(Z_S^*) &= P_S(z_S^*) + L_2(z_S^*), \ \ i.e., \ \ G_S(z_S^*, Z_S^*) = G_S(z_S^*, z_S^*), \end{aligned} \tag{7}$$

and $Z_B^* = F_2^{-1}\left(\frac{p-P_B}{h_2+p}\right)$ *and* $Z_S^* = F_2^{-1}\left(\frac{p-P_S}{h_2+p}\right)$.

Next, we determine whether an (s, S)-type policy is optimal at the regular order opportunity:

Proposition 5.4 *Under the (z_B^*, Z_B^*), (z_S^*, Z_S^*) policy, $w(y-x)+L_1(y)+\int_{-\infty}^{\infty} C_1(y-s)f_1(s)ds$ is not "left" K_O-convex.*

Thus, an (s, S)-type order policy is not generally optimal for the regular order opportunity. By induction, it is also easy to see that an (s, S)-type order policy is not optimal for any regular order opportunity in a multi-period model. In addition, for a multi-period, multi-adjustment model, except at the very last adjustment opportunity, a (z_B^*, Z_B^*), (z_S^*, Z_S^*) policy is not optimal.

5.2 Upper Bounds on the Order and Adjustment Quantities

We next consider upper bounds on the quantity that can be ordered and/or the size of the upward and downward adjustments. Let $L_O > 0$, $L_B > 0$ and $L_S > 0$ denote the upper bounds on the order quantity, the upward adjustment quantity and the downward adjustment quantity, respectively. The optimal policy is a modified buy-up-to or sell-down-to policy: If the quantity needed to raise the inventory level up to Z_B^* is greater than the maximum upward adjustment quantity, i.e., $Z_B^* - z > L_B$, then the buyer should buy L_B, the maximum amount. Similarly, if $z - Z_S^* > L_S$, it is optimal to sell L_S, the maximum amount. For the regular order opportunity, if the on-hand inventory level is less than $Y^* - L_O$, order L_O; if the on-hand inventory level is between $Y^* - L_O$ and Y^*, order to raise the inventory level up to Y^*; otherwise, do not order.

5.3 Positive Lead Time for Regular Orders

We next consider a positive constant lead time for regular orders. We assume the adjustment opportunities have zero lead time, part of the flexibility provided by the adjustment contract. We first discuss the difficulty in analyzing this problem by comparing it to a standard model with positive order lead times. Karlin and Scarf (1958) proved that an order-up-to inventory policy is optimal for a buyer facing a wholesale price contract with a constant lead time. They do so by formulating the problem in terms of the inventory position (on-hand plus on-order inventory), rather than on-hand inventory. In their case, the order decision at the start of each period impacts the on-order inventory level, not the on-hand inventory level. However, in our model we have two contracts: a wholesale price contract and an adjustment contract. Decisions made at regular order opportunities impact the on-order inventory level, while the decisions made at the adjustment opportunities impact the on-hand inventory levels. Therefore, we cannot simply define our system state as either inventory position or on-hand inventory.

The structure of the optimal policy for the positive lead time case is similar to, but more complicated than, that for the zero lead time case. At each adjustment opportunity, a buy-up-to and sell-down-to policy is still optimal. However, the optimal buy-up-to and sell-down-to levels depend on each individual regular order that has been placed, but has not yet arrived, by the adjustment time. For the regular orders, we can derive an expression for the optimal order quantity. However, the optimal policy is not an order-up-to policy with fixed order-up-to level.

6 Computational Study

In this section, we present a computational study to supplement the analytical results provided in this paper. We explore the benefit to the buyer of using the adjustment contract and consider the sensitivity of the optimal inventory policy parameters. In addition, we consider how the number of adjustment opportunities and the timing of those opportunities affect the buyer's expected cost. All of the figures discussed in this section are contained in Appendix II.

6.1 Experimental Design

We consider both single- and multi-period models (i.e., $m = 1$ and $m > 1$) with multiple (n) adjustment opportunities. We assume equal length subperiods and equal length periods. We model demand using a homogenous Poisson process with constant arrival rate λ. If the length of a subperiod is $t > 0$, the demand during that subperiod follows $Poisson(\lambda t)$. We assume that the buyer starts the planning horizon, $[0, T]$, with zero inventory, there is a constant unit holding cost, h, and constant buying and selling prices, P_B and P_S. We set $w = 1$. We used a complete factorial design, i.e., we performed experiments for all the combinations of the problem parameters m, n, P_B, P_S, h and p. We show only selected, but the most typical, results.

6.2 Value of the Adjustment Contract to the Buyer

We first study how the value of the adjustment contract to the buyer will vary as a function of the problem parameters, i.e., n, P_B, P_S, h, p and m. We define the percentage value of the adjustment contract for the buyer (PVB) to be the percentage difference between the buyer's minimum expected cost if he can procure inventory only under the wholesale price contract and the buyer's minimum expected cost if he also participates in an adjustment contract. Since we assume the buyer starts the planning period with zero inventory, we have $PVB = \frac{C_{NA}(0) - C(0)}{C_{NA}(0)}$.

Figure 1 shows the typical behavior of PVB as a function of the number of the adjustments, n, i.e., PVB increases with n for all h and p. The concavity of the PVB curve indicates that the buyer's marginal benefit from an additional adjustment opportunity diminishes as more adjustment opportunities are provided. Thus, having a small number of adjustment opportunities may be sufficient to capture most of the benefits of the adjustment contract.

Figure 1 shows that when P_B is large relative to P_S (e.g., $P_B = 1.9$, $P_S = 0.1$), the marginal benefit diminishes more quickly than when P_B is small relative to P_S (e.g., $P_B = 1.1$, $P_S = 0.9$). Note that the buyer will use the adjustment contract less often in the first case than in the second case. Therefore, in the first case, PVB is relatively small and its marginal value diminishes more rapidly. Figure 1 also shows the impact of the holding and penalty costs (h and p) on PVB. The figure indicates that the adjustment contract is more beneficial to the buyer when his holding cost is large (e.g., expensive items) and his penalty cost is large (e.g., unsatisfied demands are costly). The flexibility provided by the adjustment contract helps the buyer to balance his over-stocking and under-stocking costs, thus reducing total inventory cost.

Figure 2 shows the impact of P_B and P_S on PVB. Generally, for each P_S, if P_B increases, PVB decreases and the marginal PVB decreases; for each P_B, if P_S increases, PVB increases and the marginal PVB also increases. Thus, as expected, the buyer prefers a lower buying price and a higher selling price. Figure 2 also shows that, when P_B is small, increasing P_S does not have much impact on PVB. However, when P_B is large, increasing P_S has a significant impact on PVB, i.e., an increase in P_S is more beneficial to the buyer when P_B is large. Note that when P_B is large, the buyer will use the upward adjustment option less often and thus the regular order quantity will be higher. Since the buyer is more likely to have excess inventory, the benefit from using the adjustment opportunities is obtained largely from the downward adjustments. Thus, increasing P_S has a greater impact on PVB when P_B is large.

Figure 3 shows how the number of periods (m) affects the percentage value of the adjustment contract for the buyer. As can be seen from the figure, PVB is convex and decreasing in the number of periods. As the number of the periods increases, the buyer's benefit from using the adjustment contract decreases and levels off to a value that is generally significantly greater than zero. For example, when $m = 7$, for all experiments conducted, PVB is still approximately 10%, representing considerable savings over the wholesale price contract.

6.3 Sensitivity Analysis on Optimal Inventory Policy Parameters

We next examine the relationship between the optimal buy-up-to levels, $Z^*_{B1}, \ldots, Z^*_{Bn}$, and the optimal sell-down-to levels, $Z^*_{S1}, \ldots, Z^*_{Sn}$, for various P_B, P_S, h and p. Figure 4 shows the impact of the buying price on the optimal buy-up-to levels throughout the period, where i represents the adjustment opportunity, $i = 1, \ldots, n$. For a given P_B, Z^*_{Bi} decreases as we get later in the period, i.e., as i increases. Although not shown, our numerical results indicate that Z^*_{Si} decreases as i increases. Thus, the buyer should store less inventory later in the period.

Figure 4 also indicates that, as P_B increases, Z^*_{Bi} decreases or stays unchanged, for all i. Intuitively, as P_B increases, the buyer should buy less at each adjustment opportunity. Also, an increase in P_B has a greater impact on the optimal buy-up-to levels toward the end of the period than at the start of the period. Similarly, although not shown here, as P_B increases, Z^*_{Si} increases or stays unchanged, for all i. However, the impact of P_B on Z^*_{Si} is quite small, particularly towards the end of the period. Recall that, as shown in Section 3.2, for the last adjustment opportunity, the sell-down-to level depends only on the selling price and not on the buying price. Similarly, the buy-up-to level depends only on the buying price and not on the selling price. For the other adjustment opportunities, the buying price mainly impacts the buy-up-to levels and the selling price mainly impacts the sell-down-to levels.

Figure 5 shows the impact of the selling price on the optimal sell-down-to levels throughout the period. For each given P_S, Z^*_{Si} decreases as we get later in the period, i.e., as i increases. Although not shown, our numerical results also indicate that Z^*_{Bi} decreases as i increases. The results are similar to those in Figure 4. The figure also indicates that, as P_S increases, Z^*_{Si} decreases, for all i. Intuitively, as P_S increases, the buyer should sell more at each adjustment opportunity to take advantage of a larger selling price. An increase in P_S has a larger impact on the optimal sell-down-to levels toward the end of the period than at the start of the period. Similarly, although not shown, as P_S increases, Z^*_{Bi} remains essentially unchanged, for all i. Finally, although not shown here, we also studied the impact of the buying and selling prices on the optimal order-up-to level, Y^*. As expected, Y^* is increasing in both.

Figures 6 and 7 show the impact of the unit penalty cost relative to the unit holding cost, p/h, on the optimal inventory policy parameters. Figure 6 shows that as p/h increases, Z^*_{Bi} increases for all i. Figure 7 shows that as p/h increases, Z^*_{Si} increases for all i. These results imply that as the unit penalty cost becomes more dominant, the buyer should keep more inventory by buying more and selling less at each adjustment opportunity.

6.4 Effect of the Timing of a Single Adjustment Opportunity

Finally, we study how the timing of a single adjustment opportunity impacts the percentage value of the adjustment contract to the buyer. We study a single period model (with period length T) and consider three alternative timings ($t \in (0, T)$) for the adjustment opportunity: (1) early adjustment, i.e., $t = (1/4)T$, (2) middle adjustment, i.e., $t = (1/2)T$, and (3) late adjustment, i.e., $t = (3/4)T$. We let h_1 and h_2 represent the unit holding cost for the first and second subperiods, respectively. For the three adjustment timings, we set h_1 and h_2 as follows: (1) $h_1 = 0.05$ and $h_2 = 0.15$, (2) $h_1 = 0.1$ and $h_2 = 0.1$, and (3) $h_1 = 0.15$ and $h_2 = 0.05$.

Figure 8 shows that, as the timing moves from early to late, the percentage value of the adjustment contract increases from 1.1% to 4.5%. Thus the buyer prefers a later adjustment opportunity. While this is the typical result, it is not always the case. For 6% of the cases considered, the buyer preferred to have the adjustment opportunity in the middle of the period.

Figure 9 indicates that, as the adjustment opportunity moves later in the period, the optimal buy-up-to level, Z^*_B, and sell-down-to level, Z^*_S, decrease, while the optimal order-up-to level, Y^*, decreases and then increases. The impact of the timing of the adjustment opportunity on Z^*_B and Z^*_S is larger than on Y^*, i.e., Z^*_B and Z^*_S are sensitive to the timing of the adjustment opportunity, but Y^* is not. As the adjustment opportunity moves later in the period, the mean and the variance of the demand in the second subperiod decrease. Thus, the buyer should keep less inventory for this subperiod by decreasing the buy-up-to and sell-down-to levels.

7 Conclusions and Future Directions

In this paper, we studied the problem faced by a buyer who procures inventory using a portfolio of supply contracts consisting of a wholesale price contract and an adjustment contract. For a multi-period model, we proved that the optimal inventory policy combines a standard order-up-to policy for the wholesale contract with a buy-up-to, sell-down-to policy for the adjustment contract. In order to quantify the benefits of the adjustment contract to the buyer, we demonstrated that using this portfolio of contracts reduces the buyer's total expected costs for a range of buying and selling prices. We also demonstrated that the buyer's total expected costs are strictly decreasing in the number of adjustment opportunities. We performed a computational study to study the impact of the problem parameters on the optimal policy.

This paper provides several key contributions to the literature. First, there has been little research on supply contracts that allow for two-way (i.e., both upward and downward) adjustments to the inventory level. Such adjustment contracts provide clear advantages to the buyer relative to less flexible supply contracts such as the buyback or quantity flexibility contracts. In addition, the adjustment contract provides potential benefits for the supplier, i.e., under such a contract the supplier may be able to charge a high buying price and offer a low buy-back price at the adjustment times. Thus, there is a practical role for supply contracts that allow for such two-way adjustments. Second, the adjustment contract we propose also provides clear advantages to the buyer relative to spot market procurement, which is often risky (i.e., buying and selling prices are random) and may be subject to limited liquidity. Third, while our paper shares some similarities with the literature on procurement with spot market participation, we differ from the literature in some key ways. In particular, we allow the buying and selling prices to differ, which most of the current literature does not (Goel and Gutierrez (2006) are one exception). In addition, we consider a number of critical model extensions that are generally not considered in the literature on spot market procurement, e.g., lead times that may differ for the different procurement contracts, fixed order costs, and analysis of the infinite horizon problem.

Finally, there are a number of future directions for this research. One important extension is to study whether adjustment contracts can be used to achieve supply chain coordination. While this paper provides a key first step by analyzing the buyer's optimal decisions under an adjustment contract and by demonstrating the value of the adjustment contract to the buyer, a key issue that needs to be addressed is the benefit to the supplier who offers the adjustment contract. Clearly, by offering adjustment opportunities, the seller may incur higher production costs. However, as noted above, the supplier may be able to profit from the adjustment contracts through appropriate price setting. Moreover, a seller who serves many buyers could potentially take advantage of risk pooling, i.e., may use returns from one buyer to satisfy orders from another buyer. Li and Ryan (2007) discuss the benefits of the adjustment contract to the supplier and consider supply chain coordination through adjustment contracts.

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9 Appendix I: Proofs

Proof of Proposition 3.1: For notational convenience we prove the result for the single period case; the same approach can be used for the multiple period case. We first prove the optimality of the policy for the last subperiod. We show that for any z, $G_{Bn}(z, Z_B)$ is a convex function of Z_B and is minimized when $Z_B = Z^*_{Bn}$. To see this, we have $\frac{\partial}{\partial Z_B} G_{Bn}(z, Z_B) = 0$, which implies $Z^*_{Bn} = F^{-1}_{n+1}\left(\frac{p-P_{Bn}}{h_{n+1}+p}\right) > 0$, and $\frac{\partial^2}{\partial Z_B^2} G_{Bn}(z, Z_B) \geq 0$. We can also show that for any z, $G_{Sn}(z, Z_S)$ is a convex function of Z_S and is minimized when $Z_S = Z^*_{Sn} = F^{-1}_{n+1}\left(\frac{p-P_{Sn}}{h_{n+1}+p}\right)$. Furthermore, $p > P_{Bn} \geq P_{Sn}$ implies that $Z^*_{Sn} \geq Z^*_{Bn} > 0$. We then have the following expressions for $G_{ln}(z) \triangleq \min_{Z_l \leq z} G_{ln}(z, Z_l)$, where $l = B, S$:

$$G_{Bn}(z) = \begin{cases} G_{Bn}(z, Z^*_{Bn}) = P_{Bn}(Z^*_{Bn} - z) + L_{n+1}(Z^*_{Bn}) & \text{if } z < Z^*_{Bn}, \\ G_{Bn}(z, z) = L_{n+1}(z) & \text{if } z \geq Z^*_{Bn}, \end{cases} \tag{8}$$

$$G_{Sn}(z) = \begin{cases} G_{Sn}(z, z) = L_{n+1}(z) & \text{if } z \leq Z^*_{Sn}, \\ G_{Sn}(z, Z^*_{Sn}) = -P_{Sn}(Z^*_{Sn} - z) + L_{n+1}(Z^*_{Sn}) & \text{if } z > Z^*_{Sn}. \end{cases} \tag{9}$$

Utilizing the above results, together with the convexity of $G_{ln}(z, Z_l)$, $l = B, S$, we have:

$$C_n(z) = G_{Bn}(z) \wedge G_{Sn}(z) = \begin{cases} P_{Bn}(Z^*_{Bn} - z) + L_{n+1}(Z^*_{Bn}) & \text{if } z < Z^*_{Bn}, \\ L_{n+1}(z) & \text{if } Z^*_{Bn} \leq z \leq Z^*_{Sn}, \\ -P_{Sn}(Z^*_{Sn} - z) + L_{n+1}(Z^*_{Sn}) & \text{if } z > Z^*_{Sn}. \end{cases} \tag{10}$$

The expression for $C_n(z)$ indicates that the buy-up-to Z^*_{Bn} or sell-down-to Z^*_{Sn} policy is optimal. We can show that $C_n(z)$ is convex by checking the continuity of the first derivative and the sign of the second derivative. The continuity of the first derivative at points $z = Z^*_{Bn}$ and $z = Z^*_{Sn}$

is due to the first order conditions that Z_{Bn}^* and Z_{Sn}^* satisfy.

We complete the proof by induction. We assume that the proposition holds true for subperiod $i \leq n$ and analyze subperiod $i-1$. First, the assumption that $C_i(\cdot)$ is a convex function implies that $G_{l(i-1)}(z, Z_l)$, is convex function of Z_l, $l = B, S$, for any z. Next, we let $Z_{l(i-1)}^*$, $l = B, S$, denote the solution to $\frac{\partial}{\partial Z_l} G_{l(i-1)}(z, Z_l) = 0$ and thus it satisfies:

$$P_{l(i-1)} + (h_i + p)F_i(Z_{l(i-1)}^*) - p + \int_0^\infty C_i'(Z_{l(i-1)}^* - s) f_i(s) ds = 0, l = B, S, \tag{11}$$

where $C_i'(\xi - s) \triangleq \frac{d}{dz} C_i(z-s)\Big|_{z=\xi}$, for any $\xi \in \mathbb{R}$. We can easily show that $Z_{S(i-1)}^* \geq Z_{B(i-1)}^* > 0$ using the fact that $F_i(\cdot)$ and $C_i'(\cdot)$ are both non-decreasing function and $C_i'(\cdot) \in [-P_{Si}, -P_{Bi}]$.

Following (8)-(9), we next derive expressions for $G_{B(i-1)}(z)$ and $G_{S(i-1)}(z)$:

$$G_{B(i-1)}(z) = \begin{cases} P_{B(i-1)}(Z_{B(i-1)}^* - z) + L_i(Z_{B(i-1)}^*) + \int_0^\infty C_i(Z_{B(i-1)}^* - s) f_i(s) ds & \text{if } z \leq Z_{B(i-1)}^*, \\ L_i(z) + \int_0^\infty C_i(z - s) f_i(s) ds & \text{if } z \geq Z_{B(i-1)}^*. \end{cases} \tag{12}$$

$$G_{S(i-1)}(z) = \begin{cases} L_i(z) + \int_0^\infty C_i(z - s) f_i(s) ds & \text{if } z \leq Z_{S(i-1)}^*, \\ -P_{S(i-1)}(z - Z_{S(i-1)}^*) + L_i(Z_{S(i-1)}^*) + \int_0^\infty C_i(Z_{S(i-1)}^* - s) f_i(s) ds & \text{if } z \geq Z_{s(i-1)}^*. \end{cases} \tag{13}$$

Following the arguments for the convexity of $G_{Bn}(z)$ and $G_{Sn}(z)$, we can easily obtain the convexity for $G_{B(i-1)}(z)$ and $G_{S(i-1)}(z)$. Finally, using (12)-(13) we rewrite $C_{i-1}(z)$:

$$C_{i-1}(z) = \begin{cases} P_{B(i-1)}(Z_{B(i-1)}^* - z) + L_i(Z_{B(i-1)}^*) + \int_0^\infty C_i(Z_{B(i-1)}^* - s) f_i(s) ds & \text{if } z \leq Z_{B(i-1)}^*, \\ L_i(z) + \int_0^\infty C_i(z - s) f_i(s) ds & \text{if } z \in [Z_{B(i-1)}^*, Z_{S(i-1)}^*], \\ -P_{S(i-1)}(z - Z_{S(i-1)}^*) + L_i(Z_{S(i-1)}^*) + \int_0^\infty C_i(Z_{S(i-1)}^* - s) f_i(s) ds & \text{if } z \geq Z_{s(i-1)}^*, \end{cases}$$

which obviously implies the optimality of the buy-up-to or sell-down-to policy. The convexity of $C_{i-1}(z)$ with respect to z can be proved by using the same arguments we used for $C_n(z)$. ∎

Proof of Proposition 3.2: We will prove (i)-(iii) in order. First, note from Proposition 3.1 that Z_B^* only depends on P_B and Z_S^* only depends on P_S. Therefore, Z_B^* and Z_S^* will change only if P_B and P_S change, respectively. Note that Y^* satisfies the corresponding first order condition, i.e., $w - p + (h_1 + p)F_1(Y^*) + +(h_2 + p)\int_{Z_B^*}^{Z_S^*} f_2(s)F_1(Y^* - s)ds = 0$. Differentiating both sides of the equation with respect to P_B and P_S, we find that Y^* does not decrease as P_B increases or as P_S increases. Furthermore, as P_B increases, it is clear that $Z_B^* = F_2^{-1}\left(\frac{p - P_B}{h_2 + p}\right)$

does not increase. As P_S increases, $Z_S^* = F_2^{-1}\left(\frac{p-P_S}{h_2+p}\right)$ does not increase. Thus (ii) and (iii) are proved. It is easy to see that Y^* does not increase as w increases and thus (i) is proved. ∎

Proof of Proposition 3.3: We first prove that the optimality of the buy-up-to or sell-down-to policy preserves for the infinite horizon case. Suppose the optimal policy is of any format. It is easy to show that $|\int_{-\infty}^{\infty} C'_{i+1}(z-s)f(s)ds| \leq P_B$ for all i and $z \in \mathbb{R}$. This implies that $\lim_{m\to\infty} \alpha^m \int_{-\infty}^{\infty} C'_m(z-s)f(s)ds = 0$ for all $z \in \mathbb{R}$, where $C_m(\cdot)$ represents the minimum expected cost for the remainder horizon after the $(m+1)$st inventory decision point, which can be a regular order or an adjustment order. Using this result, we can show that $\int_{-\infty}^{\infty} C'_{i+1}(z-s)f(s)ds$ is convex and thus the optimality of the buy-up-to and sell-down-to policy.

To prove the proposition, we first note that Z_{B0}^* and Z_{Bi}^*, $i = 1, \ldots, n$, satisfy:

$$\begin{cases} w + L'(Z_{B0}^*) + \alpha \int_{-\infty}^{\infty} C'_1(Z_{B0}^* - s)f(s)ds = 0 \\ P_B + L'(Z_{Bi}^*) + \alpha \int_{-\infty}^{\infty} C'_{i+1}(Z_{Bi}^* - s)f(s)ds = 0 \end{cases} \tag{14}$$

The solution, $Z_{B0}^*, \ldots, Z_{Bn}^*$, to equation (14) is unique. It suffices to show that $Z_{B0}^* = Z_{B1}^* = \cdots = Z_{Bn}^* = F^{-1}\left(\frac{p-(1-\alpha)w}{h+p}\right)$ satisfies (14). If $P_B = w$, we have for $i = 0$: $C'_0(z) = -w$ if $z \leq Z_{B0}^*$; $L'(z) + \alpha \displaystyle\int_{-\infty}^{\infty} C'_1(z-s)f(s)ds$ if $z \geq Z_{B0}^*$. For $i = 1, \ldots, n$, $C'_i(z) = -w$ if $z \leq Z_{Bi}^*$; $L'(z) + \alpha \displaystyle\int_{-\infty}^{\infty} C'_{i+1}(z-s)f(s)ds$ if $Z_{Bi}^* \leq z \leq Z_{Si}^*$; $-P_S$ if $z \geq Z_{Si}^*$, where $C'_{n+1}(z-s) = C'_0(z-s)$. Now we can verify that $Z_{B0}^* = Z_{B1}^* = \cdots = Z_{Bn}^* = F^{-1}\left(\frac{p-(1-\alpha)w}{h+p}\right)$ satisfies (14). ∎

Proof of Propositions 4.1 and 4.2: We prove Proposition 4.1; Proposition 4.2 can be shown similarly. Utilizing the buy-up-to or sell-down-to policy, we see that $C(x) = C_{NA}(x)$ if and only if the optimal action at each adjustment opportunity is neither buy nor sell. In other words, $C(x) = C_{NA}(x)$ if and only if $P\left\{\max\{x, y_{NA}^*\} - \sum_{k=1}^{i} D_k \in [Z_{Bi}^*, Z_{Si}^*], \text{ for all } i\right\} = 1$. Since $C(x) \leq C_{NA}(x)$, the result follows. ∎

Proof of Proposition 5.3: Using "left" K_B-convexity and "right" K_S-convexity, we have:

$$G_B(z) = \begin{cases} G_B(z, Z_B^*) = K_B + P_B(Z_B^* - z) + L_2(Z_B^*) & \text{if } z < z_B^*, \\ G_B(z, z) = L_2(z) & \text{if } z \geq z_B^*. \end{cases}$$

$$G_S(z) = \begin{cases} G_S(z, z) = L_2(z) & \text{if } z \leq z_S^*, \\ G_S(z, Z_S^*) = K_S - P_S(z - Z_S^*) + L_2(Z_S^*) & \text{if } z > z_S^*, \end{cases} \tag{15}$$

which, together with further comparison, implies the following:

$$C_1(z) = \begin{cases} G_B(z, Z_B^*) = K_B + P_B(Z_B^* - z) + L_2(Z_B^*) & \text{if } z < z_B^*, \\ G_B(z, z) = G_S(z, z) = L_2(z) & \text{if } z_B^* \le z \le z_S^*, \\ G_S(z, Z_S^*) = K_S - P_S(z - Z_S^*) + L_2(Z_S^*) & \text{if } z > z_S^*, \end{cases} \tag{16}$$

where $z_B^* \le z \le z_S^*$ is feasible due to the fact that $z_B^* < z_S^*$, which is implied by the facts: $Z_B^* < Z_S^*$, $z_B^* < Z_B^*$, and $z_S^* > Z_S^*$. This concludes the proof. ∎

Proof of Proposition 5.4: (16), together with the definitions of z_B^*, Z_B^*, z_S^* and Z_S^*, implies that $C_1(z)$ is continuous and differentiable. However, $C_1'(z)$, is not continuous when $z = z_B^*$ and $z = z_S^*$. But $\int_{-\infty}^{\infty} C_1(y-s)f_1(s)ds$ is second differentiable everywhere and $\frac{d^2}{dy^2}\int_{-\infty}^{\infty} C_1(y-s)f_1(s)ds = [P_B + L_2'(z_B^*)]\, f_1(y-z_B^*) - [P_S + L_2'(z_S^*)]\, f_1(y-z_S^*) + \int_{y-z_S^*}^{y-z_B^*} L_2''(y-s)f_1(s)ds$, which is continuous in y but may not be positive. Thus, the second derivative of $w(y-x) + L_1(y) + \int_{-\infty}^{\infty} C_1(y-s)f_1(s)ds$, as in (5), over y is continuous in y for any x. The curve $w(y-x)+L_1(y)+\int_{-\infty}^{\infty} C_1(y-s)f_1(s)ds$ as a function of y may have multiple local minimums and maximums.

Next, we will show that "left" K_O-convexity of $C_1(z)$, a sufficient condition for (s, S) policy, cannot be guaranteed, i.e., we will show that when $z \in (Z_S^*, z_S^*)$ and $z + a \in [z_S^*, \infty)$, as a is large enough, we will have $K_O + C_1(z+a) - C_1(z) - aC_1'(z) < 0$. Using (16), we have $K_O + C_1(z+a) - C_1(z) - aC_1'(z) = K_O + K_S + P_S(Z_S^* - z) + L_2(Z_S^*) - L_2(z) - a[P_S + L_2'(z)]$, where $z \in (Z_S^*, z_S^*)$ and $z + a \in [z_S^*, \infty)$. Since $L_2(\cdot)$ is a convex function, $z \in (Z_S^*, z_S^*)$ implies that $L_2'(z) > -P_S$, which implies $[P_S + L_2'(z)] > 0$. Thus, for any fixed $z \in (Z_S^*, z_S^*)$, we have that if $a > \max\left\{z_S^* - z, \frac{K_O+K_S+P_S(Z_S^*-z)+L_2(Z_S^*)-L_2(z)}{P_S+L_2'(z)}\right\}$, then $K_O + C_1(z+a) - C_1(z) - aC_1'(z) < 0$. ∎

Appendix II: Figures

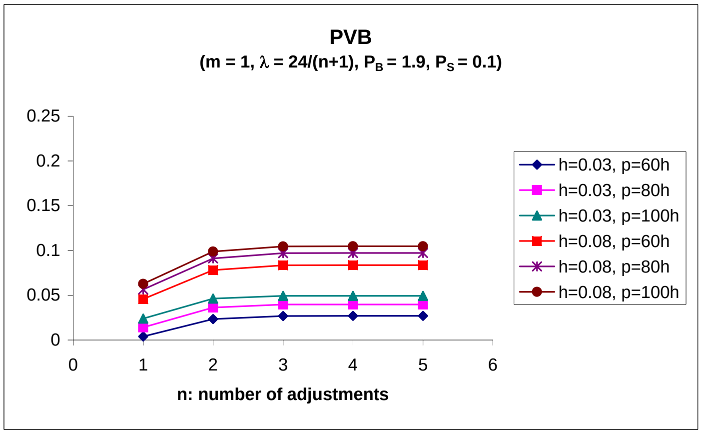

Figure 1a

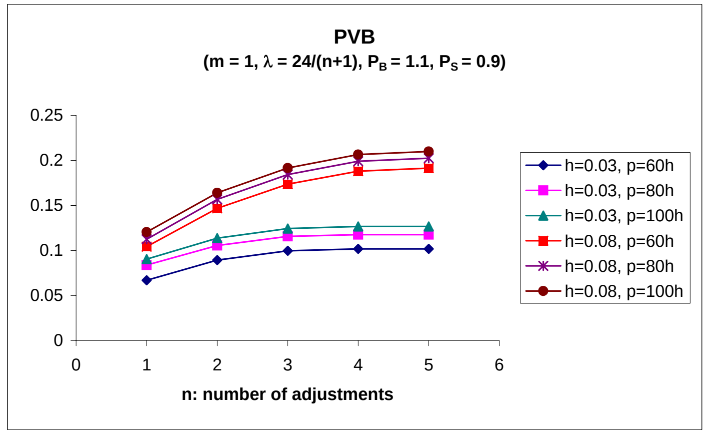

Figure 1b

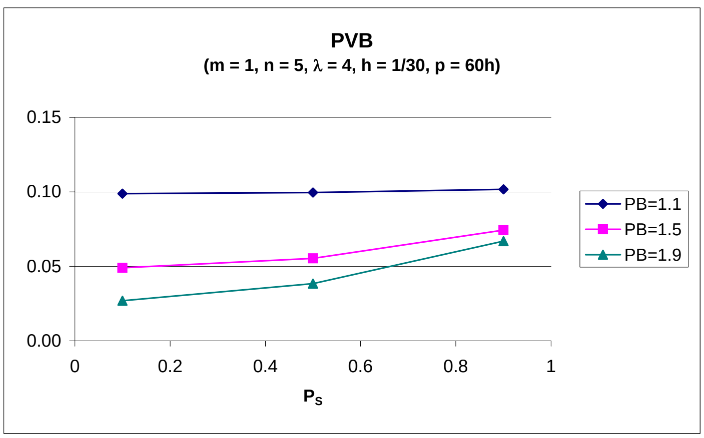

Figure 2

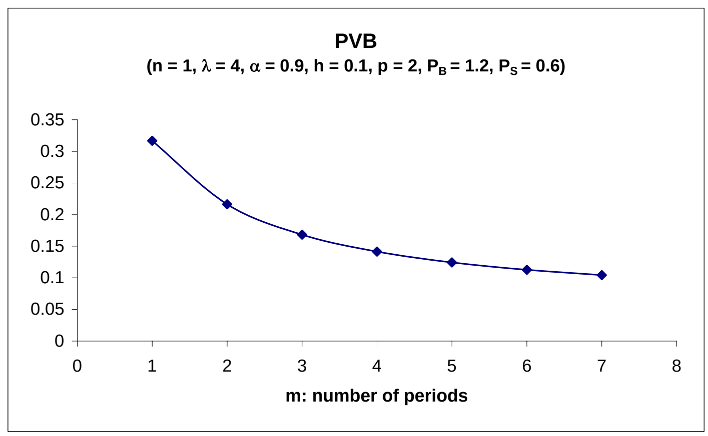

Figure 3

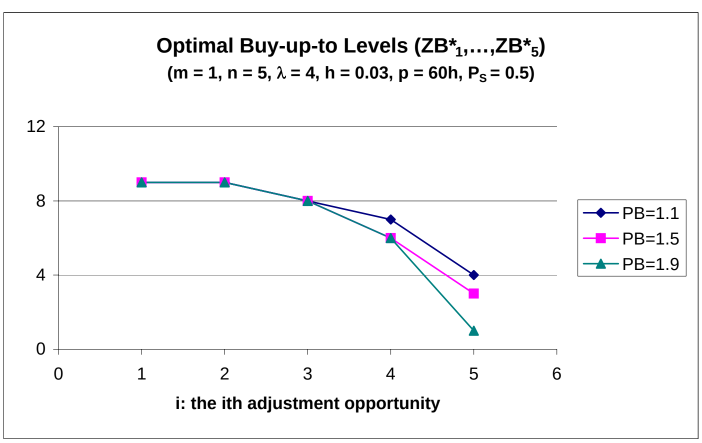

Figure 4

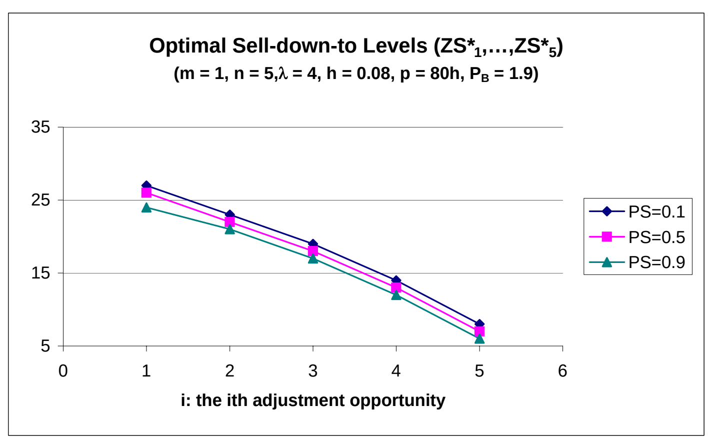

Figure 5

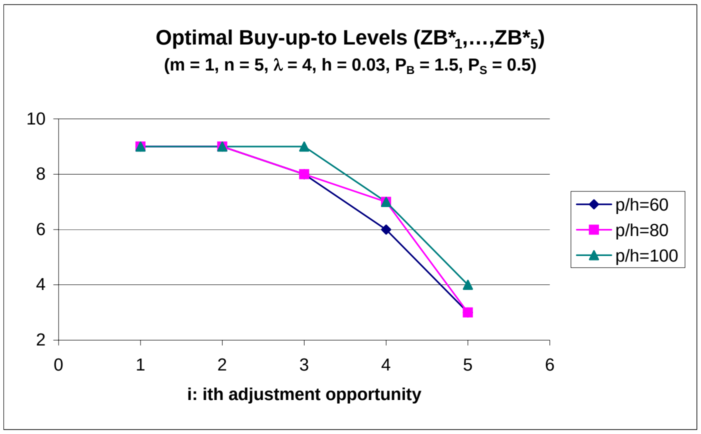

Figure 6

Figure 7

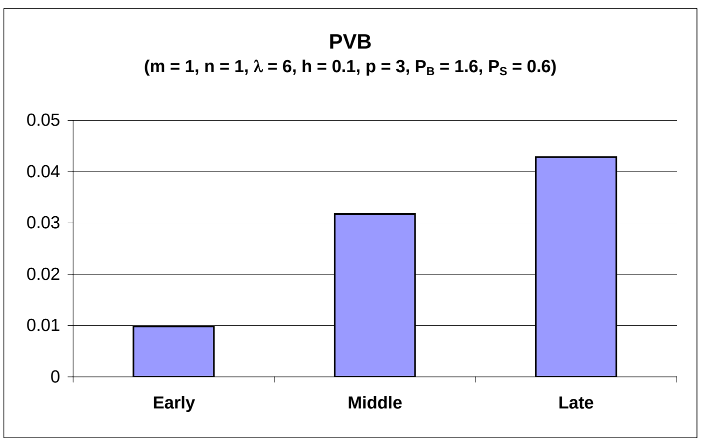

Figure 8

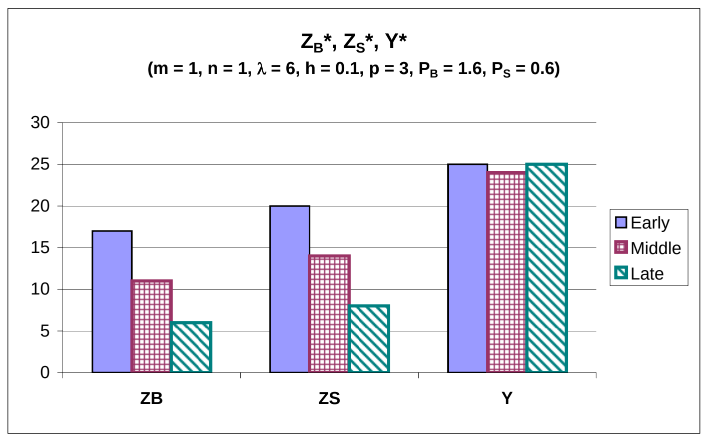

Figure 9